Chapter 3

Basic Floquet Theory

3.1 General Results

If we have a problem of the form

\[ x' = A(t)x \]  

(3.1)

where \( A(t) \) is periodic with period \( T \), then \( x \) need not be periodic, however it must be of the form

\[ e^{\mu t} p(t) \]  

(3.2)

where \( p(t) \) has period \( T \). Additionally, it has \( n \) such \( \mu_j \) and together they satisfy

\[ e^{\mu_1 T} e^{\mu_2 T} \cdots e^{\mu_n T} = \exp \left( \int_0^T \text{tr} (A(s)) \, ds \right). \]  

(3.3)

The following theorems prove those results. We follow Ward [28].

Definition (Fundamental Matrix). Let \( x^1(t), \ldots, x^n(t) \) be \( n \) solutions of \( x' = A(t)x \). Let

\[ X(t) = \begin{bmatrix} x^1 & \cdots & x^n \end{bmatrix} \]  

(3.4)

so that \( X(t) \) is an \( n \times n \) matrix solution of \( X' = AX \).

If \( x^1(t), \ldots, x^n(t) \) are linearly independent, then \( X(t) \) is non-singular and is called a fundamental matrix. If \( X(t_0) = I \), then \( X(t) \) is the principal fundamental matrix.

Lemma 3.1. If \( X(t) \) is a fundamental matrix then so is \( Y(t) = X(t)B \) for any non-singular constant matrix \( B \).

Proof. Since \( X(t) \) and \( B \) are non-singular then the inverse of \( Y(t) \) is \( B^{-1}X^{-1}(t) \) and so \( Y(t) \) is non-singular. Also,

\[ Y' = X'B = AXB = AY \]  

(3.5)
so that $Y'(t) = AY(t)$.

Lemma 3.2. Let the Wronskian $W(t)$ of $X(t)$ be the determinant of $X(t)$. Then

$$W(t) = W(t_0) \exp \left( \int_{t_0}^{t} \text{tr} (A(s)) \, ds \right).$$  \hfill (3.6)

Proof. Let $t_0$ be some time. Expanding in a Taylor series,

$$X(t) = X(t_0) + (t - t_0) X'(t_0) + O \left( (t - t_0)^2 \right)$$  \hfill (3.7)

$$= X(t_0) + (t - t_0) A(t_0) X(t_0) + O \left( (t - t_0)^2 \right)$$  \hfill (3.8)

$$= \left[ I + (t - t_0) A(t_0) \right] X(t_0) + O \left( (t - t_0)^2 \right)$$  \hfill (3.9)

so that

$$\det (X(t)) = \det \left[ I + (t - t_0) A(t_0) \right] \det (X(t_0))$$  \hfill (3.10)

$$W(t) = \det \left[ I + (t - t_0) A(t_0) \right] W(t_0).$$  \hfill (3.11)

Now since

$$\det (I + \epsilon C) = 1 + \epsilon \text{tr} (C) + O \left( \epsilon^2 \right),$$  \hfill (3.12)

we have that

$$W(t) = W(t_0) \left( 1 + (t - t_0) \text{tr} (A(t_0)) \right).$$  \hfill (3.13)

Now by expanding $W(t)$ in a Taylor series, we obtain that

$$W(t) = W(t_0) + (t - t_0) W'(t_0) + O \left( (t - t_0)^2 \right)$$  \hfill (3.14)

so that

$$W'(t_0) = W(t_0) \text{tr} (A(t_0)).$$  \hfill (3.15)

Since we have not made any assumptions about $t_0$, we can the write

$$W'(t) = W(t) \text{tr} (A(t)).$$  \hfill (3.16)

We know that the solution to this equation is

$$W(t) = W(t_0) \exp \left( \int_{t_0}^{t} \text{tr} (A(s)) \, ds \right)$$  \hfill (3.17)

Theorem 3.3. Let $A(t)$ be a $T$-periodic matrix. If $X(t)$ is a fundamental matrix then so is $X(t + T)$ and there exists a non-singular constant matrix $B$ such that

i. $X(t + T) = X(t)B$ for all $t$

ii. $\det (B) = \exp \left( \int_{0}^{T} \text{tr} (A(s)) \, ds \right)$
Proof. Begin by showing that $X(t+T)$ is also a fundamental matrix. Let $Y(t) = X(t+T)$. Then

$$Y'(t) = X'(t+T) = A(t+T)X(t+T) = A(t)X(t+T) = A(t)Y(t) \quad (3.18)$$

and so $X(t+T)$ is a fundamental matrix.

i. Let $B(t) = X^{-1}(t)Y(t)$. Then

$$Y(t) = X(t)X^{-1}(t)Y(t) \quad (3.19)$$

$$= X(t)B(t) \quad (3.20)$$

Let $B_0 = B(t_0)$. We know by lemma 3.1 that $Y_0(t) = X(t)B_0$ is a fundamental matrix, where, by definition, $Y_0(t_0) = Y(t_0)$. Since these are both solutions to $X' = AX$, by the uniqueness of the solution, we must then have $Y_0(t) = Y(t)$ for all time. As a result, $B_0 = B(t)$ and so $B$ is time-independent.

ii. From lemma 3.2, we have that

$$W(t) = W(t_0) \exp \left( \int_{t_0}^{t} \text{tr} (A(s)) \, ds \right) \quad (3.21)$$

$$W(t + T) = W(t_0) \exp \left( \int_{t_0}^{t} \text{tr} (A(s)) \, ds + \int_{t}^{t+T} \text{tr} (A(s)) \, ds \right) \quad (3.22)$$

$$W(t + T) = W(t) \exp \left( \int_{t}^{t+T} \text{tr} (A(s)) \, ds \right) \quad (3.23)$$

$$W(t + T) = W(t) \exp \left( \int_{0}^{T} \text{tr} (A(s)) \, ds \right). \quad (3.24)$$

We also know that

$$X(t + T) = X(t)B \quad (3.25)$$

$$\det \left( X(t + T) \right) = \det \left( X(t) \right) \det \left( B \right) \quad (3.26)$$

$$W(t + T) = W(t) \det \left( B \right) \quad (3.27)$$

and so

$$\det \left( B \right) = \exp \left( \int_{0}^{T} \text{tr} (A(s)) \, ds \right) \quad (3.28)$$

Remark. Since $B$ is time-independent, it can be computed by setting $t = 0$, so that $B = X^{-1}(0)X(T)$. If we took the initial conditions $X(0) = I$, then $B = X(T)$.

