Chapter 2

Hysteresis

2.1 Introduction

Consider the problem

$$y' = y - \frac{1}{3}y^3 + \lambda. \tag{2.1}$$

This has equilibriums when

$$\lambda = -y + \frac{1}{3}y^3, \tag{2.2}$$

which are shown in figure 2.1. The reason why we obtain hysteresis is that as λ increases from some large negative value, it stays on the negative stable branch until $\lambda = \lambda_c = 2/3$, whereas as λ decreases from some large positive value, it stays on the positive branch until $\lambda = -\lambda_c$. If we let

$$\lambda = \lambda_c \cdot 1.2 \cdot \sin(0.1t), \tag{2.3}$$

then the solution will continuously go through this hysteresis loop. This is shown in figure 2.2.

2.2 Delay in Bifurcation

Note that in figure 2.2, there is a small delay between the disappearance of the, for example, positive stable stationary solution and the time when the solution moves to the negative stable stationary solution. The object of this section is to find the approximate length of that delay.

Let $\lambda = \lambda_c + \epsilon t$. Expanding everything nearby, $y = y_c + \epsilon^p y_1$ and $t = \epsilon^q \tau$. As a result,

$$y' = f(y, \lambda) \tag{2.4}$$

$$= f|_{y_c,\lambda_c} + f_y|_{y_c,\lambda_c}(y - y_c) + f_{yy}|_{y_c,\lambda_c}\frac{(y - y_c)^2}{2} + f_\lambda|_{y_c,\lambda_c}(\lambda - \lambda_c). \tag{2.5}$$

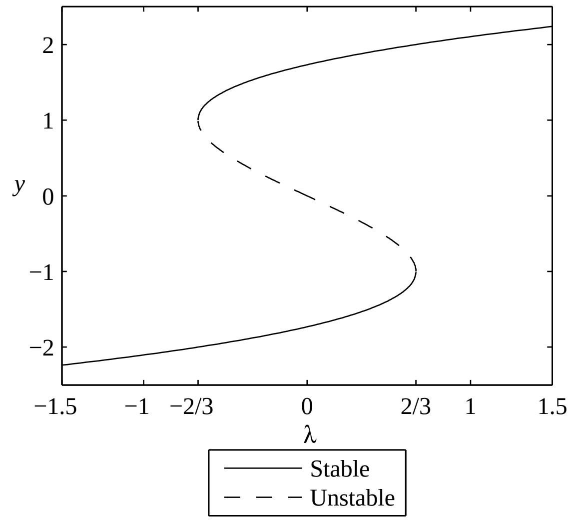

Figure 2.1: The stable and unstable branches of the bifurcation diagram for equation 2.1.

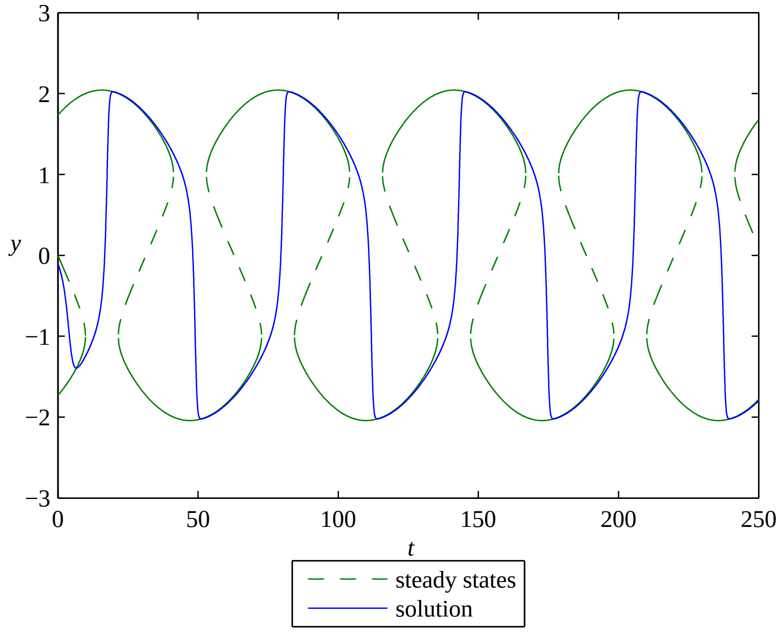

Figure 2.2: The solution to the equation 2.1 where λ varies in time as in equation 2.3. The initial condition that was used is $y(0) = -0.1$.

Now by definition, when the stable stationary solution first 'disappears', $f = 0$ since we're still at the stationary solution. Locally at the point where the stable stationary solution first 'disappears', the curve determined by $f(y, \lambda) = 0$ is vertical in the λ-y plane, and so $f_y = 0$. As a result we obtain

$$y' = f_{yy}\big|_{y_c,\lambda_c} \frac{(y - y_c)^2}{2} + f_\lambda\big|_{y_c,\lambda_c} (\lambda - \lambda_c) \tag{2.6}$$

$$\epsilon^{p-q} y_{1\tau} = \epsilon^{2p} \frac{f_{yy}}{2} y_1^2 + \epsilon^{1+q} f_\lambda \tau. \tag{2.7}$$

In order for these to all be of the same order, we must have

$$p - q = 2p = 1 + q \tag{2.8}$$

and as a result, we must have $p = 1/3$, $q = -1/3$. We then obtain that

$$y_{1\tau} = \frac{f_{yy}}{2} y_1^2 + f_\lambda \tau, \tag{2.9}$$

which is an equation of order 1. We must now solve this equation. If we let $y_1 = \beta v$ and $\tau = \delta s$, then we obtain

$$\frac{\beta}{\delta} v' = \frac{f_{yy}\beta^2}{2} v^2 + f_\lambda \delta s \tag{2.10}$$

$$v' = \frac{f_{yy}\beta\delta}{2} v^2 + \frac{f_\lambda \delta^2}{\beta} s. \tag{2.11}$$

Now we choose to specify that

$$\frac{f_{yy}\beta\delta}{2} = -1, \qquad \frac{f_\lambda \delta^2}{\beta} = 1, \tag{2.12}$$

which means that

$$\delta = -\left(\frac{2}{f_{yy} f_\lambda}\right)^{1/3}, \qquad \beta = \left(\frac{4 f_\lambda}{f_{yy}^2}\right)^{1/3}, \tag{2.13}$$

and so

$$v' = -v^2 + s. \tag{2.14}$$

If we then let $v(s) = \varphi'(s)/\varphi(s)$, we obtain that

$$\varphi''(s) = s\varphi(s), \tag{2.15}$$

and so

$$\varphi(s) = a_0 \operatorname{Ai}(s) + a_1 \operatorname{Bi}(s), \tag{2.16}$$

$$v(s) = \frac{a_0 \operatorname{Ai}'(s) + a_1 \operatorname{Bi}'(s)}{a_0 \operatorname{Ai}(s) + a_1 \operatorname{Bi}(s)}. \tag{2.17}$$

