Problem 1: Consider the diffusive logistic model for $u = u(x,t)$

$$
\begin{align*}
\partial_t^2 u + \partial_x^2 f(u) &> 0 < 1 \\
\partial_x^2 u(0,t) = u(1,t) &= 0, \quad u(x,0) = u_0(x) > 0.
\end{align*}
$$

(i) Show that the bifurcating solution branch of nontrivial equilibria near $\alpha = \pi/2$ is linearly stable.

**Hint:** The eigenvalue problem is

$$
\begin{align*}
\phi_{xx} + \alpha^2 f'(u) \phi &= \sigma \phi \\
\phi_x(0) = \phi(1) &= 0
\end{align*}
$$

The nontrivial branch of equilibria near $\alpha = \pi/2$ is

$$
u = \varepsilon \cos(\pi x/2)$$

when $\alpha = \frac{\pi}{2} \left[ 1 + \frac{4\varepsilon}{3\pi} + \cdots \right]$, $\varepsilon << 1$. Show that the first eigenvalue of $(x)$ is $\sigma_1 = \varepsilon \lambda_1$, for some $\lambda_1 > 0$ to be found.

(ii) Now introduce a small parameter $\varepsilon > 0$ by

$$
\alpha^2 = \frac{\pi^2}{4} \left( 1 + \varepsilon \right)
$$

Find an approximation to the time-dependent problem $(x,x)$ for $\varepsilon << 1$ in the form

$$
u(x,t) = \varepsilon u_0(x,T) + \varepsilon^2 u_1(x,T)$$

with $T = \varepsilon t$.

You will obtain that $u_0(x,T) = A(T) \cos(\pi x/2)$ where $A(T)$ satisfies a first order ODE obtained by applying a solvability condition to the problem for $u_1$. What are the steady states for this ODE for $A(T)$?
Problem 2 Consider the fish-harvesting model with a weak Allee effect modeled by

\[
\begin{align*}
\begin{cases}
    u_t &= u_{xx} + \alpha^2 f(u), & 0 < x < 1 \\
    u_x(0,t) &= u(1,t) = 0, & u(x,t) \geq 0
\end{cases}
\end{align*}
\]

where we assume that \( f(u) \) is smooth with \( f(0) = 0, f'(0) > 0 \) and \( f''(0) > 0 \).

(i) Determine the bifurcation point from linearizing around the zero solution. Label the bifurcation point \( \alpha = \alpha_c \) for some \( \alpha_c > 0 \).

(ii) Determine a local approximation for the bifurcating solution branch near the bifurcation point in (i), and plot this branch qualitatively.

(Hint: Let \( \alpha = \alpha_c (1 + \varepsilon \alpha_1 + ...) \), \( u = \varepsilon w_0 + \varepsilon^2 w_1 + ... \) into the steady-state problem and determine \( \alpha_1 \) in terms of \( f'(0) \) and \( f''(0) \).

(iii) Let \( f(u) = u(1-u)(u+\alpha) \). Plot using MATLAB a global bifurcation diagram of \( w(\alpha) \) versus \( \alpha \) for the choice \( \alpha = 1/10 \). (Hint: You need to find an integral relation between \( \alpha^2 \) and \( w(\alpha) \).)

(iv) In a few sentences, explain what you would expect to see regarding the dynamics of \( u(x,t) \) starting from some initial condition.

(v) A strong Allee effect is where \( f(u) \) has the form

\[ f(u) = u(1-u)(u+\alpha) \]

would there be a bifurcation from the trivial equilibrium \( u=0 \) for this choice of \( f(u) \)?
Problem 3 Consider the Allen-Cahn or Ginzburg-Landau equation
for \( u(x,t) \) given by
\[
\frac{\partial u}{\partial t} = u_{xx} + \Lambda(u - u^3)
\]
for \( 0 < x < \pi \), \( t > 0 \)
with \( u(0,t) = u(\pi,t) = 0 \), and \( \Lambda > 0 \) a parameter.

(i) Determine the bifurcation values of \( \Lambda \) corresponding to the trivial solution.

(ii) Determine the linear stability properties of the trivial solution and encode this on the bifurcation diagram.

(iii) Determine an integral characterizing the global branch of equilibria with \( w_x(\pi/2) = 0 \), \( w(\pi/2) = w_0 > 0 \) and \( w(x - \pi/2) > w(\pi/2 - x) \) and \( w > 0 \) on \( 0 < x < \pi \). Plot \( w_0 \) vs \( \Lambda \) from a numerical quadrature.