Definition (Characteristic Multipliers and Exponents). The eigenvalues $\rho_1, \ldots, \rho_n$ of $B$ are called the characteristic multipliers for $X'(t) = A(t)X(t)$. The characteristic exponents or Floquet exponents are $\mu_1, \ldots, \mu_n$ satisfying

$$\rho_1 = e^{\mu_1 T}, \quad \rho_2 = e^{\mu_2 T}, \quad \ldots \quad \rho_n = e^{\mu_n T}. \quad (3.29)$$

Note that $\mu_j$ for $j \in \mathbb{N}$ may be complex.
Properties.

i. The characteristic multipliers (eigenvalues) $\rho_1, \ldots, \rho_n$ of $B = X(T)$ with $X(0) = I$ satisfy

$$\text{det } (B) = \rho_1 \rho_2 \cdots \rho_n = \exp \left( \int_0^T \text{tr } (A(s)) \, ds \right). \quad (3.30)$$

This follows from theorem 3.3ii.

ii. Since the trace is the sum of the eigenvalues, we also have

$$\text{tr } (B) = \rho_1 + \rho_2 + \cdots + \rho_n. \quad (3.31)$$

iii. The characteristic exponents are not unique since if $\rho_j = e^{\mu_j T}$, then $\rho_j = e^{(\mu_j + 2\pi i/T)T}$.

iv. The characteristic multipliers $\rho_j$ are an intrinsic property of the equation $X'(t) = AX$ and do not depend on the choice of the fundamental matrix.

Proof. Suppose $\hat{X}(t)$ is another fundamental matrix. Then

$$\hat{X}(t + T) = \hat{X}(t)\hat{B}. \quad (3.32)$$

We have showed in the proof of theorem 3.3 that since $X(t)$ and $\hat{X}(t)$ are fundamental matrices then there is a constant non-singular matrix $C$ such that

$$\hat{X}(t) = X(t)C \quad (3.33)$$

so that

$$\hat{X}(t + T) = X(t + T)C \quad (3.34)$$

$$\left(\hat{X}(t)\hat{B}\right) = (X(t)B)C \quad (3.35)$$

$$X(t)CB = X(t)BC \quad (3.36)$$

$$CB = BC \quad (3.37)$$

$$C^2 = CB$$

so the eigenvalues of $B$ and $\hat{B}$ are the same. \hfill \Box

Theorem 3.4. Let $\rho$ be a characteristic multiplier and let $\mu$ be the corresponding characteristic exponent so that $\rho = e^{\mu T}$. Then there exists a solution $x(t)$ of $x' = A(t)x$ such that

i. $x(t + T) = \rho x(t)$

ii. There exists a periodic solution $p(t)$ with period $T$ such that $x(t) = e^{\mu t}p(t)$.

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Proof.

i. Let \( b \) be an eigenvector of \( B \) corresponding to eigenvalue \( \rho \). Let \( x(t) = X(t)b \). Then \( x' = Ax \) and

\[
\begin{align*}
x(t + T) &= X(t + T)b \\
&= X(t)Bb \\
&= \rho X(t)b \\
&= \rho x(t)
\end{align*}
\]

so that \( x(t + T) = \rho x(t) \).

ii. Let \( p(t) = x(t)e^{-\mu t} \). We now need to show that \( p(t) \) is \( T \)-periodic.

\[
\begin{align*}
p(t + T) &= x(t + T)e^{-\mu (t+T)} \\
&= \rho x(t)e^{-\mu (t+T)} \\
&= \frac{\rho}{e^{\mu T}} x(t)e^{-\mu t} \\
&= x(t)e^{-\mu t} \\
&= p(t)
\end{align*}
\]

As a result, we have a solution of the form \( x(t) = e^{\mu t}p(t) \) where \( p(t) \) is periodic with period \( T \).

Remarks.

i. If \( \mu \) is replaced by \( \mu + 2\pi i/T \), then we get

\[
x(t) = e^{\mu t}p(t)e^{2\pi i t/T}
\]

where \( p(t)e^{2\pi i t/T} \) is still periodic with period \( T \). As a result, the fact that \( \mu \) is not unique does not alter our results.

ii. We have that

\[
\begin{align*}
x_j(t + T) &= \rho_j x_j(t) \\
x_j(t + NT) &= \rho_j^N x_j(t).
\end{align*}
\]

Each characteristic multipliers falls into one of the following categories:

(a) If \( |\rho| < 1 \), then \( \text{Re}(\mu) < 0 \) and so \( x(t) \xrightarrow{t \to \infty} 0 \).

(b) If \( |\rho| = 1 \), then \( \text{Re}(\mu) = 0 \) and so we have a pseudo-periodic solution. If \( \rho = \pm 1 \), then the solution is periodic with period \( T \).

(c) If \( |\rho| > 1 \), then \( \text{Re}(\mu) > 0 \) and so \( x(t) \sim \infty \) as \( t \to \infty \).

The entire solution is stable if all the characteristic multipliers satisfy \( |\rho_j| \leq 1 \).
As for the general solution, suppose that $b_1, \ldots, b_n$ are $n$ linearly independent eigenvectors of $B$ corresponding to distinct eigenvalues $\rho_1, \ldots, \rho_n$. Then there are $n$ linearly independent solutions to $x' = Ax$, which by the above theorem are given by

$$x_j(t) = e^{\mu_j t} p_j(t)$$

where $p_j(t)$ is $T$-periodic. As a result, we can define

$$X_0(t) = \begin{bmatrix} x_1 \cdots x_n \end{bmatrix}, \quad P_0(t) = \begin{bmatrix} p_1 \cdots p_n \end{bmatrix},$$

$$D_0(t) = \begin{bmatrix} \mu_1 & 0 \\ \vdots & \ddots \\ 0 & \mu_n \end{bmatrix}, \quad Y_0(t) = \begin{bmatrix} e^{\mu_1 t} & 0 \\ \vdots & \ddots \\ 0 & e^{\mu_n t} \end{bmatrix},$$

such that

$$X_0 = P_0 Y_0, \quad Y_0' = D_0 Y_0.$$  

Now consider what happens if $\rho < 0$. Suppose $\rho < 0$ real, so that we can write

$$\rho = e^{(\nu + i\pi/T)T}$$

where

$$\rho = -e^{\nu T}.$$  

Then we obtain

$$x(t) = e^{\mu t} p(t)$$

$$= e^{\nu t} e^{i\pi t/T} p(t)$$

$$= e^{\nu t} q(t),$$

where $q(t)$ has period $T$ since $p(t)$ has period $T$. Since we can choose $x$ to be real, without loss of generality, we can also choose $q$ to be real. For the general solution, if $\rho_j < 0$, we can replace $p_j$ with $q_j$ and $\mu_j$ with $\nu_j$ so that

$$P_0 = \begin{bmatrix} p_1 \cdots q_j \cdots p_n \end{bmatrix}, \quad Y_0 = \begin{bmatrix} e^{\nu_1 T} & 0 \\ \vdots & \ddots \\ 0 & e^{\nu_n T} \end{bmatrix}$$