Now as $t \to -\infty$, by definition, $s \to \infty$. Now as $s \to \infty$, $\mathrm{Ai}(s) \to 0$ and $\mathrm{Bi}(s) \to \infty$, and so

$$v(s) \xrightarrow{s\to\infty} \begin{cases} \mathrm{Bi}'(s)/\,\mathrm{Bi}(s) & \text{if} \quad a_1 \neq 0 \\ \mathrm{Ai}'(s)/\,\mathrm{Ai}(s) & \text{if} \quad a_1 = 0 \end{cases} \tag{2.18}$$

$$\xrightarrow{s\to\infty} \begin{cases} \sqrt{s} & \text{if} \quad a_1 \neq 0 \\ -\sqrt{s} & \text{if} \quad a_1 = 0 \end{cases} \tag{2.19}$$

Now in our case, our initial conditions (that we started from $\lambda = -\infty$, $y = -\infty$) require that as $s \to \infty$, $v(s)$ must be negative. As a result, we conclude that $a_1 = 0$. We then obtain that

$$y = y_c + \epsilon^p y_1 \tag{2.20}$$

$$= y_c + \epsilon^{1/3}\beta v(s) \tag{2.21}$$

$$= y_c + \epsilon^{1/3}\beta v(s) \tag{2.22}$$

$$= y_c + \epsilon^{1/3}\left(\frac{4f_\lambda}{f_{yy}^2}\right)^{1/3}\frac{\mathrm{Ai}'(s)}{\mathrm{Ai}(s)}, \tag{2.23}$$

where

$$s = \frac{\tau}{\delta} \tag{2.24}$$

$$= -\tau\left(\frac{f_\lambda f_{yy}}{2}\right)^{1/3} \tag{2.25}$$

$$= -t\epsilon^{-q}\left(\frac{f_\lambda f_{yy}}{2}\right)^{1/3} \tag{2.26}$$

$$= -t\epsilon^{1/3}\left(\frac{f_\lambda f_{yy}}{2}\right)^{1/3}. \tag{2.27}$$

Now we expect this to be invalid when y_1 becomes large. As a result, we predict that the solution will approach the opposite stationary branch when $\mathrm{Ai}(s) = 0$ for the first time, at which point y_1 blows up. This occurs when $s \sim -2.33811$.

No Time Dependence

Set $\lambda = \lambda_c + \epsilon$ (with no time dependence). Then expand y and t using $y = y_c + \epsilon^p y_1$ and $t = \epsilon^q \tau$. Then, performing analysis as before,

$$f(y,\lambda) = y - \frac{1}{3}y^3 + \lambda \tag{2.28}$$

$$y' = \frac{f_{yy}}{2}\left(\epsilon^p y_1\right)^2 + f_\lambda \epsilon \tag{2.29}$$

$$\epsilon^{p-q} y_{1\tau} = \epsilon^{2p}\frac{f_{yy}y_1^2}{2} + f_\lambda \epsilon \tag{2.30}$$

so that $p - q = 2p = 1$ means that $p = 1/2$, $q = -1/2$ and

$$y_{1\tau} = \frac{f_{yy}}{2} y_1^2 + f_\lambda. \tag{2.31}$$

Solving this, we find that

$$y = y_c + \sqrt{\frac{2f_\lambda}{f_{yy}}} \tan\left(\tau\sqrt{\frac{f_{yy}f_\lambda}{2}}\right), \tag{2.32}$$

which we expect to be valid until

$$\tau \sim \frac{\pi}{\sqrt{2f_{yy}f_\lambda}} \tag{2.33}$$

$$t \sim \frac{\pi}{\sqrt{2f_{yy}f_\lambda}} \cdot \frac{1}{\sqrt{\epsilon}} \tag{2.34}$$

As a result, even when we speed up this process by letting $\lambda - \lambda_c$ increase linearly with time, this natural delay causes the solution to take a while before it moves over to the opposite stationary branch.

2.2.1 Simple Example

If we consider the example from the introduction (§2.1), that

$$y' = y - \frac{1}{3}y^3 + \lambda, \tag{2.35}$$

then $f_\lambda = 1$ and $f_{yy} = -2y_c = 2$. From equations 2.23 and 2.27 we have that

$$y = -1 + \epsilon^{1/3}\frac{\mathrm{Ai}'(s)}{\mathrm{Ai}(s)}, \tag{2.36}$$

$$s = -t\epsilon^{1/3}. \tag{2.37}$$

We expect a transition to occur when $\mathrm{Ai}(s) = 0$, which is when $s \sim -2.33811$, or

$$t \sim \epsilon^{-1/3} \cdot 2.33811 \tag{2.38}$$

This approximate solution is shown in figure 2.4 for $\epsilon = 0.01$, in comparison with the numerical solution also shown in figure 2.3.

2.2.2 Insect Infestation

Introduction

A model for the spruce budworm infestation is

$$\frac{dN}{dt} = RN\left(1 - \frac{N}{k}\right) - P(N) \tag{2.39}$$

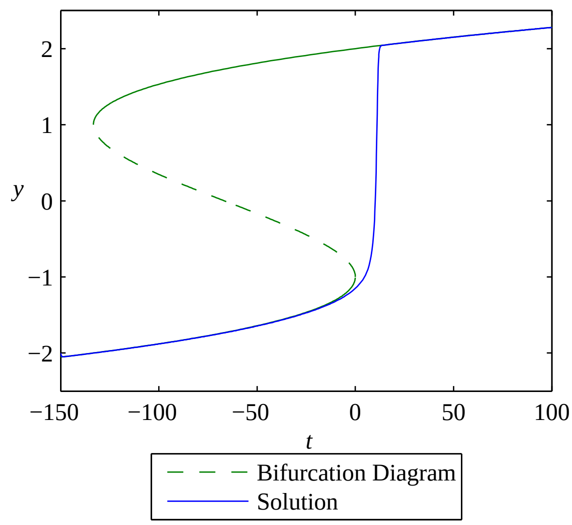

Figure 2.3: The numerical solution to equation 2.1 with $\epsilon = 0.01$ and $\lambda = \lambda_c + \epsilon t$. The initial condition is $y = -2$ at $t = -150$.