(iv) Prove that the branch of solutions in (iii) is linearly stable.
Problem 1
\[ a^2 \frac{U_t}{t} = U_{XX} + a^2 f(U), \quad 0 < X < 1, \quad f(U) = U(1-U) \]
\[ U_X(0,t) = 0, \quad U(1,t) = 0, \quad U(X,0) = h(X) > 0 \]

(i) From the notes there is an equilibrium solution with
\[ U_e = e \cos \left( \frac{\pi X}{2} \right) + \ldots \quad \alpha = \frac{\pi}{2} \left( 1 + \frac{4e}{3\pi} + \ldots \right) \]

The linearized eigenvalue problem is
\[ \phi_{XX} + a^2 f'(U_e) \phi = \sigma \phi \quad \text{with} \quad \sigma = \alpha^2 \Delta \]
\[ \phi_X(0) = 0, \quad \phi(1) = 0. \]

Now we calculate
\[ a^2 f'(U_e) = a^2 \left( 1 - 2U_e \right) = \frac{\pi^2}{4} \left( 1 + \frac{8e}{3\pi} + \ldots \right) \left( 1 - 2e \cos \left( \frac{\pi X}{2} \right) \right) \]
\[ = \frac{\pi^2}{4} \left[ 1 + e \left( \frac{8}{3\pi} - 2 \cos \left( \frac{\pi X}{2} \right) \right) \right] \]

Now we expand \( \phi = \phi_0 + e \phi_1 + \ldots \), \( \sigma = e \sigma_1 + \ldots \)

This yields that
\[ \left( \phi_{0,XX} + e \phi_{1,XX} + \ldots \right) + \frac{\pi^2}{4} \left[ 1 + e \left( \frac{8}{3\pi} - 2 \cos \left( \frac{\pi X}{2} \right) \right) \right] \left[ \phi_0 + e \phi_1 + \ldots \right] = e \sigma_1 \phi_0 \]

This yields that
\[ a \phi_0 = \phi_{0,XX} + \frac{\pi^2}{4} \phi_0 = 0 \quad ; \quad \phi_0(0) = 0, \quad \phi_0(1) = 0 \]
\[ a \phi_1 = -\frac{\pi^2}{4} \left( \frac{8}{3\pi} - 2 \cos \left( \frac{\pi X}{2} \right) \right) \phi_0 + \sigma_1 \phi_0 \quad ; \quad \phi_1(0) = \phi_1(1) = 0. \]

We obtain \( \phi_0 = \cos \left( \frac{\pi X}{2} \right) \). The solvability condition for \( \phi_1 \) is that
\[ (\phi_0, a \phi_1) = \int_0^1 \phi_0 a \phi_1 \, dx = 0. \]
This yields that
\[ \sigma_1 \int_0^1 \phi_0^2 \, dx = \frac{2\pi}{3} \int_0^1 \phi_0^2 \, dx - \frac{\pi^2}{3} \int_0^1 \phi_0^3 \, dx. \]

Thus
\[ \sigma_1 = \frac{2\pi}{3} - \frac{\pi^2}{3} \left( \int_0^1 \cos^3 \left( \frac{\pi X}{2} \right) \, dx \right) = \frac{2\pi}{3} - \frac{\pi^2}{3} \left( \frac{8}{3\pi} \right) = -\frac{2\pi}{3}. \]

Thus, since the principal eigenvalue satisfies \( \sigma < -2\pi e/3 < 0 \), hence locally the bifurcating solution branch is stable.
\( \partial^2 u_t = \partial^2 xx + \partial^2 f(u) \quad f(u) = u(1-u) \)

\( \partial x (0, t) = 0, \quad u (1, t) = 0. \)

We define \( v \) by \( \partial^2 v = \partial^2/4 (1 + \varepsilon) \).

We now expand \( u = \varepsilon u_0 (x, \tau) + \varepsilon^2 u_1 (x, \tau) + \ldots \quad \tau = \varepsilon t \)

We obtain

\[
\partial^2 f(u) = \frac{\partial}{\partial \varepsilon} \left( \varepsilon f(0) \right) \left[ f'(0) \left( \varepsilon u_0 + \varepsilon^2 u_1 \right) + \frac{f''(0)}{2} \varepsilon^2 u_0^2 \right] + \ldots
\]

\[
= \varepsilon \left\{ \frac{\partial}{\partial \varepsilon} f'(0) u_0 \right\} + \varepsilon^2 \left\{ \frac{\partial^2}{\partial \varepsilon^2} f'(0) u_0 + \frac{\partial}{\partial \varepsilon} f''(0) u_0^2 \right\}
\]