and

$$X_0(t) = P_0(t) Y_0(t).$$
v. Suppose now that $\rho$ is complex. Then since $\rho$ is an eigenvalue of the real matrix $B$, $\overline{\rho}$ is as well. The characteristic exponents are $\mu$ and $\overline{\mu}$. Let

$$\mu = \nu + i\sigma,$$

and

$$p(t) = q(t) + i r(t)$$

(3.62)

where $q(t)$ and $r(t)$ must both have period $T$ since $p(t)$ does. Since $x(t) = e^{\mu t} p(t)$ is a solution to $\dot{x}' = A(t)x$, then by taking the complex conjugate, so is $\overline{x}(t) = e^{\overline{\mu} t} \overline{p}(t)$. We can write these as

$$x(t) = e^{(\nu + i\sigma)t} (q(t) + i r(t))$$

(3.63)

and

$$\overline{x}(t) = e^{(\nu - i\sigma)t} (q(t) - i r(t))$$

(3.65)

(3.64)

We can alternately write the linearly independent real solutions

$$x_R = \text{Re} \left[ e^{\nu t} p(t) \right] = e^{\nu t} \left[ \cos (\sigma t) q(t) - \sin (\sigma t) r(t) \right],$$

$$x_I = \text{Im} \left[ e^{\nu t} p(t) \right] = e^{\nu t} \left[ \sin (\sigma t) q(t) + \cos (\sigma t) r(t) \right],$$

(3.67)

(3.68)

so that

$$X_0 = \begin{bmatrix} x_1 & \cdots & x_R & x_I & \cdots & x_n \end{bmatrix},$$

(3.69)

$$P_0 = \begin{bmatrix} p_1 & \cdots & q & r & \cdots & p_n \end{bmatrix},$$

(3.70)

$$Y_0 = \begin{bmatrix} 0 & e^{\mu_1 T} & e^{\mu_2 T} & \cdots & e^{\mu_n T} \\
\vdots & e^{\nu t} \cos (\sigma t) & e^{\nu t} \sin (\sigma t) & \cdots & 0 \\
0 & -e^{\nu t} \sin (\sigma t) & e^{\nu t} \cos (\sigma t) & \cdots & 0 \\
\end{bmatrix}$$

(3.71)

and

$$X_0(t) = P_0(t) Y_0(t).$$

(3.72)

### 3.1.1 Example

For example, consider

$$x_1' = \left(1 + \frac{\cos(t)}{2 + \sin(t)}\right) x_1$$

(3.73)

$$x_2' = x_1 - x_2.$$  

(3.74)
Here, we know that the solution is in general
\[ x_1 = c_1 e^{t} (2 + \sin (t)) \]  
\[ x_2 = c_1 e^{t} \left( 2 + \frac{1}{2} \sin (t) - \frac{1}{2} \cos (t) \right) + c_2 e^{-t} \]
which we can write as
\[ x = c_1 e^{t} \left[ 2 + \frac{2 + \sin (t)}{2 + \frac{1}{2} \sin (t) - \frac{1}{2} \cos (t)} \right] + c_2 e^{-t} \left[ \begin{array}{c} 0 \\ 1 \end{array} \right]. \]

Using all the above definitions, the fundamental matrix is
\[ X(t) = \begin{bmatrix} e^{t} (2 + \sin (t)) & 0 \\ e^{-t} (2 + \frac{1}{2} \sin (t) - \frac{1}{2} \cos (t)) & e^{-t} \end{bmatrix} \]
so that
\[ B = X^{-1}(0)X(2\pi) \]
\[ = \begin{bmatrix} 2 & 0 \\ \frac{3}{2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 e^{2\pi} & 0 \\ \frac{3}{2} e^{2\pi} & e^{-2\pi} \end{bmatrix} \]
\[ = \begin{bmatrix} 2 e^{2\pi} & 0 \\ \frac{3}{2} e^{2\pi} & e^{-2\pi} \end{bmatrix} \]
\[ = \begin{bmatrix} e^{2\pi} & 0 \\ 0 & e^{-2\pi} \end{bmatrix} \]
As a result \( \rho_1 = e^{2\pi}, \rho_2 = e^{-2\pi} \) and so \( \mu_1 = 1 \) and \( \mu_2 = -1 \). Theorem 3.4 then tells us that there is a solution of the form
\[ x_1(t) = e^{t} p_1(t), \quad x_2(t) = e^{-t} p_2(t) \]
where \( p_1(t) \) and \( p_2(t) \) are periodic with period \( 2\pi \). We know that in fact
\[ p_1(t) = \begin{bmatrix} 2 + \sin (t) \\ 2 + \frac{1}{2} \sin (t) - \frac{1}{2} \cos (t) \end{bmatrix}, \quad p_2(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]

### 3.1.2 Periodic Solution
Consider a problem of the form \( x' = f(x) \) with \( x \in \mathbb{R}^n \) where there is a periodic solution \( x(t) = \phi(t) \) with period \( T \). Linearise the solution about \( \phi \) by writing \( x = \phi + v \). We then obtain
\[ v' = A(t)v \]
where \( A(t) \) is the Jacobian of \( f \) (so \( A_{ij}(t) = \frac{\partial f_i}{\partial x_j} |_{\phi(t)} \)). Since \( \phi(t) \) has period \( T \), so does \( A(t) \). Now let \( X(t) \) be the principal fundamental matrix of \( v' = Av \) (so that \( X(0) = I \)). Then \( B = X(T) \).
Now by definition,
\[ \phi'(t) = f(\phi(t)) \] (3.86)
so
\[ \phi''(t) = \frac{\partial f_i}{\partial x_j} \phi'(t) \] (3.87)
\[ \phi''(t) = A(t)\phi'(t) \] (3.88)

If we let \( v = \phi' \), then
\[ v'(t) = A(t)v(t) \] (3.89)
where, since \( \phi(t) \) has period \( T \) by assumption, \( v(t) \) must also, and so the corresponding characteristic multiplier is 1. As a result, for a nonlinear system with a periodic solution, one characteristic multiplier is always \( \rho = 1 \).