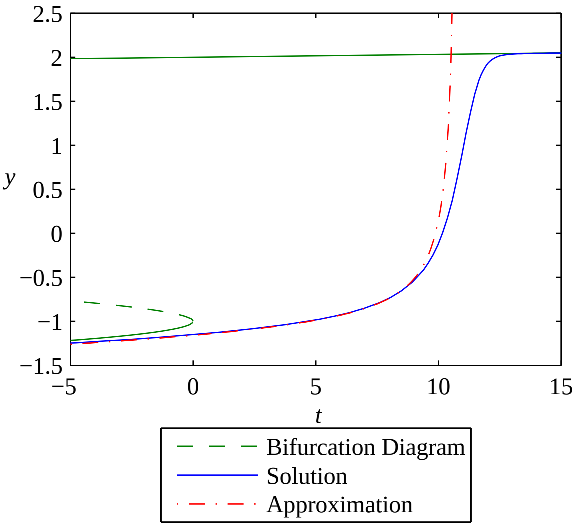

Figure 2.4: A zoomed-in version of the numerical solution to equation 2.1 with $\epsilon = 0.01$ and $\lambda = \lambda_c + \epsilon t$, as in figure 2.3. This is shown in comparison with the approximate solution given by equation 2.36.

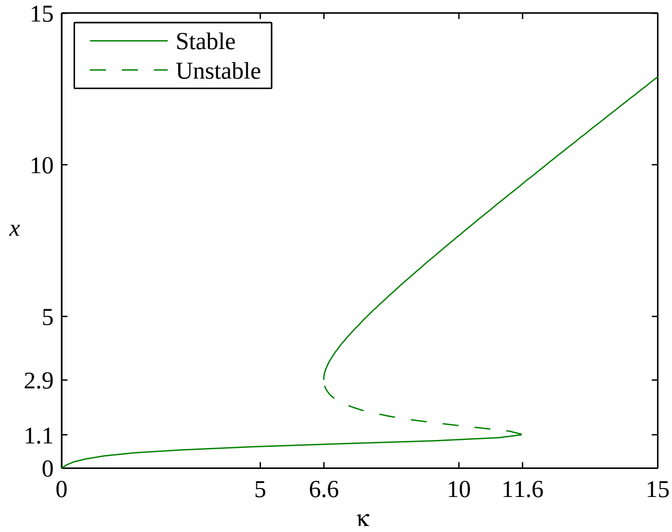

Figure 2.5: The stable and unstable branches of the bifurcation diagram for equation 2.42 for $r = 0.55$.

where N is the size of the population, R is the maximum rate of growth of the population, k is the carrying capacity of the environment and $P(N)$ is the rate of death (predation rate) of the population. The predation rate is given by

$$P(N) = \frac{BN^2}{A^2 + N^2}, \tag{2.40}$$

which goes through the origin and has a maximum as $N \to \infty$ of B. If we let

$$x = \frac{1}{A}N, \qquad \tau = \frac{B}{A}t, \qquad r = \frac{A}{B}R, \qquad \kappa = \frac{1}{A}k \tag{2.41}$$

then our problem reduces to the non-dimensional problem

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{\kappa}\right) - \frac{x^2}{1 + x^2}. \tag{2.42}$$

The equilibrium points for this are shown for $r = 0.55$ in figure 2.5. For r fixed we vary κ according to

$$\kappa = \left(\frac{\kappa_1 + \kappa_2}{2}\right) + 2 \times \left(\frac{\kappa_2 - \kappa_1}{2}\right) \sin(\epsilon t), \tag{2.43}$$

where κ_1 and κ_2 are the lower and upper bounds on the region where there is an unstable equilibrium point. We then obtain the solutions shown in figures 2.6 and 2.7. Observe that even though in figure 2.7 there are regions of time

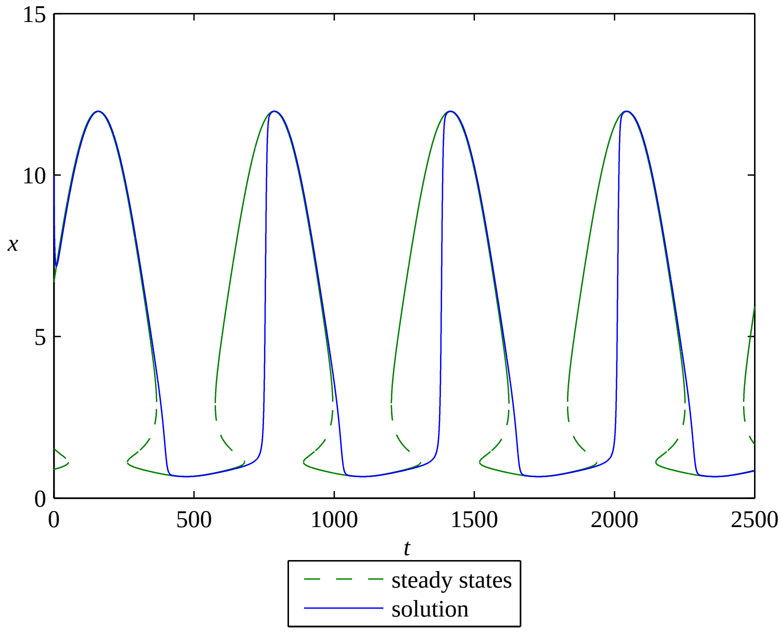

Figure 2.6: The solution to equation 2.42 for $r = 0.55$ and where κ varies in time as in equation 2.43. Here, $\epsilon = 0.01$ and the initial condition that was used is $x(0) = 10$.

in which the region of low population is no longer stable, due to the time delay required to reach the region of high population, the solution is forced to remain in the low-population state.