This yields that

\[
\frac{\partial}{\partial \varepsilon} e^2 u_0 + \ldots = \varepsilon \left[ \frac{\partial}{\partial \varepsilon} f'(0) u_0 \right] + \varepsilon^2 \left[ \frac{\partial^2}{\partial \varepsilon^2} f'(0) u_0 + \frac{\partial}{\partial \varepsilon} f''(0) u_0^2 \right]
\]

This yields that

\[
\frac{\partial}{\partial \varepsilon} f'(0) u_0 = 0 \quad 0 < x < 1 \quad u_{0x} (0) = u_0 (1) = 0
\]

\[
\frac{\partial}{\partial \varepsilon} f''(0) u_0^2 = 0
\]

Thus \( u_0 = A (7) \cos \left( \frac{\pi x}{2} \right) \). Then defining \( \phi_0 = \cos \left( \frac{\pi x}{2} \right) \), the solvability condition for \( u_1 \) is

\[
0 = \int_0^1 \phi_0 a_1 u_1 dx = \frac{\pi}{2} f'(0) \int_0^1 \phi_0^2 dx - \frac{\pi^2}{4} \frac{f''(0)}{A} \phi_0^3 dx
\]

Now, since \( f(u) = u(1-u) \), \( f'(0) = 1 \), \( f''(0) = -2 \). Hence,

\[
\frac{\pi}{2} A' = \frac{\pi^2}{4} \left[ A - 2 A^2 \int_0^1 \phi_0^3 \right] = \frac{\pi^2}{4} A \left( 1 - 2 A \left( 8/3 \pi \right) \right)
\]

Thus yields \( A' = A \left( 1 - 8 A/3 \pi \right) \)
The equilibria are

\[ A = 0 \text{ unstable, } A = \frac{3\pi}{8} \text{ stable.} \]

Thus, we predict a stable equilibrium solution with

\[ u \propto \frac{3\pi \epsilon}{8} \cos \left( \frac{\pi x}{2} \right) + \ldots \text{ when } \alpha^2 = \frac{\pi^2}{4} \left(1 + \epsilon \right). \]

Remark: For \( \epsilon \ll 1 \) we can write this as

\[ u \propto \frac{3\pi \epsilon}{8} \cos \left( \frac{\pi x}{2} \right) \text{ with } \alpha \propto \frac{\pi}{2} \left(1 + \frac{\epsilon}{2} \right). \]

Now define \( \tilde{\epsilon} \) by

\[ \frac{4 \tilde{\epsilon}}{3\pi} = \frac{\epsilon}{2} \rightarrow \epsilon = 3\pi \tilde{\epsilon}/4. \]

This gives

\[ u \propto \tilde{\epsilon} \cos \left( \frac{\pi x}{2} \right) \text{ and } \alpha \propto \frac{\pi}{2} \left(1 + \frac{4 \tilde{\epsilon}}{3\pi} \right), \]

which is the equilibrium result quoted in (i). So the picture 'near' \( \alpha = \pi/2 \) is

\( O(\epsilon) \)

\( \frac{\pi}{2} \)

Time-depended solution goes to nontrivial branch on a long time-scale

\[ t = O \left( \frac{1}{\epsilon} \right). \]
\[ u_x = u_{xx} + \alpha^2 f(u), \quad 0 < x < 1 \]
\[ u_x(0, t) = 0, \quad u(t) = 0 \]
\[ f(0) = 0, \quad f'(0) > 0, \quad f''(0) > 0. \]

(i) The steady-state problem is for \( w(x) \):

\[ w'' + \alpha^2 f(w) = 0, \quad 0 < x < 1 \]
\[ w(0) = 0, \quad w(1) = 0 \]

Notice that \( w = 0 \) is a solution for all \( \alpha \). Now linearizing around \( w = 0 \) we obtain with \( f(w) = f'(0) w + \ldots \)

\[ \tilde{w}'' + \alpha^2 f'(0) \tilde{w} = 0 \]
\[ \tilde{w}'(0) = 0, \quad \tilde{w}'(1) = 0 \]

Thus, the bifurcation point is \( \alpha_C = \frac{\tilde{w}}{2 \sqrt{f'(0)}} \), \( \tilde{w} = 1 \left( \frac{wx}{2} \right) \).

Since \( f'(0) > 0 \) \( \Rightarrow \) \( \alpha_C \) exists.

Now if \( f(u) = u (1 - u) (u + a) = u + (1 - a) u^2 - u^3 \).

So \( f'(0) = a, \quad f''(0) = 2(1 - a) \).

If \( 0 < a < 1 \) \( \Rightarrow \) \( f'(0) > 0, \quad f''(0) > 0 \).