3.2 General Results for \( n = 2 \)

3.2.1 Stability of Periodic Solution

Consider a problem of the form \( x' = f(x) \) with \( x \in \mathbb{R}^2 \) where there is a periodic solution \( x(t) = \phi(t) \) with period \( T \). We know from §3.1.2 that we must have \( \rho_1 = 1 \) and we know from theorem 3.3ii that
\[ \rho_1 \rho_2 = \exp \left( \int_0^T \text{tr} (A(s)) \, ds \right) \] (3.90)
\[ \rho_2 = \exp \left( \int_0^T \text{tr} (A(s)) \, ds \right). \] (3.91)

From remark (ii) on page 53, we know that for the perturbation to be bounded and hence for the solution to be stable, we must have \( \rho_1 \leq 1 \) and \( \rho_2 \leq 1 \) and so, since we know \( \rho_1 = 1 \) and we wish \( \rho_1 \) and \( \rho_2 \) to be distinct, we must have
\[ 0 > \int_0^T \text{tr} (A(s)) \, ds \] (3.92)
\[ 0 > \int_0^T \text{tr} \left( \frac{\partial f_i}{\partial x_j} \phi(s) \right) \, ds \] (3.93)
\[ 0 > \int_0^T \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) \phi(s) \, ds \] (3.94)
\[ 0 > \int_0^T \nabla \cdot f |_{x=\phi} \, ds. \] (3.95)
We get instability when
\[ 0 < \int_0^T \nabla \cdot f |_{x=\phi} \, ds. \] (3.96)
3.2.2 Example

Consider

\[
\begin{align*}
x' &= x - y - x(\frac{x^2 + y^2}{k}) \\
y' &= x + y - y(\frac{x^2 + y^2}{k}).
\end{align*}
\] (3.97)

Let

\[
\begin{align*}
x &= r(t) \cos(\theta(t)) \\
y &= r(t) \sin(\theta(t))
\end{align*}
\] (3.99)

so that our problem becomes

\[
\begin{align*}
sin(\theta) (r - r\theta') &= cos(\theta) (r - r^3 - r') \\
cos(\theta) (r - r\theta') &= -sin(\theta) (r - r^3 - r').
\end{align*}
\] (3.101)

By squaring and adding these equations, we obtain that

\[
(r - r\theta')^2 = (r - r^3 - r')^2
\] (3.103)

so we can write

\[
\begin{align*}
a &= r - r\theta' \\
sa &= r - r^3 - r'
\end{align*}
\] (3.104)

where \( s = \pm 1 \). Our equations then become

\[
\begin{align*}
a \sin(\theta) &= sa \cos(\theta) \\
a \cos(\theta) &= -sa \sin(\theta)
\end{align*}
\] (3.106)

which can be rewritten as

\[
\begin{align*}
a \sin(\theta) &= sa \cos(\theta) \\
-s^2a \sin(\theta) &= sa \cos(\theta)
\end{align*}
\] (3.108)

so that we must have

\[
\begin{align*}
a \sin(\theta) &= -a \sin(\theta) \\
a \sin(\theta) &= 0
\end{align*}
\] (3.110)

As a result, we have that

\[
a \sin(\theta) = sa \cos(\theta) = 0
\] (3.112)

so that we must have \( a = 0 \). This means that

\[
r - r\theta' = r - r^3 - r' = 0.
\] (3.113)
We have that
\[ r' = r (1 - r^3) \]  
and so we have a solution of constant radius when \( r = 0 \) (the trivial case) and \( r = \pm 1 \). Without loss of generality, choose \( r = 1 \). Then since
\[ r\theta' = r, \]  
we have that \( \theta' = 1 \), so \( \theta = t + C \). As a result, our solution has period \( T = 2\pi \).

Now
\[
\nabla \cdot f\bigg|_{r=1} = \left[ \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} \right]_{r=1}
\]
\[
= \left[ (1 - 3x^2 - y^2) + (1 - x^2 - 3y^2) \right]_{r=1}
\]
\[
= [2 - 4r^2]_{r=1}
\]
\[
= -2
\]
so that
\[
\rho_2 = \exp \left( \int_0^T \tr (A(s)) \, ds \right)
\]
\[
= \exp \left( \int_0^{2\pi} -2 \, ds \right)
\]
\[
= e^{-4\pi}
\]
\[
< 1.
\]
As a result, the limit cycle with radius \( r = 1 \) is stable.

### 3.2.3 Stability of Second-Order ODE

Consider the second-order ODE
\[
x'' + a(t)x = 0
\]
where \( a(t) \) is periodic with period \( T \). Letting \( x_1 = x \) and \( x_2 = x' \), this can be rewritten as
\[
\begin{bmatrix}
x'_1 \\
x'_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-a(t) & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]
By choosing the initial condition
\[
\begin{bmatrix}
x_1(0) \\
x_2(0)
\end{bmatrix} = \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]
we obtain a solution of the form
\[
\begin{bmatrix}
x^{(1)}_1(t) \\
x^{(1)}_2(t)
\end{bmatrix}.
\]
Likewise by choosing the initial condition
\[
\begin{bmatrix}
  x_1(0) \\
  x_2(0)
\end{bmatrix} = \begin{bmatrix}
  0 \\
  1
\end{bmatrix}
\]
we obtain a solution of the form
\[
\begin{bmatrix}
  x_1^{(2)}(t) \\
  x_2^{(2)}(t)
\end{bmatrix}.
\]
(3.129)

As a result, we have chosen \( \mathbf{X}(0) = \mathbf{I} \) so that
\[
B = \mathbf{X}(T) = \begin{bmatrix}
  x_1^{(1)}(T) & x_1^{(2)}(T) \\
  x_1^{(1)}(T) & x_1^{(2)}(T)
\end{bmatrix}.
\]
(3.130)

Now we have from property (i) on page 52 that
\[
\rho_1 \rho_2 = \exp \left( \int_0^T \text{tr} (\mathbf{A}(s)) \, ds \right)
\]
\[
= \exp \left( \int_0^T 0 \, ds \right)
\]
\[
= 1
\]
and from property (ii) that
\[
\rho_1 + \rho_2 = \text{tr} (\mathbf{B})
\]
\[
= x_1^{(1)}(T) + x_1^{(2)}(T).
\]
(3.135)

Let \( \phi = \text{tr} (\mathbf{B}) / 2 \) so that
\[
\rho_1 \rho_2 = 1
\]
\[
\rho_1 + \rho_2 = 2 \phi.
\]
(3.137)

Solving these, we obtain that
\[
\rho = \phi \pm \sqrt{\phi^2 - 1}.
\]
(3.138)

We can rewrite \( \rho_i \) as \( \exp(\mu_i T) \), so that
\[
\mu_1 + \mu_2 = 0
\]
(3.139)

and so
\[
e^{\mu_1 T} + e^{\mu_2 T} = 2 \phi
\]
(3.140)
\[
e^{\mu_1 T} + e^{-\mu_1 T} = 2 \phi
\]
(3.141)
\[
e^{\mu_1 T} + e^{-\mu_1 T} = \phi
\]
(3.142)
\[
= \phi
\]
\[
= \phi.
\]
(3.143)

Consider the following cases.
I. Let $-1 < \phi < 1$. We can then define $\sigma$ by

$$\rho = \phi \pm \sqrt{\phi^2 - 1}$$

(3.144)

$$= \cos (\sigma T) \pm i \sin (\sigma T)$$

(3.145)

$$= e^{\pm i \sigma T}$$

(3.146)

As in remark (v) on page 55, we can write the general solution as

$$x(t) = c_1 \Re \left( e^{i \sigma t} p_1(t) \right) + c_2 \Im \left( e^{i \sigma t} p_1(t) \right)$$

(3.147)

and since $|\rho_1| = 1$ and $|\rho_2| = 1$, then from remark (ii) on page 53, the solution is stable and pseudo-periodic.