Number of Solutions

The equilibriums of equation 2.42 occur when

$$0 = rx\left(1 - \frac{x}{\kappa}\right) - \frac{x^2}{1+x^2} \tag{2.44}$$

$$r\left(1 - \frac{x}{\kappa}\right) = \frac{x}{1+x^2}. \tag{2.45}$$

By letting

$$y = \frac{x}{1+x^2} = r\left(1 - \frac{x}{\kappa}\right), \tag{2.46}$$

we can plot these and see where they intersect. See figure 2.8. For the value of r shown, there are two values of κ, labelled κ_1 and κ_2 in the figure, such that there are two equilibriums. As a result, a bifurcation occurs at κ_1 and κ_2 for this value of r. For $\kappa_1 < \kappa < \kappa_2$, there are three equilibriums, and for either

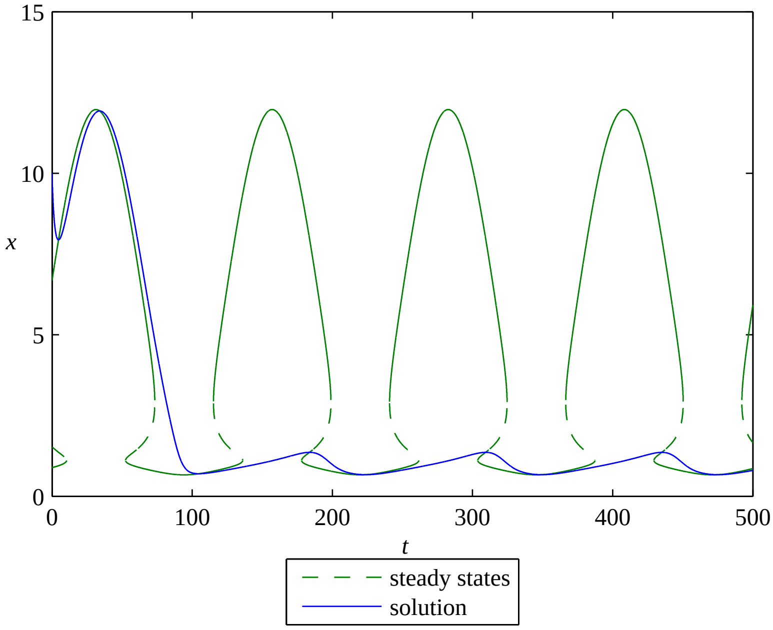

Figure 2.7: The solution to equation 2.42 for $r = 0.55$ and where κ varies in time as in equation 2.43. Here, $\epsilon = 0.05$ and the initial condition that was used is $x(0) = 10$.

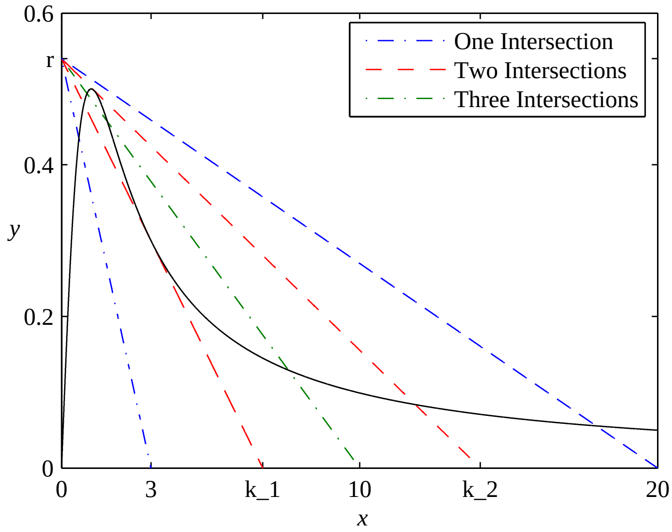

Figure 2.8: The curve and line from equation 2.46, with r fixed and different values of κ. This shows the possibility for 1, 2 or 3 equilibrium points.

$\kappa < \kappa_1$ or $\kappa > \kappa_2$, there is only one equilibrium. In order to determine the region in the r-κ plane in which there are three equilibriums, we then search for the values of r and κ such that the line is tangent to the curve. This will form the border of the region in question.

If the line is tangent to the curve at some point (x_0, y_0), then r and κ must satisfy

$$\frac{x_0}{1+x_0^2} = r\left(1 - \frac{x_0}{\kappa}\right), \tag{2.47}$$

$$\frac{1-x_0^2}{\left(1+x_0^2\right)^2} = \frac{r}{\kappa}. \tag{2.48}$$

By solving for these, we obtain the curve parametrically in terms of x_0:

$$r = \frac{2x_0^3}{\left(x^2+1\right)^2}, \qquad \kappa = \frac{2x^3}{x^2-1}. \tag{2.49}$$

By graphing this, we obtain figure 2.9.

Observe that, as in figure 2.10, that the smallest value of x_0 for which a line can be (almost) tangent to the curve is 1, where for a line to be tangent to the curve, it would need to have $r = 0.5, \kappa = \infty$. Since the curve here is concave down, as x_0 increases past 1, the value of r will rise until x_0 reaches the point of inflection, x_c, after which r will descend again. As a result, the maximum value of r occurs when the line is tangent to the curve at the point of inflection.

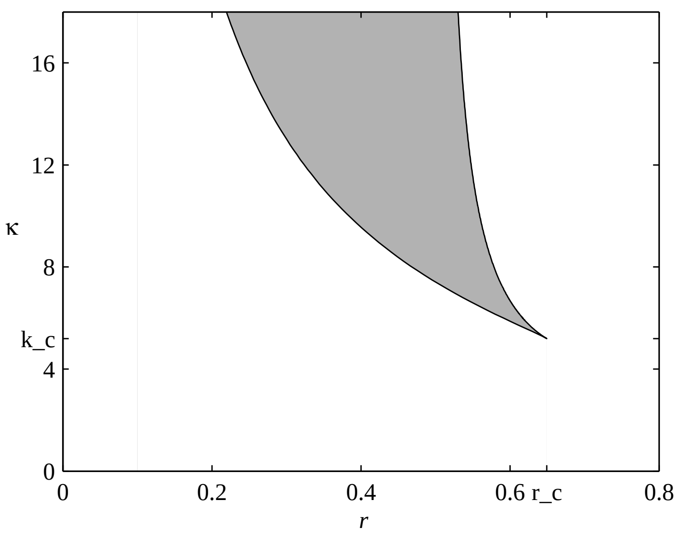

Figure 2.9: The region in the r-κ plane in which equation 2.42 has three equilibrium points.

We can calculate that this occurs when

$$x_c = \sqrt{3}, \qquad r_c = \frac{3\sqrt{3}}{8}, \qquad \kappa_c = 3\sqrt{3}. \tag{2.50}$$

Approximation to Delay

Here,

$$f_\kappa = \frac{rx^2}{\kappa^2} = 0.0051 \tag{2.51}$$

$$f_{xx} = \frac{-2r}{\kappa} - \frac{2-6x^2}{(1+x^2)^3} = 0.39 \tag{2.52}$$

If we let $\kappa = \kappa_c + \epsilon t$ for r fixed, the above analysis holds, with y replaced with x and λ replaced with κ. We then expect a transition to occur when $\mathrm{Ai}(s) = 0$, which is when $s \sim -2.33811$, or

$$t \sim \epsilon^{-1/3} \cdot 23.4768 \tag{2.53}$$

The approximate solution obtained from equation 2.23 is shown in figure 2.12 for $\epsilon = 0.01$, in comparison with the numerical solution also shown in figure 2.11.