The bifurcation point is at \( \alpha_C = \frac{\tilde{w}}{2 \sqrt{a}} \) with \( \alpha < 1 \).
(ii) Now we calculate a local approximation for the steady-state solution in $(+)$ that emerges at $\alpha_c = \frac{\bar{n}}{2 \sqrt{F'(0)}}$

We let

$$\alpha = \alpha_c \left( 1 + \varepsilon \alpha_1 \right)$$

$$W = \varepsilon W_0 + \varepsilon^2 W_1 + \ldots$$

We calculate

$$\alpha^2 F(W) = \alpha_c^2 \left( 1 + 2 \varepsilon \alpha_1 \right) \left[ F'(0) \left( \varepsilon W_0 + \varepsilon^2 W_1 \right) + \frac{F''(0)}{2} \varepsilon^2 W_0 + \ldots \right]$$

$$= \frac{\bar{n}^2}{4 F'(0)} \left[ \varepsilon F'(0) W_0 + \varepsilon^2 \left( W_1 F'(0) + \frac{W_0^2 F''(0)}{2} + 2 \alpha_1 F'(0) W_0 \right) \right]$$

Thus

$$\varepsilon W_0'' + \varepsilon^2 W_1'' + \frac{\bar{n}^2}{4 F'(0)} \left[ \varepsilon F'(0) W_0 + \varepsilon^2 \left( W_1 F'(0) + \frac{W_0^2 F''(0)}{2} + 2 \alpha_1 F'(0) W_0 \right) \right] = 0$$

EQUATING POWERS OF $\varepsilon$:

$$W_0'' + \frac{\bar{n}^2}{4} W_0 = 0; \quad W_0'(0) = 0, \quad W_0(1) = 0 \rightarrow W_0 = \cos \left( \frac{n \pi x}{2} \right).$$

$$W_1'' + \frac{\bar{n}^2}{4} W_1 = -\frac{\bar{n}^2}{4 F'(0)} \left[ \frac{W_0^2}{2} F''(0) + 2 \alpha_1 F'(0) W_0 \right]; \quad W_1'(0) = W_1(1) = 0.$$

Now from the solvability condition on the second equation (in view of $W_0 = 0$) we have

$$\frac{F''(0)}{2} \int_0^1 W_0^3 \, dx + 2 \alpha_1 \int F'(0) \left| \frac{W_0^2}{2} \right| \, dx = 0.$$

Thus

$$\alpha_1' = -\frac{F''(0)}{2 F'(0)} \int_0^1 W_0^3 \, dx = -\frac{F''(0)}{4 F'(0)} \int_0^1 \cos^3 \left( \frac{n \pi x}{2} \right) \, dx$$

$$= \frac{\bar{n}^2}{4 F'(0)} \left[ \int_0^{\pi/2} \cos^3 y \, dy \right] = \frac{\bar{n}^2}{4 F'(0)} \left[ \int_0^{\pi/2} \cos^3 y \, dy \right] = \frac{\bar{n}^2}{4 F'(0)} \left[ \frac{1}{3} \right] = \frac{2}{3} \bar{n} F''(0) = \frac{2 \bar{n} F''(0)}{F'(0)}.$$
We conclude that the local branch of bifurcating solution satisfies

\[ \alpha \approx \alpha_c \left( 1 + \varepsilon \alpha' + \ldots \right) \]

\[ W(0) = \varepsilon \]

\[ \alpha_c = \frac{\pi}{2 \sqrt{f'(0)}} \]

with

\[ \alpha' = -\frac{2}{3\pi} \frac{f''(0)}{f'(0)} \]

Since \( f'(0) > 0 \) but \( f''(0) > 0 \) we have a subcritical bifurcation near \( \alpha = \alpha_c \).
Now consider the weak Allee effect model:

\[ W'' + a^2 f(W) = 0, \quad f(W) = W(W-I)(W+a) \]

\[ W'(0) = 0, \quad W(I) = 0. \]

We multiply by \( W' \) to obtain

\[ W' W'' + a^2 W' f(W) = 0. \]

This yields

\[ \frac{1}{2} \frac{d}{dx} (W'^2) + \frac{a^2}{2} \frac{d}{dx} G(W) = 0, \quad G(W) = 2 \int_0^W f(s) \, ds. \]

Now we calculate,

\[ W'^2 = a^2 [G(W_o) - G(W)] \quad W(0) = W_0. \]

Then since \( W' < 0 \) on \( 0 < x < 1 \), we obtain

\[ \frac{dW}{dx} = -\sqrt{G(W_0) - G(W)} \, dx. \]

This yields

\[ \frac{W_0}{W} = -\int_{0}^{W_0} \left[ G(W_0) - G(\lambda) \right]^{-1/2} d\lambda. \]

