Problem 1. Consider the diffusive logistic model for $u = u(x, t)$

$$
\begin{cases}
\frac{\partial^2 u}{\partial t^2} = u_{xx} + \alpha^2 f(u), & 0 < x < 1 \\
\partial_x u(0, t) = u(1, t) = 0, & \alpha, \beta > 0.
\end{cases}
$$

(i) Show that the bifurcating solution branch of nontrivial equilibria near $\alpha = \pi/2$ is linearly stable.

Hint: The eigenvalue problem is

$$
\begin{cases}
\phi_{xx} + \alpha^2 f'(u) \phi = \sigma \phi \\
\phi_x (0) = \phi (1) = 0
\end{cases}
$$

The nontrivial branch of equilibria near $\alpha = \pi/2$ is

$$
\lambda \sim \epsilon \cos \left( \frac{\pi x}{2} \right) \text{ when } \alpha = \frac{\pi}{2} \left( 1 + \frac{4 \epsilon}{3 \pi} + \ldots \right), \epsilon \ll 1. \text{ Show that the first eigenvalue of } (x) \text{ is } \sigma \sim \epsilon \lambda, \text{ for some } \lambda > 0 \text{ to be found.}
$$

(ii) Now introduce a small parameter $\epsilon > 0$ by

$$
\alpha^2 = \frac{\pi^2}{4} (1 + \epsilon)
$$

Find an approximation to the time-dependent problem $(x)$ for $\epsilon \ll 1$ in the form $u(x, t) = \epsilon u_0(x, T) + \epsilon^2 u_1(x, T)$ with $T = \epsilon t$

You will obtain that $u_0(x, T) = A(T) \cos \left( \frac{\pi x}{2} \right)$ where $A(T)$ satisfies a first-order ODE obtained by applying a solvability condition to the problem for $u$. What are the steady-states for this ODE for $A(T)$?
Problem 2 Consider the fish-harvesting model with a weak Allee effect modeled by

\[
\begin{cases}
    u_t = u_{xx} + \alpha^2 f(u), & 0 < x < 1 \\
    u_x(0,t) = u(1,t) = 0, & u(x,t) \geq 0
\end{cases}
\]

where we assume that \( f(u) \) is smooth with \( f(0) = 0, \ f'(0) > 0 \) and \( f''(0) > 0. \)

(i) Determine the bifurcation point from linearizing around the zero solution. Label the bifurcation point \( \alpha = \alpha_c \) for some \( \alpha_c > 0. \)

(ii) Determine a local approximation for the bifurcating solution branch near the bifurcation point in (i), and plot this branch qualitatively. (Hint: Let \( \alpha = \alpha_c (1 + \varepsilon, \ldots) \), \( u = \varepsilon w_0 + \varepsilon^2 w, \ldots \) into the steady-state problem and determine \( \alpha, \) in terms of \( f'(0) \) and \( f''(0). \)

(iii) Let \( f(u) = u(1-u)(u+\alpha). \) Plot using MATLAB a global bifurcation diagram of \( w(0) \) versus \( \alpha, \) for the choice \( \alpha = 1/10. \) (Hint: you need to find an integral relation between \( \alpha^2 \) and \( w(0). \))

(iv) In a few sentences, explain what you would expect to see regarding the dynamic of \( u(x,t) \) starting from some initial condition?

(v) A strong Allee effect is where \( f(u) \) has the form \( f(u) \approx u \). Would there be a bifurcation from the trivial equilibrium \( u=0 \) for this choice of \( f(u). \)
Problem 3. Consider the Allen-Cahn or Ginzburg-Landau equation for $u(x,t)$ given by

$$u_t = u_{xx} + \lambda (u - u^3) \quad \text{on} \quad 0 < x < \pi, \quad t > 0$$

with $u(0,t) = u(\pi,t) = 0$, and $\lambda > 0$ a parameter.

(i) Determine the bifurcation value of $\lambda$ corresponding to the trivial solution.

(ii) Determine the linear stability properties of the trivial solution and encode this on the bifurcation diagram.

(iii) Determine an integral characterizing the global branch of equilibria with $w_x(\pi/2) = 0$, $w(\pi/2) = w_0 > 0$ and $w(x - \pi/2) > w(\pi/2 - x)$ and $w > 0$ on $0 < x < \pi$. Plot $w_0$ vs $\lambda$ from a numerical quadrature.

(iv) Prove that the branch of solutions in (iii) is linearly stable.