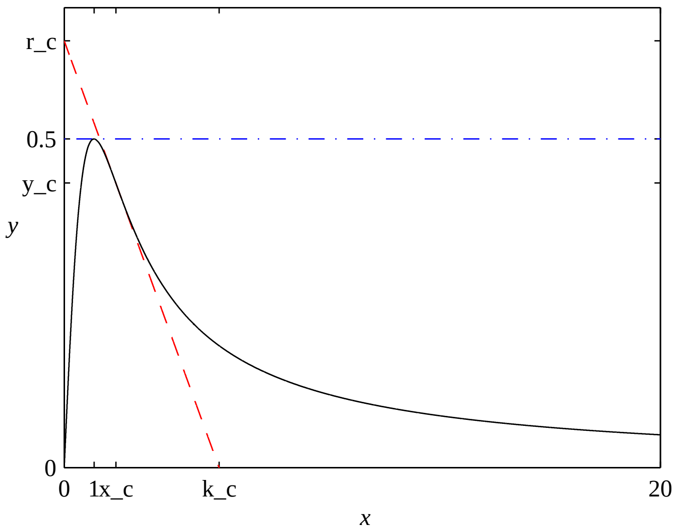

Figure 2.10: The curve and line from equation 2.46, with the critical values of r and κ given in equation 2.50.

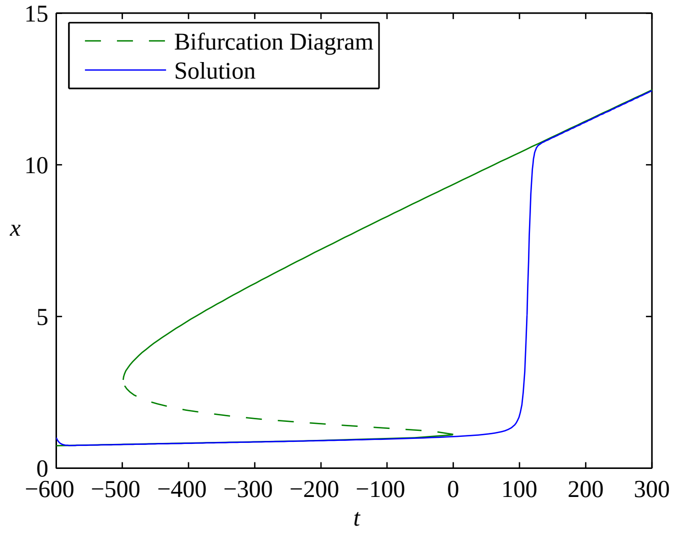

Figure 2.11: The numerical solution to equation 2.42 with $\epsilon = 0.01$, $r = 0.55$ and $\kappa = \kappa_c + \epsilon t$. The initial condition is $y = 1$ at $t = -600$.

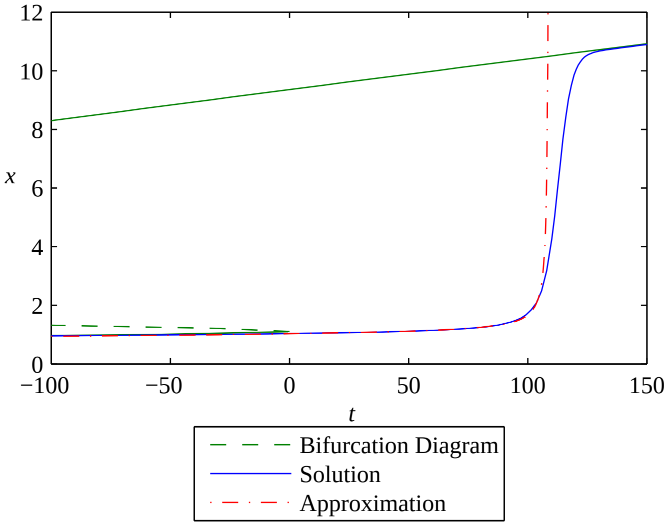

Figure 2.12: A zoomed-in version of the numerical solution to equation 2.42 with $\epsilon = 0.01$, $r = 0.55$ and $\lambda = \lambda_c + \epsilon t$, as in figure 2.11. This is shown in comparison with the approximate solution given by equation 2.23.

2.3 Time Delay in Second-Order Equation

Consider the equation

$$\frac{d^2x}{dt^2} + \kappa\frac{dx}{dt} = x\left(F - 2x^2\right). \tag{2.54}$$

We have equilibriums when

$$0 = x\left(F - 2x^2\right) \tag{2.55}$$

$$x = 0, \pm\sqrt{\frac{F}{2}}. \tag{2.56}$$

As a result, we get a bifurcation when $x = 0$, $F = 0$.

Note that even though these solutions can now be oscillatory, they still exhibit the delay between their original and final states. See figures 2.13–2.15.

Let us attempt to do a similar analysis as in the previous section. Let $F = 0 + \epsilon t$ and let us expand

$$x = 0 + \epsilon^p x_1, \qquad t = \epsilon^q \tau. \tag{2.57}$$

Then we obtain

$$\frac{d^2x_1}{d\tau^2}\epsilon^{p-2q} + \kappa\frac{dx_1}{d\tau}\epsilon^{p-q} = \tau x_1\epsilon^{1+p+q} - 2x_1^3\epsilon^{3p}. \tag{2.58}$$

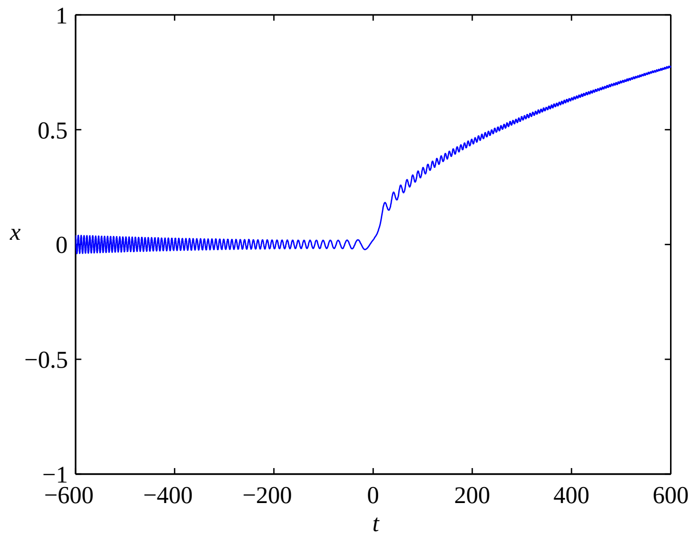

Figure 2.13: The solution to equation 2.54, using the following parameters: $x(-600) = 0.04$, $x'(-600) = 0$, $\epsilon = 0.002$, $\kappa = 0.005$.