Now set \( x = 1 \) with \( W = W_0 \). This yields that

\[ \alpha = \int_{0}^{W_0} \left[ G(W_0) - G(\lambda) \right]^{-1/2} d\lambda = \frac{W_0}{W} \int_{0}^{1} \left[ G(W_0) - G(W_0 s) \right]^{-1/2} ds. \]

Now we have

\[ G(W) = 2 \int_{0}^{W} \left[ W^2 + a \right] d\lambda = 2 \int_{0}^{W} (W - W^2 - a W) \, dW \]

\[ G(W) = 2 \int_{0}^{W} \left[ \frac{(1-a)}{3} W^3 + \frac{a}{2} W^2 - \frac{a}{3} W^4 \right] dW \]

\[ G(W) = 2 \int_{0}^{W} \left[ \frac{1-a}{3} W^3 + \frac{a}{2} W^2 - \frac{a}{4} W^4 \right] \]

\[ G(W_0) - G(W_0 s) = 2 \left[ \frac{(1-a)}{3} \left[ W_0^3 - W_0^2 s^3 \right] + \frac{a}{2} \left( W_0^2 - W_0^2 s^2 \right) \right. \]

\[ - \frac{1}{4} \left( W_0^4 - W_0^4 s^4 \right) \]
This yields that

\[ G(W_0) - G(W_0S) = 2W_0^2 \left[ \frac{(1-a)}{3} (1-S^3)W_0 - \frac{1}{4} (1-S^4)W_0^2 + \frac{a}{2} (1-S^2) \right] \]

Then

\[ \delta = \frac{1}{\sqrt{2}} \int_0^1 \left[ \frac{a}{2} (1-S^2) + (1-a) (1-S^3)W_0 - \frac{1}{4} (1-S^4)W_0^2 \right]^{1/2} dS. \]

We let \( W_0 \rightarrow 0 \) to obtain from (**) that

\[ \delta \sim \frac{1}{\sqrt{a}} \int_0^1 \frac{1}{\sqrt{1-S^2}} dS = \frac{\pi}{2\sqrt{a}}. \]

Thus, the bifurcation point should be at \( \delta^2 = \frac{\pi^2}{4a} \).

Remark 1) If we linearize around \( U = 0 \) we get

\[ U = e^{\sigma t} \phi \]

and

\[ \sigma \phi = \phi_{xx} + \delta^2 F'(0) \phi, \quad \phi_x(0) = 0, \quad \phi(1) = 0. \]

Set \( \sigma = 0 \) for exchange of stability and

\[ \delta^2 F'(0) = \frac{\pi^2}{4} \rightarrow \phi = \phi(0) \left( \frac{\pi x}{2} \right) \]

Thus with \( F'(0) = a \) we obtain

\[ \delta^2 = \frac{\pi^2}{4a} \rightarrow \delta = \frac{\pi}{2\sqrt{a}}, \text{which} \]

agrees with the above.

The plot of \( \delta^2 \) on next page.
(x) is the bifurcation curve with weak Allee effect. We took $a = \frac{1}{10}$.

$$\alpha = \frac{1}{\sqrt{2}} \int_0^1 \left[ \frac{3}{2} (1-x^2) + \frac{1}{3} (1-a)(1-x^3) \right] W_0 - \frac{1}{4} (1-x^4) W_0^2 \right]^{1/2} ds$$

The other curve $U$ for logistic case $f(U) = U(1-U)$.

(iv) The anticipated stability diagram is as shown. There is a range of $\alpha$ where bistability occurs, and there is clearly the possibility of hysteresis. Notice that the extinction threshold is now at the saddle node bifurcation point as shown. Thus if $0 < \alpha < \frac{\pi}{2\sqrt{a}}$ with $a = \frac{1}{10}$ the time dependent solution should either approach the extinct solution branch or the upper branch depending on $U(0)$.
(v) Now consider a strong Allee effect where \( f'(0) < 0 \).

Linearizing around \( w = 0 \) we obtain the linearized problem:

\[
\begin{align*}
\tilde{w}_{xx} + \alpha^2 f'(0) \tilde{w} &= 0, \quad 0 < x < 1 \\
\tilde{w}_x(0) &= 0, \quad \tilde{w}(1) = 0.
\end{align*}
\]

Since \( \alpha^2 > 0 \) and \( f'(0) < 0 \) there is no value of \( \alpha^2 \) for which a non-trivial solution for \( \tilde{w} \) exists.
WE BEGIN WITH \[ u_t = u_{xx} - \lambda f(u), \quad 0 < x < \pi, \quad (\lambda > 0) \]
\[ u(0, t) = u(\pi, t) = 0. \]

