Problem 1. Consider the deflection $u = u(r)$ of the MEMS membrane with a power law permittivity profile $\varepsilon_0 / \varepsilon_2 = r^{-\mu}$ with $\mu > 0$ so that in dimensions $N = 1$ (1-D) or $N = 2$ (2-D disk) we have

$$u_{rr} + \frac{(N-1)}{r} u_r = \frac{A r^\mu}{(1 + \mu)^2} \quad \text{on} \quad 0 < r < 1$$

with $u(1) = 0$ and $u(0) = 0$. \(\star\)

(i) Use the scale invariance technique (coupled to an ODE solver) to plot $|u(0)|$ versus $A$ for $\mu = 0.5$, $\mu = 1.2$, and $\mu = 2.0$ for $N=1$ (slab) and $N=2$ (disk). (Two plots needed: one for $N=1$ the other for $N=2$).

(ii) In the ODE $w(p)$ from scale invariance technique of (i), let $A = \log p$ and $w(p) = e^{8A} v(A)$ for some $B$ to derive the following autonomous ODE for $v(A)$:

$$v'' + \left( N - \frac{2}{3} + \frac{2 \mu}{3} \right) v' + \left( \frac{2 + \mu}{3} \right) \left( N - \frac{4}{3} + \frac{\mu}{3} \right) v = \frac{1}{\sqrt{v}}. \quad \star$$

(iii) By studying the fixed point of $\star$ and its linear stability properties show that:

(A) For $N=1$ and $1 < \mu < \mu_1$, that $A \rightarrow A^*$ monotonically as $|u(0)| \rightarrow 1^-$ for some $A_1$ and $A^*$.

(B) For $N=1$ and $\mu > \mu_1$, show that $A \rightarrow A^*$ with decreasing oscillations as $|u(0)| \rightarrow 1^-$.  

(C) Now let $2 \leq N \leq 7$ (general $N$), show that $A \rightarrow A^*$ with oscillations as $|u(0)| \rightarrow 1^-$ for any $\mu > 0$, show that this may fail (for some $\mu$) if $N = 8$. 


(iv) Notice for \( n = 2 \) (disk) that the solution to \( (\star) \) at the first fold point seems to have a common value of \( u(0) \) for all \( \lambda > 0 \). To explain this theoretically show for \( n = 2 \) that for \( (\star) \) the change of variables \( \gamma^B = \rho \), so \( u(\gamma) \rightarrow \gamma^B \) \( \left[ \gamma^B \right] \) yield the BVP

\[
\ddot{u} + \frac{1}{\rho} \dot{u} = \frac{\Lambda}{u^2}
\]

where \( \Lambda = \left( \frac{\lambda}{2} + 1 \right)^{-2} \lambda \), for some \( B \) to be found. How can this transformation be used for \( n = 2 \) instead of the approach used in (i)?

**Problem 2**

Consider the radially symmetric MEMS BVP for \( u(\gamma) \):

(10 points)

\[
\ddot{u} + \frac{1}{\gamma} \dot{u} = \frac{\Lambda}{(1+u)^2}, \quad 0 < \gamma < 1
\]

\( u(1) = 0 \).

(i) By using scale invariance technique and defining \( S = -u(0) \), plot \( \frac{du}{ds} \) at the first and second fold points. What does this tell us about the eigenvalue of the linearization of \( \frac{u_t}{\gamma} = u_{\gamma \gamma} + \frac{1}{\gamma} u_\gamma - \frac{\Lambda}{(1+u)^2} \) at these two fold points?

(ii) By using scale invariance technique plot \( \int_0^1 \frac{1}{\gamma^B} d\gamma \) versus \( \Lambda \): "the preferred bifurcation diagram." What does this tell us about the stability of the solution branches near the fold point? Explain your answer.
CONSIDER A SPHERICAL DOMAIN AND LET \( u(r) \) SATISFY THE GELFAND PROBLEM

\[
\begin{aligned}
&u'' + \frac{2}{r} u' + \lambda e^u = 0, \\&0 \leq r \leq 1
\end{aligned}
\]

\[ (\star) \]

\[
\begin{aligned}
u'(0) = 0, \\&u(1) = 0.
\end{aligned}
\]

(i) Prove that there is no solution to (\( \star \)) if \( \lambda > \pi^2 e^{-1} \).

(ii) Use a scale invariance technique to plot \( u(0) \) versus \( \lambda \). Numerically compute the value of \( \lambda \) at the first fold point.