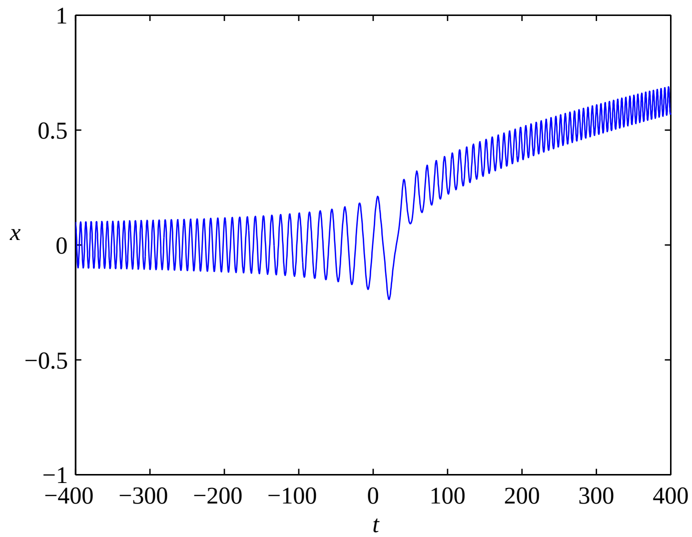

Figure 2.14: The solution to equation 2.54, using the following parameters: $x(-400) = 0.1$, $x'(-400) = 0$, $\epsilon = 0.002$, $\kappa = 0$.

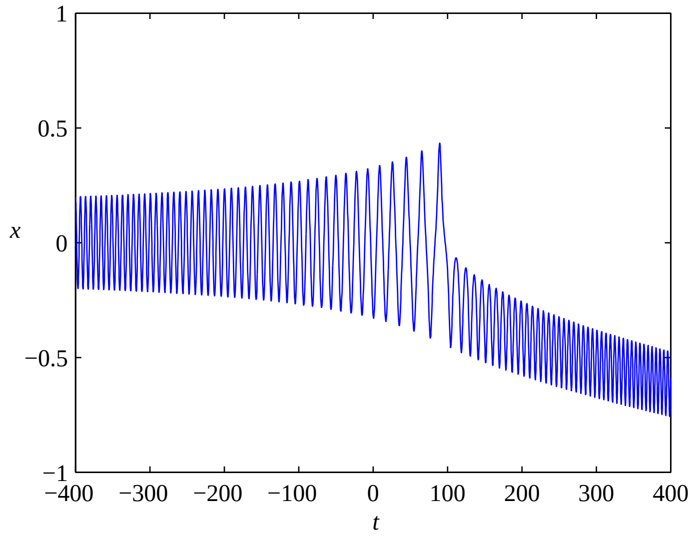

Figure 2.15: The solution to equation 2.54, using the following parameters: $x(-400) = 0.2$, $x'(-400) = 0$, $\epsilon = 0.002$, $\kappa = 0$.

To simplify the problem, we will make some assumptions about the size of κ. In the first case, we will assume κ is of order 1. In the second, we will assume κ is approximately 0 (i.e., small in comparison with ϵ) and x_1 is small. In both of these cases, the problem will not feature large oscillations. Finally, we outline what one can do for κ of order ϵ with no assumptions about the size of x_1, which involves extending out the range of validity of the case when κ is approximately 0 and x_1 is small.

2.3.1 Approximation for κ of order 1

For this approximation, we assume that κ is of order 1. As a result, we must have either

$$p - 2q = 3p, \qquad p - 1 = 1 + p + q \tag{2.59}$$

or

$$p - 2q = 1 + p + q, \qquad p - q = 3p. \tag{2.60}$$

Case I

In the case

$$p - 2q = 3p, \qquad p - 1 = 1 + p + q, \tag{2.61}$$

we obtain

$$q = -\frac{1}{2}, \qquad\qquad p = \frac{1}{2}. \tag{2.62}$$

As a result, our differential equation becomes

$$\frac{d^2x_1}{d\tau^2}\epsilon^{3/2} + \kappa\frac{dx_1}{d\tau}\epsilon = \tau x_1\epsilon - 2x_1^3\epsilon^{3/2}. \tag{2.63}$$

Now under the assumption that κ is of order 1, and only keeping the highest terms, we obtain

$$\kappa\frac{dx_1}{d\tau} = \tau x_1 \tag{2.64}$$

$$x_1 = Ce^{\tau^2/2\kappa} \tag{2.65}$$

$$x = C\sqrt{\epsilon}e^{t^2\epsilon/2\kappa} \tag{2.66}$$

$$x = x(0)e^{t^2\epsilon/2\kappa}. \tag{2.67}$$

Case II

In the case

$$p - 2q = 1 + p + q, \qquad\qquad p - q = 3p, \tag{2.68}$$

we obtain

$$q = -\frac{1}{3}, \qquad\qquad p = \frac{1}{6}. \tag{2.69}$$

As a result, our differential equation becomes

$$\frac{d^2x_1}{d\tau^2}\epsilon^{3/4} + \kappa\frac{dx_1}{d\tau}\epsilon^{1/2} = \tau x_1\epsilon^{3/4} - 2x_1^3\epsilon^{1/2}. \tag{2.70}$$

Again under the assumption that κ is of order 1, and only keeping the highest terms, we obtain

$$\kappa\frac{dx_1}{d\tau}\epsilon^{1/2} = -2x_1^3\epsilon^{1/2} \tag{2.71}$$

$$x_1 = \frac{\pm 1}{\sqrt{\frac{4}{\kappa}\tau + C}}. \tag{2.72}$$

Notice, however, that as $\tau \to \infty$, this goes to 0. We expect that as $\tau \to \infty$, our solution should also go to infinity, since we want a transition to the stable states. As a result, we must reject this solution.