WE HAVE \[ f(u) = -(u - u^3). \] \[ f'(0) = 0, \quad f''(0) = 0, \quad f'''(0) > 0. \]

(i) THE EQUILIBRIUM PROBLEM IS
\[ w_{xx} - \lambda f(w) = 0, \quad 0 < x < \pi \]
\[ w(0) = w(\pi) = 0. \]

WE HAVE \[ \delta = 0 \] IF A SOLUTION \[ \forall \lambda > 0. \] NOW LINEARIZING NEAR \[ \delta = 0. \]
WE GET
\[ \hat{w}_{xx} - \lambda \hat{f}'(0) \hat{w} = 0 \]
\[ \hat{w}(0) = \hat{w}(\pi) = 0. \]

WITH \[ f'(0) = -1. \] THEN \[ \hat{w} = \sin \left( \lambda x \right) \] WITH \[ \sin (\lambda \pi) = 0 \] OR \[ \lambda \pi = n \pi \]
WITH \[ n = 1, 2, \ldots, \] WE OBTAIN THAT \[ \lambda_n = n, \quad n = 1, 2, \ldots. \]

NOW FROM A WEAKLY NONLINEAR ANALYSIS WE OBTAIN THAT (FROM THE NOTES)
\[ \delta \sim \varepsilon \sin (\lambda x) + O(\varepsilon^2) \]
\[ \lambda \sim n^2 + \varepsilon^2 \frac{n^2 f'''(0)}{8 [f'(0)]^2} + \cdots \]
WHERE \[ f'''(0) = 6, \quad f'(0) = 1 \rightarrow \lambda \sim n^2 + 3 \varepsilon^2 \frac{n^2}{4} + \cdots \]

(ii) NOW WE LINEARIZE: \[ u(x, t) = w(x) + v(x, t) \] WITH \[ v(x, t) \ll 1 \] TO OBTAIN THE LINEARIZED PROBLEM:
\[ v_t = v_{xx} - \lambda f'(w) v, \quad 0 < x < \pi, \quad t > 0 \]
\[ v(0, t) = v(\pi, t) = 0. \]

WE THEN WRITE \[ v(x, t) = e^{\sigma t} \tilde{v}(x) \] TO OBTAIN THE EIGENVALUE PROBLEM:
\[
\begin{cases}
\tilde{v}_{xx} - [\sigma + \lambda f'(w)] \tilde{v} = 0, & 0 < x < \pi \\
\tilde{v}(0) = \tilde{v}'(\pi) = 0
\end{cases}
\]

STRUML-LOUIVILLE EIGENFUNCTIONS \( \Phi_1, \Phi_2, \Phi_3 \) SATISFYING
\[ \sigma_1 < \sigma_2 < \sigma_3 \ldots, \quad \text{with eigenfunction } \tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \text{ satisfying } \]
\[ \int_0^\pi \Phi_j(x) \tilde{v}_i(x) \, dx = 0, \quad j \neq i, \quad \sigma_j \rightarrow -\infty, \quad \lambda \rightarrow +\infty, \]
AND \( \tilde{v}_1(x) > 0 \) \forall \quad 0 < x < \pi.
Now for linear stability of steady-state we must have $\sigma_1 < 0$.

Thus, $W(x)$ is linearly stable iff $\sigma_1 < 0$.

Now let $W = 0$. Then

$$\tilde{D} = \sin \left[ \sqrt{-\sigma - 4f'(0)} x \right] \text{ with } f'(0) = -1.$$

Thus,

$$\pi \sqrt{-\sigma + \lambda} = j\pi, \quad j = 1, 2, \ldots$$

We obtain that

$$\sigma_j = -j^2 + \lambda, \quad j = 1, 2, \ldots$$

We conclude that $\sigma_1 < 0$ iff $-1 + \lambda < 0$ or $\lambda < 1$. Thus, we have

$$\sigma_1 \text{ in the spectral plane}$$

(iii) Now look for a steady-state $W(x)$ as shown

$$W(\frac{\pi}{2}) = W_0, \quad W'(\frac{\pi}{2}) = 0, \quad W(x) > 0 \text{ on } 0 < x < \pi,$$

$$W' < 0 \text{ on } \frac{\pi}{2} < x < \pi.$$  

$$W \left( x - \frac{\pi}{2} \right) = W \left( \frac{\pi}{2} - x \right).$$

Now

$$W'' + \lambda (W - W^3) = 0.$$  

Multiply by $W'$ and integrate to get

$$\frac{W'^2}{2} + \lambda \left( \frac{W^2}{4} - \frac{W_0^2}{4} \right) = 0 \quad \text{where we put } W' = 0 \text{ at } x = \frac{\pi}{2} \text{ where } W \left( \frac{\pi}{2} \right) = W_0.$$