Now $e^{i \sigma t}$ has period $\hat{T} = \frac{2\pi}{\sigma}$. Now since $\phi \neq 1$ and $\phi \neq -1$, we must have

$$\sigma T \neq m\pi$$

(3.148)

$$\frac{2\pi}{\hat{T}} \neq m\pi$$

(3.149)

$$\frac{2T}{m} \neq \hat{T}$$

(3.150)

so that $\hat{T} \neq 2T, T, \frac{2T}{3}, \ldots$.

Note that for $\hat{T}$ to equal $nT$, we must have

$$\sigma = \frac{2\pi}{n\hat{T}}$$

(3.151)

for $n \neq 1, 2$ from above.

II. Let $\phi > 1$. Then since $\rho = \phi \pm \sqrt{\phi^2 - 1}$, we must have $\rho_1 > 1$ and since $\rho_1 \rho_2 = 1$, we must have $\rho_1 > 1 > \rho_2 > 0$ and $\rho_2 = \frac{1}{\rho_1}$ means $\mu_2 = -\mu_1$.

Our solution must therefore be of the form

$$x(t) = c_1 e^{\mu_1 t} p_1(t) + c_2 e^{-\mu_1 t} p_2(t)$$

(3.152)

where $p_1(t)$ and $p_2(t)$ are both periodic with period $T$. As a result, the solution is unstable.

III. Let $\phi = 1$. Then $\rho_1 = \rho_2 = 1$. Here, theorem 3.4 only guarantees that we will have one solution $x(t)$ of the form $e^{\mu t} p(t)$. If $B$ has two linearly independent eigenvectors, we can find two linearly independent $p_1(t)$ and $p_2(t)$ so that the two solutions are both in the standard form. However, if $B$ only has one eigenvector, we will end up with one solution of the form $p_1(t)$ (since $\rho = 1$ in this case) and the other of the form $tp_1(t) + p_2(t)$.

To see this, we replace

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

(3.153)
with the Jordan block

\[
\begin{bmatrix}
\lambda & 1 \\
0 & \lambda
\end{bmatrix}.
\] (3.154)

As a result, instead of our solution being of the form

\[
X(t) = P(t) \exp\left( \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix} t \right)
\] (3.155)

\[
= P(t) \begin{bmatrix}
e^{\lambda_1 t} & 0 \\
0 & e^{\lambda_2 t}
\end{bmatrix}
\] (3.156)

\[
= \begin{bmatrix}
P_1 e^{\lambda_1 t} & P_2 e^{\lambda_2 t} \\
P_3 e^{\lambda_1 t} & P_4 e^{\lambda_2 t}
\end{bmatrix},
\] (3.157)

it will be of the form

\[
X(t) = P(t) \exp\left( \begin{bmatrix}
\lambda & 1 \\
0 & \lambda
\end{bmatrix} t \right)
\] (3.158)

\[
= P(t) \begin{bmatrix}
e^{\lambda t} & te^{\lambda t} \\
0 & e^{\lambda t}
\end{bmatrix}
\] (3.159)

\[
= \begin{bmatrix}
P_1 e^{\lambda t} & P_1 te^{\lambda t} + P_2 e^{\lambda t} \\
P_3 e^{\lambda t} & P_3 te^{\lambda t} + P_4 e^{\lambda t}
\end{bmatrix}.
\] (3.160)

See the papers by Akhmedov [1] and Wiesel and Pohlen [30].

IV. Let \( \phi < -1 \). Since \( \rho = \phi \pm \sqrt{\phi^2 - 1} \), we must have \( \rho_1 < -1 \) and since \( \rho_1 \rho_2 = 1 \), we must have \( \rho_1 < -1 < \rho_2 < 0 \) and \( \rho_2 = \frac{1}{\rho_1} \) means \( \mu_2 = -\mu_1 \). Now we can write \( \mu_1 = \frac{\phi}{T} + \gamma \) so that our solution must be of the form

\[
x(t) = c_1 e^{\gamma t} e^{\pi t/T} p_1(t) + c_2 e^{-\gamma t} e^{\pi t/T} p_2(t)
\] (3.161)

where \( p_1(t) \) and \( p_2(t) \) are both periodic with period \( T \) and so \( e^{\pi t/T} p_1(t) \) and \( e^{\pi t/T} p_2(t) \) are both periodic with period \( 2T \). As a result, the solution is unstable.

V. Let \( \phi = -1 \). Then \( \rho_1 = \rho_2 = -1 \). As in the case when \( \phi = 1 \), we have one solution which is periodic (this time with period \( 2T \)),

\[
x_1(t) = e^{\pi t/T} p_1(t)
\] (3.162)

and the other which grows linearly with time,

\[
x_2(t) = te^{\pi t/T} p_1(t) + e^{\pi t/T} p_2(t).
\] (3.163)

We summarise these results in figure 3.1. For \( \phi > 1 \), we have an unstable solution of the form

\[
x(t) = c_1 e^{\mu_1 t} p_1(t) + c_2 e^{-\mu_1 t} p_2(t).
\] (3.164)
For $\phi = 1$, we have an unstable solution of the form
\[ x(t) = (c_1 + tc_2) p_1(t) + c_2 p_2(t). \] (3.165)

For $-1 < \phi < 1$, we have a stable pseudo-periodic solution of the form
\[ x(t) = c_1 \text{Re}(e^{i\sigma t} p(t)) + c_2 \text{Im}(e^{i\sigma t} p(t)). \] (3.166)

For $\phi = -1$, we have an unstable solution of the form
\[ x(t) = (c_1 + tc_2) q_1(t) + c_2 q_2(t). \] (3.167)

Finally, for $\phi < -1$, we have an unstable solution of the form
\[ x(t) = c_1 e^{\gamma t} q_1(t) + c_2 e^{-\gamma t} q_2(t) \] (3.168)
where $p_i(t)$ represents a function that has period $T$ and $q_i(t)$ represents a function that has period $2T$.