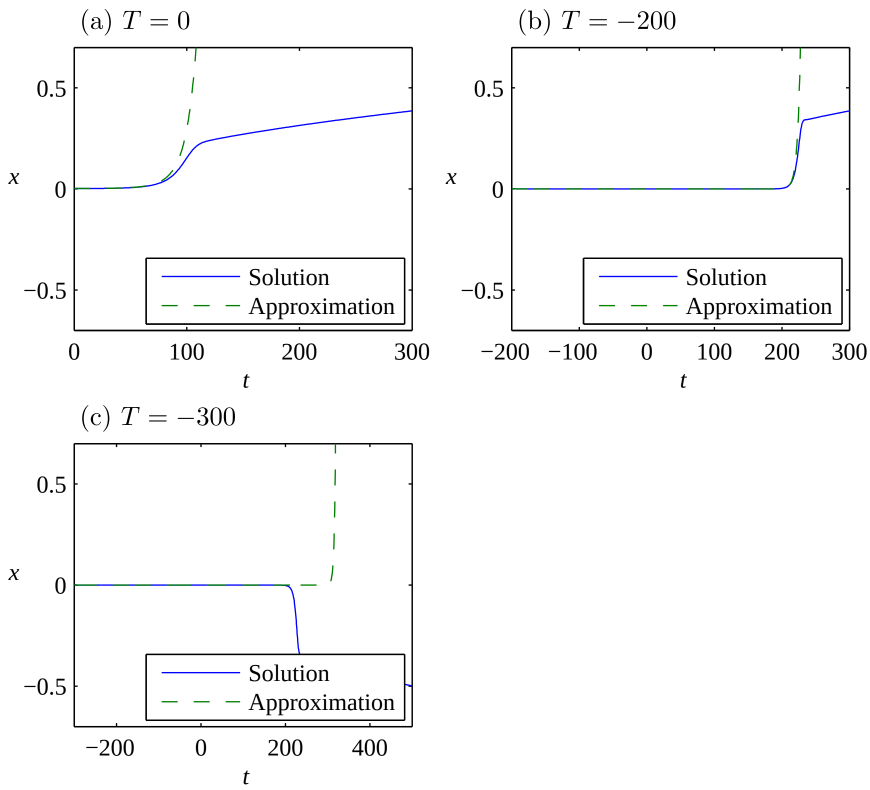

Figure 2.16: The solution to equation 2.54 in comparison to the approximation in equation 2.73, using the values in equation 2.74 except that the starting time T is varied.

Comparison Between Approximation and Numerical Simulations

Under the assumption that κ is of order 1, we have then that our solution takes the form

$$x = x(0)e^{t^2\epsilon/2\kappa}. \tag{2.73}$$

We use the values

$$T = 0, \qquad x(T) = 0.002, \qquad x'(T) = 0, \qquad \epsilon = 0.001, \qquad \kappa = 1 \tag{2.74}$$

and compare this approximation to numerical simulations for different variations in the parameters. We observe from figure 2.16 that the approximation doesn't work well if $-T$ is too large, and so we used $T = 0$ in all the other figures. Note that we observe the approximation failing when $x(T)$ is too big in figure 2.17 and when κ is too small in figure 2.18.

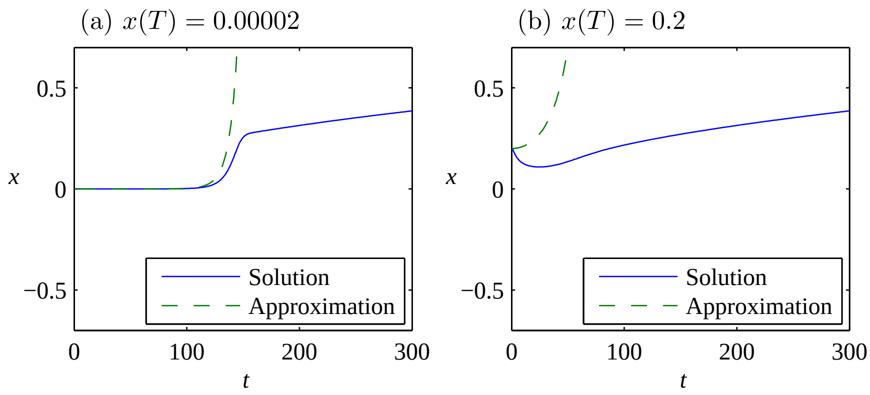

Figure 2.17: The solution to equation 2.54 in comparison to the approximation in equation 2.73, using the values in equation 2.74 except that the starting position $x(T)$ is varied.

2.3.2 Approximation for $\kappa = 0$ and x_1 Small

If $\kappa = 0$, then our problem reduces to

$$\frac{d^2x_1}{d\tau^2}\epsilon^{p-2q} = \tau x_1 \epsilon^{1+p+q} - 2x_1^3\epsilon^{3p}. \tag{2.75}$$

As a result, we must have

$$p - 2q = 1 + p + q = 3p, \tag{2.76}$$

so that $p = 1/3$, $q = -1/3$ and our problem becomes

$$\frac{d^2x_1}{d\tau^2} = \tau x_1 - 2x_1^3. \tag{2.77}$$

If we further assume that x_1 is small, this is approximately

$$\frac{d^2x_1}{d\tau^2} = \tau x_1 \tag{2.78}$$

$$x_1 = C_1 \operatorname{Ai}(\tau) + C_2 \operatorname{Bi}(\tau), \tag{2.79}$$

where C_1 and C_2 can be determined by requiring that $x(t)$ satisfies the given initial conditions at $t = T$ for some $T < 0$. We can use the values

$$T = -100, \quad x(T) = 0.01, \quad x'(T) = 0, \quad \epsilon = 0.001, \quad \kappa = 0 \tag{2.80}$$

and compare this approximation to numerical simulations for different variations in the parameters. Note that we observe the approximation failing when κ is too big (which is natural since our approximation has no κ dependence) in figure 2.19 and when $x(T)$ is too big in figure 2.20. Note that in figure 2.20a, the solution appears to become large at approximately the right time, however in the wrong direction. In figure 2.20b, the solution becomes large at the wrong time, too.