Then,

$$W'' + \lambda \left( W^2 - W_0^2 \right) = 0.$$  

Now

$$W_x = -\sqrt{\lambda} \int W_0^2 - W^2 + \frac{1}{2} \left( W^4 - W_0^4 \right)^{1/2} \text{ on } \frac{\pi}{2} < x < \pi.$$  

Separating variables and integrating:

$$\frac{dW}{\left[ W_0^2 - W^2 + \frac{1}{2} (W^4 - W_0^4) \right]^{1/2}} = -\sqrt{\lambda} \, dx$$

Integrating:

$$\left[ \frac{W}{W_0^2 - W^2 + \frac{1}{2} (W^4 - W_0^4)} \right]^{1/2} = -\sqrt{\lambda} \int_{\frac{\pi}{2}}^{x} ds = -\sqrt{\lambda} \left( x - \frac{\pi}{2} \right)$$
Flipping limits:
\[
\int_{W_0}^{W} \frac{d\lambda}{W_0 - \lambda + \frac{1}{2} (\lambda^2 - W_0^2)} \frac{1}{\lambda^2} = \sqrt{\lambda} \left( \frac{X - \frac{\pi}{2}}{\lambda} \right) \text{ for } X > \frac{\pi}{2}.
\]

Now set \( X = \frac{\pi}{2} \) and \( W = 0 \) to get
\[
\sqrt{\lambda} = \sqrt{W_0} \int_{0}^{W_0} \frac{d\lambda}{W_0 - \lambda + \frac{1}{2} (\lambda^2 - W_0^2)} \frac{1}{\lambda^2}.
\]

Now let \( \lambda = W_0 s \). Then
\[
\sqrt{\lambda} = \frac{2}{\pi} \int_{0}^{1} \frac{ds}{1 - s - \frac{W_0}{2} (1 - s^4)} \frac{1}{s^2}.
\]

This determines \( \lambda \) as a function of \( W_0 \). As \( W_0 \to 0 \) we have
\[
\sqrt{\lambda} \sim \frac{2}{\pi} \int_{0}^{1} \frac{ds}{\sqrt{1 - s^2}} = \frac{2}{\pi} \arcsin \left| \frac{1}{s} \right| \to 1 \to 1.
\]

By differentiating,
\[
\frac{d}{dW_0} \sqrt{\lambda} = \frac{2}{\pi} \int_{0}^{1} \frac{W_0}{2} \frac{(1 - s^4)}{1 - s^2 - \frac{W_0^2}{2} (1 - s^4)} \frac{1}{s^2} > 0
\]

so \( \lambda \to \lambda_0 \). Now define
\[
J(s, W_0) = 1 - s^2 - \frac{W_0^2}{2} (1 - s^4).
\]

We have near \( s = 1 \),
\[
J(s, W_0) \approx 2 (s^2 - 1) (s - 1) + \frac{(W_0^2 - 2)}{2} (s - 1)^2
\]

Thus \( \lambda, W_0 \to 1 \) we have a non-integrable singularity as
\[
J(s, W_0) \approx O((s - 1)^1).
\]

The numerics is as shown:

![Numerical Graph](image-url)

S: linearly stable
U: unstable
(iv) Now consider the primary non-trivial solution branch of (iii).

We have \( W > 0 \). Consider the first eigenpair \( \Phi, \sigma \geq 0 \) on \( 0 < x < \pi \)

\[
\Phi_{xx} - [\sigma + \lambda f'(w)] \Phi = 0
\]

\[
\Phi(0) = \Phi(\pi) = 0
\]

\[
W_{xx} - \lambda f(w) = 0
\]

\[
W(0) = W(\pi) = 0
\]

We calculate

\[
0 = \int_0^\pi \left( \Phi_{xx} W - W_{xx} \Phi, \right) \, dx = \left[ W \left[ (\sigma + \lambda f'(w)) \Phi, \right] - \lambda f(w) \Phi, \right]_0^\pi \, dx.
\]

Thus

\[
0 = \sigma \int_0^\pi W \Phi, \, dx + \int_0^\pi \left( \lambda W f'(w) \Phi, - \lambda f(w) \Phi, \right) \, dx
\]

Thus become

\[
\sigma \int_0^\pi W \Phi, \, dx = \lambda \int_0^\pi W f'(w) - W f(w) \Phi, \, dx.
\]

Now \( f(w) := (w - w^3) \) so \( f(w) - W f'(w) = -W + W^3 + W \left[ 1 - 3 w^2 \right] = -2w^3 \).