### 3.2.4 Application to Hill’s Equation

Consider Hill’s equation
\[ x'' + (\delta + \epsilon b(t)) x = 0 \] (3.169)
where \( b(t) \) has period \( T \). If \( \epsilon = 0 \), the solution is stable, however, there are some values of \( \delta \) for which the solution is only marginally stable, according to the above criteria. As a result, we expect that for \( \epsilon \) small but nonzero near those values of \( \delta \), we will get the beginning of a region of instability. We wish to find those values of \( \delta \).

For \( \epsilon = 0 \), if \( X(0) = I \), then
\[
X(t) = \begin{bmatrix}
\cos\left(\sqrt{\delta} t\right) & \frac{1}{\sqrt{\delta}} \sin\left(\sqrt{\delta} t\right) \\
-\sqrt{\delta} \sin\left(\sqrt{\delta} t\right) & \cos\left(\sqrt{\delta} t\right)
\end{bmatrix}
\] (3.170)
and so
\[
B = X(T) = \begin{bmatrix}
\cos\left(\sqrt{\delta} T\right) & \frac{1}{\sqrt{\delta}} \sin\left(\sqrt{\delta} T\right) \\
-\sqrt{\delta} \sin\left(\sqrt{\delta} T\right) & \cos\left(\sqrt{\delta} T\right)
\end{bmatrix}.
\] (3.171)
As a result,
\[
\phi = \frac{\text{tr} (B)}{2} = \cos\left(\sqrt{\delta} T\right).
\] (3.172)
If \( \phi = 1 \), then
\[
\sqrt{\delta} T = 2m\pi
\] (3.173)
\[
\delta = \left(2m \frac{\pi}{T}\right)^2
\] (3.174)
where \( m \) is a positive integer since \( \sqrt{\delta} > 0 \). If \( \phi = -1 \), then
\[
\sqrt{\delta} \pi = (2m + 1)\pi
\] (3.175)
\[
\delta = \left((2m + 1) \frac{\pi}{T}\right)^2
\] (3.176)
Now we have from the previous section that \( \phi = 1 \) corresponds to the existence of a periodic solution of period \( T \) and \( \phi = -1 \) corresponds to the existence of a periodic solution of period \( 2T \). As a result, we will have the border between stability and instability breaking off from \( \epsilon = 0 \) at
\[
\delta = \left(2m \frac{\pi}{T}\right)^2
\] (3.177)
corresponding to solutions with period \( T \) and breaking off from \( \epsilon = 0 \) at
\[
\delta = \left((2m + 1) \frac{\pi}{T}\right)^2
\] (3.178)
corresponding to solutions with period \( 2T \).
3.3 Stability Boundary of Mathieu’s Equation

3.3.1 Undamped Case

We have from §3.2.3 and §3.2.4 that on the edge of the region of stability, we have either \( \phi = 1 \) or \( \phi = -1 \). The former corresponds to the existence of a periodic solution with period \( T \) and the latter to a periodic solution with period \( 2T \). In order to determine the region of stability of the Mathieu equation in the \( \delta - \epsilon \) plane, we then need to determine the conditions on \( \delta \) and \( \epsilon \) required in order to have a solution which is periodic with either period \( \pi \) or \( 2 \pi \). We follow McLachlan [17] and Ward [28].

Functions of Period \( \pi \)

We can write a general function of period \( \pi \) as

\[
x = \sum_{n=0}^{\infty} a_n \cos (2nt) + \sum_{n=1}^{\infty} b_n \sin (2nt).
\]  

(3.179)

We then obtain

\[
0 = x'' + (\delta + \epsilon \cos (2t)) x
\]  

(3.180)

\[
0 = \sum_{n=0}^{\infty} (\delta - 4n^2) a_n \cos (2nt) + \sum_{n=1}^{\infty} (\delta - 4n^2) b_n \sin (2nt)
\]

\[
+ \epsilon \sum_{n=0}^{\infty} a_n \cos (2nt) \cos (2t) + \epsilon \sum_{n=1}^{\infty} b_n \sin (2nt) \cos (2t).
\]  

(3.181)

Using the identities

\[
\cos (A) \cos (B) = \frac{1}{2} (\cos (A - B) + \cos (A + B))
\]  

(3.182)

\[
\sin (A) \cos (B) = \frac{1}{2} (\sin (A - B) + \sin (A + B))
\]  

(3.183)

this becomes

\[
0 = \sum_{n=0}^{\infty} (\delta - 4n^2) a_n \cos (2nt) + \sum_{n=1}^{\infty} (\delta - 4n^2) b_n \sin (2nt)
\]

\[
+ \frac{\epsilon}{2} \sum_{n=0}^{\infty} a_n (\cos (2(n+1)t) + \cos (2(n-1)t))
\]

\[
+ \frac{\epsilon}{2} \sum_{n=1}^{\infty} b_n (\sin (2(n+1)t) + \sin (2(n-1)t)).
\]  

(3.184)
and so we must have

\[
0 = \sum_{n=0}^{\infty} \left( \delta - 4n^2 \right) a_n \cos (2nt) \\
+ \frac{\epsilon}{2} \sum_{n=0}^{\infty} a_n \left( \cos (2(n+1)t) + \cos (2(n-1)t) \right) \\
0 = \left( \delta a_0 + \frac{\epsilon}{2} a_1 \right) \cos (0) + \left( (\delta - 4) a_1 + \frac{\epsilon}{2} (2a_0 + a_2) \right) \cos (2t) \\
+ \sum_{n=2}^{\infty} \left( \left( \delta - 4n^2 \right) a_n + \frac{\epsilon}{2} (a_{n-1} + a_{n+1}) \right) \cos (2nt) \tag{3.185}
\]

and

\[
0 = \sum_{n=1}^{\infty} \left( \delta - 4n^2 \right) b_n \sin (2nt) \\
+ \frac{\epsilon}{2} \sum_{n=1}^{\infty} b_n \left( \sin (2(n+1)t) + \sin (2(n-1)t) \right) \\
0 = \left( (\delta - 4) b_1 + \frac{\epsilon}{2} b_2 \right) \sin (2t) \\
+ \sum_{n=2}^{\infty} \left( \left( \delta - 4n^2 \right) b_n + \frac{\epsilon}{2} (b_{n-1} + b_{n+1}) \right) \sin (2nt). \tag{3.186}
\]