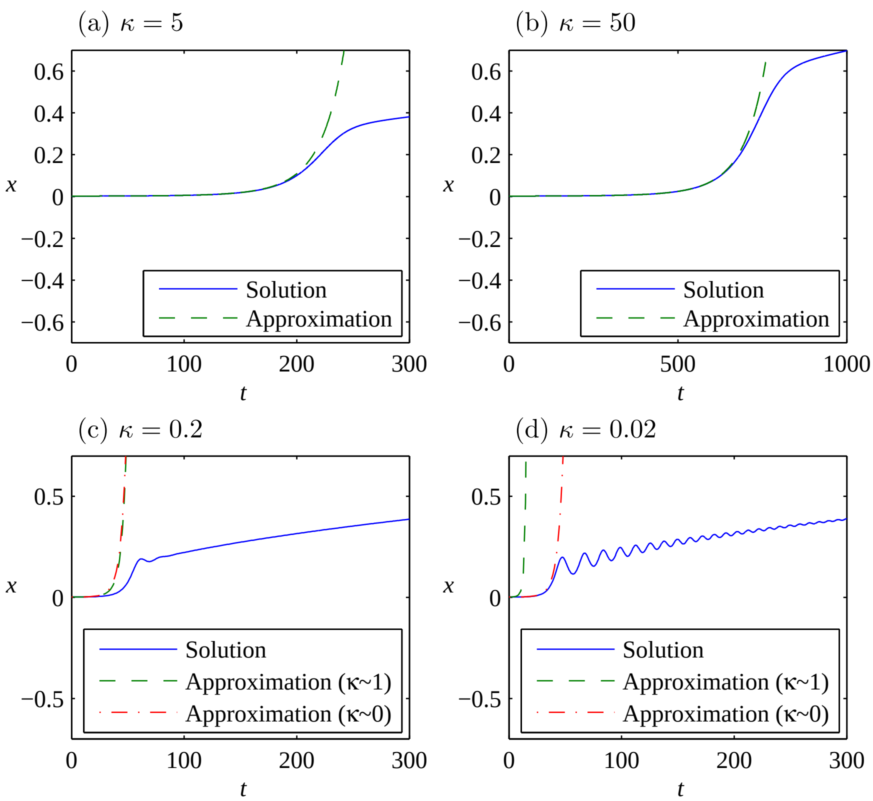

Figure 2.18: The solution to equation 2.54 in comparison to the approximations in equations 2.73 and 2.79, using the values in equation 2.74 except that κ is varied.

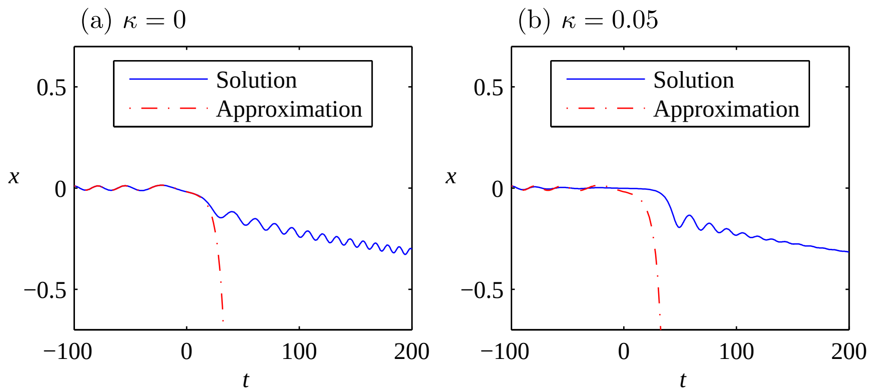

Figure 2.19: The solution to equation 2.54 in comparison to the approximation in equation 2.79, using the values in equation 2.80 except that κ is varied.

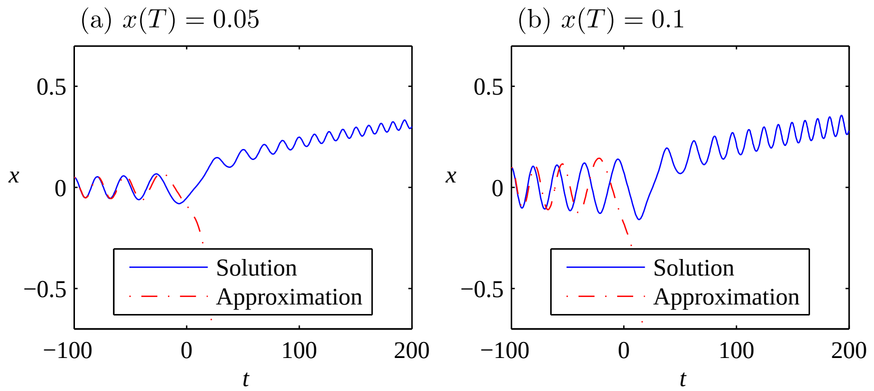

Figure 2.20: The solution to equation 2.54 in comparison to the approximation in equation 2.79, using the values in equation 2.80 except that $x(T)$ is varied.

2.3.3 Approximation for κ order ϵ

If we assume κ is of order ϵ, it is possible to, after much work (see [16]), obtain a very good approximation for this problem. We give here only a very brief outline of the key steps involved.

First, the solution to equation 2.54 can be approximated for times outside a region near zero of the order of $\epsilon^{-1/3}$, for negative time and for positive time (Corollaries 3.3 and 4.2 in the paper by Marée [16]). We then search for an approximation for time near zero to join these two approximations together. These three regions will represent the time a bit before the bifurcation, the time near the bifurcation (which is where the transition to a different state occurs) and the time a bit after the bifurcation. Performing the usual transformation, we end up with (since we assume $\kappa = \epsilon k$)

$$\frac{d^2x_1}{d\tau^2}\epsilon^{p-2q} + k\frac{dx_1}{dt}\epsilon^{1+p-q} = \tau x_1\epsilon^{1+p+q} - 2x_1^3\epsilon^{3p} \tag{2.81}$$

where we're looking for a solution valid for small time, and so $q < 0$. We can see that the ϵ^{1+p-q} term is certainly smaller than the ϵ^{1+p+q} term, and so we must have to highest order

$$\frac{d^2x_1}{d\tau^2}\epsilon^{p-2q} = \tau x_1\epsilon^{1+p+q} - 2x_1^3\epsilon^{3p} \tag{2.82}$$

so that

$$p - 2q = 1 + p + q = 3p. \tag{2.83}$$

We then obtain that $q = -1/3$ and $p = 1/3$, so that

$$\frac{d^2x_1}{d\tau^2} = \tau x_1 - 2x_1^3, \tag{2.84}$$

where this is the second Painlevé equation. Note that we get the time scaling ($t = \epsilon^{-1/3}\tau$) which is the exact one required to join our two large-time approximations together. Also note that this is the same result as we obtained in §2.3.2 under the assumption that $\kappa = 0$.

The crucial result is that we have a theorem (Theorem 6.1 in the paper by Marée [16]) that gives us an approximation for the solution to equation 2.84 for large positive time and for large negative time, where *the coefficients in the approximation for large positive time can be explicitly determined from the coefficients in the approximation for large negative time.* We can then equate the coefficients in the approximation to equation 2.54 for large negative time (which are determined from the initial conditions) to those in the approximation to equation 2.84 for large negative time. From these, we can determine the coefficients in the approximation to equation 2.84 for large positive time, which we can finally use to determine those in the approximation to equation 2.54 for large positive time. Hence we have determined the coefficients to the approximation to 2.54 after the transition, in terms of the initial conditions. This approximation will tell us in which stable state our solution will end up, whether the positive x state or the negative x state.