So \( f(w) = (1 - 3 w^2) \)

we conclude that

\[
\sigma \int_0^\pi W \Phi, \, dx = \lambda \int_0^\pi \left[ 2 \Phi, w^3 \right] \, dx.
\]

Since \( \Phi, \sigma \geq 0 \) and \( W > 0 \), \( \sigma, \lambda \) are real. Thus \( \sigma_j < 0 \) for \( j = 2, 3, 4 \ldots \).

We conclude that the primary non-trivial solution branch is linearly stable.
WE CONSIDER \[ U^2 = U_{xx} + \Lambda (U - U^3) \]

\[ U(0, t) = U(\pi, t) = 0. \]

THE BIFURCATION POINT IS \( \Lambda = 1 \) AND WE LET \( \Lambda = 1 + \epsilon^2 \)

SO THAT \( \Lambda > 1 \). WE THEN LOOK FOR A SOLUTION WITH \[ \gamma = \epsilon^2 t \quad \text{AND} \quad U = \epsilon U_1 + \epsilon^2 U_2 + \epsilon^3 U_3 + \cdots \]

WHERE \( U_j = U_j(x, \gamma) \). WE HAVE

\[
\epsilon^3 U_{\gamma} + \cdots = \epsilon U_{xx} + \epsilon^2 U_{2xx} + \epsilon^3 U_{3xx} + (1 + \epsilon^2) \left[ \epsilon U_1 + \epsilon^2 U_2 + \epsilon^3 U_3 - \epsilon^3 U_1^3 + \cdots \right]
\]

\[
= \epsilon \left[ U_{xx} + U_1 \right] + \epsilon^2 \left[ U_{2xx} + U_2 \right] + \epsilon^3 \left[ U_{3xx} + U_3 - U_1^3 + U_1 \right]
\]

COLLECTING POWER OF \( \epsilon \):

\[ O(\epsilon): \quad U_{xx} + U_1 = 0, \quad 0 < x < \pi \] \[ U_1(0): U_1(\pi) = 0 \]

\[ O(\epsilon^2): \quad U_{2xx} + U_2 = 0, \quad 0 < x < \pi \] \[ U_2(0): U_2(\pi) = 0 \]

\[ O(\epsilon^3): \quad U_{3xx} + U_3 = U_{\gamma} + U_1^3 - U_1, \quad 0 < x < \pi \] \[ U_3(0): U_3(\pi) = 0 \]

NOW THE SOLVABILITY CONDITION IS THAT

\[
\int_0^\pi \left( U_{\gamma} + U_1^3 - U_1 \right) U_1 \, dx = 0 \quad \Rightarrow \quad A'(\gamma) \int_0^\pi \sin^4 x \, dx + A^3 \int_0^\pi \sin^4 x \, dx - \frac{3\pi}{8} \left( \frac{\pi}{2} \right) = 0
\]

WE CONCLUDE THAT

\[
A' = A - A^3 \frac{\int_0^\pi \sin^4 x \, dx}{\int_0^\pi \sin^2 x \, dx}
\]

\[
\text{but} \quad \frac{\int_0^\pi \sin^4 x \, dx}{\int_0^\pi \sin^2 x \, dx} = \frac{(3\pi/8)}{(\pi/2)} = \frac{3}{4}
\]
This yields that
\[ A'(\tau) = A - 3A^3 = h(A) \]

We conclude that \( A_e = 0 \) is unstable
and \( A_e = \frac{2}{\sqrt{3}} \) is stable.

Recall that our local description of bifurcating branch was
\[ \lambda \equiv 1 + 3\varepsilon^2, \quad U = \varepsilon \sin(x). \]

Now relabel \( \varepsilon = \frac{2}{\sqrt{3}} \varepsilon_0 \rightarrow \lambda \equiv 1 + \varepsilon_0^2 \rightarrow U = \frac{2}{\sqrt{3}} \sin(A_e) \]

which is consistent with our analysis of steady-state of
slow time dynamics \( \lambda : 1 + \varepsilon, \quad U = \frac{2}{\sqrt{3}} \sin(A_e). \)

Remark: Although other higher bifurcation points are
associated with a zero eigenvalue crossing, it is
higher eigenvalue that cross through zero;
\( \sigma_2 = 0 \) at \( \lambda = \frac{1}{2}, \sigma_1 > 0 \)
\( \sigma_3 = 0 \) at \( \lambda = \frac{1}{3}, \sigma_1 > 0, \sigma_2 > 0 \)
Thus there is no slow time dependence assumption
of the form \( U = \varepsilon A(\tau) \sin(jx) \) with \( j = 2, 3, \ldots \) at \( \lambda = j^2 \)
possible.