By orthogonality of the sine and cosine, these can be rewritten as

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} =
\begin{bmatrix}
\delta & \frac{\epsilon}{2} & \frac{\epsilon}{2} & \frac{\epsilon}{2} & 0 \\
\frac{\epsilon}{2} & \delta - 4 \cdot 1^2 & \frac{\epsilon}{2} & 0 & 0 \\
\frac{\epsilon}{2} & \frac{\epsilon}{2} & \delta - 4 \cdot 2^2 & \frac{\epsilon}{2} & 0 \\
\frac{\epsilon}{2} & \frac{\epsilon}{2} & \frac{\epsilon}{2} & \delta - 4 \cdot 3^2 & \frac{\epsilon}{2} \\
0 & \frac{\epsilon}{2} & \frac{\epsilon}{2} & \frac{\epsilon}{2} & \frac{\epsilon}{2} \\
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
\vdots
\end{bmatrix} \tag{3.189}
\]

and

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} =
\begin{bmatrix}
\delta - 4 \cdot 1^2 & \frac{\epsilon}{2} & \frac{\epsilon}{2} & \frac{\epsilon}{2} & 0 \\
\frac{\epsilon}{2} & \delta - 4 \cdot 2^2 & \frac{\epsilon}{2} & 0 & 0 \\
\frac{\epsilon}{2} & \frac{\epsilon}{2} & \delta - 4 \cdot 3^2 & \frac{\epsilon}{2} & 0 \\
\frac{\epsilon}{2} & \frac{\epsilon}{2} & \frac{\epsilon}{2} & \delta - 4 \cdot 4^2 & \frac{\epsilon}{2} \\
0 & \frac{\epsilon}{2} & \frac{\epsilon}{2} & \frac{\epsilon}{2} & \frac{\epsilon}{2} \\
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
\vdots \\
\end{bmatrix} \tag{3.190}
\]

In order to have a non-zero solution, the determinant of at least one of these (infinite) matrices must be zero. This gives us the requirement that \( \epsilon \) and \( \delta \) must satisfy in order to be on the borderline between stability and instability.

We can approximate the determinants of these matrices by the determinants of the finite \( n \times n \) matrices of the same form. The resultant curves in the \( \delta-\epsilon \) plane for different values of \( n \) are shown in figure 3.2.
Figure 3.2: The approximation to the border of the region of stability of the Mathieu equation (determined by equations 3.189, 3.190, 3.198, 3.199) where each infinite matrix is approximated by its $n \times n$ counterpart.
Functions of Period $2\pi$

We now perform a similar analysis for functions of period $2\pi$. We can write a general function of period $2\pi$ as

$$x = \sum_{n=0}^{\infty} a_n \cos (nt) + \sum_{n=1}^{\infty} b_n \sin (nt). \quad (3.191)$$

We then remove from this all the terms which also have period $\pi$ since we have already dealt with those. If we included them, we would obtain the lines in the $\delta$-$\epsilon$ plane where we obtain solutions that either have period $\pi$ or have period $2\pi$. As a result, we have

$$x = \sum_{n=1 \text{ odd}}^{\infty} a_n \cos (nt) + \sum_{n=1 \text{ odd}}^{\infty} b_n \sin (nt). \quad (3.192)$$

so that we obtain

$$0 = x'' + (\delta + \epsilon \cos (2t)) x \quad (3.193)$$

$$0 = \sum_{n=1 \text{ odd}}^{\infty} (\delta - n^2) a_n \cos (nt) + \sum_{n=1 \text{ odd}}^{\infty} (\delta - n^2) b_n \sin (nt)$$

$$+ \epsilon \sum_{n=1 \text{ odd}}^{\infty} a_n \cos (nt) \cos (2t) + \epsilon \sum_{n=1 \text{ odd}}^{\infty} b_n \sin (nt) \cos (2t) \quad (3.194)$$

$$0 = \sum_{n=1 \text{ odd}}^{\infty} (\delta - n^2) a_n \cos (nt) + \sum_{n=1 \text{ odd}}^{\infty} (\delta - n^2) b_n \sin (nt)$$

$$+ \frac{\epsilon}{2} \sum_{n=1 \text{ odd}}^{\infty} a_n \cos ((n+2)t) + \cos ((n-2)t))$$

$$+ \frac{\epsilon}{2} \sum_{n=1 \text{ odd}}^{\infty} b_n \sin ((n+2)t) + \sin ((n-2)t)) \quad (3.195)$$

We must then have

$$0 = \sum_{n=1 \text{ odd}}^{\infty} \left( (\delta - 1) a_1 + \frac{\epsilon}{2} (a_1 + a_3) \right) \cos (t)$$

$$+ \sum_{n=1 \text{ odd}}^{\infty} \left( (\delta - n^2) a_n + \frac{\epsilon}{2} (a_{n-2} + a_{n+2}) \right) \cos (nt) \quad (3.196)$$
\[ 0 = \sum_{n \in \mathbb{N}} \left( (\delta - 1) b_1 + \frac{\epsilon}{2} (-b_1 + b_3) \right) \sin(t) \]
\[ + \sum_{n \in \mathbb{N}} \left( (\delta - n^2) b_n + \frac{\epsilon}{2} (b_{n-2} + b_{n+2}) \right) \sin(nt) \]  
(3.197)

which we can write as
\[
\begin{bmatrix}
0 \\
0 \\
0 \\
\vdots
\end{bmatrix} = \begin{bmatrix}
\delta - 1^2 + \frac{\epsilon}{2} & \frac{\epsilon}{2} & \frac{\epsilon}{2} & 0 \\
\frac{\epsilon}{2} & \delta - 3^2 & \frac{\epsilon}{2} & 0 \\
\frac{\epsilon}{2} & \frac{\epsilon}{2} & \delta - 5^2 & \frac{\epsilon}{2} \\
0 & \frac{\epsilon}{2} & \frac{\epsilon}{2} & \delta - 7^2 & \frac{\epsilon}{2} \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} \begin{bmatrix}
\alpha_1 \\
\alpha_3 \\
\alpha_5 \\
\vdots
\end{bmatrix} \]  
(3.198)

and
\[
\begin{bmatrix}
0 \\
0 \\
0 \\
\vdots
\end{bmatrix} = \begin{bmatrix}
\delta - 1^2 - \frac{\epsilon}{2} & \frac{\epsilon}{2} & \frac{\epsilon}{2} & 0 \\
\frac{\epsilon}{2} & \delta - 3^2 & \frac{\epsilon}{2} & 0 \\
\frac{\epsilon}{2} & \frac{\epsilon}{2} & \delta - 5^2 & \frac{\epsilon}{2} \\
0 & \frac{\epsilon}{2} & \frac{\epsilon}{2} & \delta - 7^2 & \frac{\epsilon}{2} \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} \begin{bmatrix}
\beta_1 \\
\beta_3 \\
\beta_5 \\
\vdots
\end{bmatrix} \]  
(3.199)

As before, in order to obtain a nonzero solution, we must have the determinant of at least one of the matrices being zero. This constrains \(\delta\) and \(\epsilon\).

The resultant region of stability is shown in figure 3.3

### 3.3.2 Undamped Case with \(\epsilon\) small

Consider now when \(\epsilon\) is small. We have from §3.2.4 that for \(\epsilon\) small, we will have the border between stability and instability near
\[ \delta = (2m)^2 \]  
(3.200)

and
\[ \delta = (2m + 1)^2. \]  
(3.201)

As a result, we seek periodic solutions near \(\delta = n^2\) to the equation
\[ x'' + (\delta + \epsilon \cos(2t)) x = 0. \]  
(3.202)

Let
\[ x = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \ldots, \]  
(3.203)
\[ \delta = n^2 + \epsilon \delta_1 + \epsilon^2 \delta_2 + \ldots \]  
(3.204)
Substituting these into Mathieu’s equation, we obtain

\[ x''_0 + n^2 x_0 = 0 \]  
\[ x''_1 + n^2 x_1 = -\delta_1 x_0 - x_0 \cos (2t) \]  
\[ x''_2 + n^2 x_2 = -\delta_1 x_1 - \delta_2 x_0 - x_1 \cos (2t). \]

For \( n \neq 0 \), the solution to equation 3.205 is

\[ x_0 = a \cos (nt) + b \sin (nt). \]

Inserting this into equation 3.206, we obtain

\[ x''_1 + n^2 x_1 = -\delta_1 x_0 - x_0 \cos (2t) \]
\[ = -\delta_1 (a \cos (nt) + b \sin (nt)) \]
\[ - (a \cos (nt) + b \sin (nt)) \cos (2t) \]
\[ = -\delta_1 a \cos (nt) - \delta_1 b \sin (nt) \]
\[ - \frac{a}{2} \cos ((n + 2) t) - \frac{a}{2} \cos ((n - 2) t) \]
\[ - \frac{b}{2} \sin ((n + 2) t) - \frac{b}{2} \sin ((n - 2) t). \]

Under the assumption that \( n \neq 1 \), in order to eliminate secular terms, we must have

\[ -\delta_1 a = 0, \quad -\delta_1 b = 0. \]
As a result, in order to avoid \( x_0 \) being the zero solution, we must have \( \delta_1 = 0 \). We then have

\[
x_1'' + n^2 x_1 = -\frac{a}{2} \cos ((n + 2) t) - \frac{a}{2} \cos ((n - 2) t) \\
- \frac{b}{2} \sin ((n + 2) t) - \frac{b}{2} \sin ((n - 2) t).
\]  

Letting

\[
x_1 = \sum_{i=0}^{\infty} c_i \sin (it) + d_i \cos (it),
\]  

this becomes

\[
- \sum_{i=1}^{\infty} c_i i^2 \sin (it) + d_i i^2 \cos (it) + \sum_{i=0}^{\infty} c_i n^2 i^2 \sin (it) + d_i n^2 \cos (it)
\]

\[
= -\frac{a}{2} \cos ((n + 2) t) - \frac{a}{2} \cos ((n - 2) t) \\
- \frac{b}{2} \sin ((n + 2) t) - \frac{b}{2} \sin ((n - 2) t).
\]  

Equating coefficients of the sines and cosines, we obtain that

\[
c_{n-2} = \frac{b}{8 (-n + 1)}, \quad d_{n-2} = \frac{a}{8 (-n + 1)} \tag{3.216}
\]

\[
c_{n+2} = \frac{b}{8 (n + 1)}, \quad d_{n+2} = \frac{a}{8 (n + 1)}. \tag{3.217}
\]

We can assume that all the \( \sin (nt) \) and \( \cos (nt) \) component is already in \( x_0 \), so we can choose \( c_n = 0, d_n = 0 \). All remaining \( c_i \) and \( d_i \) are zero. As a result,

\[
x_1 = \frac{b}{8 (-n + 1)} \sin ((n - 2) t) + \frac{b}{8 (n + 1)} \sin ((n + 2) t) \\
+ \frac{a}{8 (-n + 1)} \cos ((n - 2) t) + \frac{a}{8 (n + 1)} \cos ((n + 2) t). \tag{3.218}
\]

Finally, inserting this into equation 3.207, we obtain that

\[
x_2'' + n^2 x_2 = -\delta_2 (a \cos (nt) + b \sin (nt)) \\
- \frac{b}{16 (-n + 1)} (\sin (nt) + \sin ((n - 4) t)) \\
- \frac{b}{16 (n + 1)} (\sin ((n + 4) t) + \sin (nt)) \\
- \frac{a}{16 (-n + 1)} (\cos (nt) + \cos ((n - 4) t)) \\
- \frac{a}{16 (n + 1)} (\cos ((n + 4) t) + \cos (nt)). \tag{3.219}
\]
Under the assumption that $n \neq 2$, in order to eliminate the secular terms, we must have

\begin{align*}
0 &= -\delta_2 a - \frac{a}{16 (-n + 1)} - \frac{a}{16 (n + 1)}, \quad (3.220) \\
0 &= -\delta_2 b - \frac{b}{16 (-n + 1)} - \frac{b}{16 (n + 1)}, \quad (3.221)
\end{align*}

which can be rewritten as

\begin{align*}
0 &= -a \left( \delta_2 - \frac{1}{8 (n^2 - 1)} \right), \quad (3.222) \\
0 &= -b \left( \delta_2 - \frac{1}{8 (n^2 - 1)} \right). \quad (3.223)
\end{align*}

As a result, in order to avoid a nonzero $x_0$ (i.e., making sure that we don’t simultaneously have $a = 0$ and $b = 0$), we must have

\begin{equation}
\delta_2 = \frac{1}{8 (n^2 - 1)}. \quad (3.224)
\end{equation}

**Case $n = 2$**

In the case $n = 2$, eliminating the secular terms in equation 3.219 tells us that

\begin{align*}
0 &= -\delta_2 a + \frac{a}{8} - \frac{a}{48}, \quad (3.225) \\
0 &= -\delta_2 b - 0 - \frac{b}{48}, \quad (3.226)
\end{align*}

which become

\begin{align*}
0 &= -a \left( \delta_2 - \frac{5}{48} \right), \quad (3.227) \\
0 &= -b \left( \delta_2 + \frac{1}{48} \right). \quad (3.228)
\end{align*}

As a result, for $n = 2$ we must have either

\begin{equation}
a = 0, \quad \delta_2 = -\frac{1}{48} \quad (3.229)
\end{equation}

or

\begin{equation}
b = 0, \quad \delta_2 = \frac{5}{48}. \quad (3.230)
\end{equation}

As a result, for $n = 2$, we either have

\begin{equation}
\delta = 4 - \epsilon^2 \frac{1}{48} + O (\epsilon^3) \quad (3.231)
\end{equation}

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or

\[ \delta = 4 + \varepsilon^2 \frac{5}{48} + O(\varepsilon^3). \] (3.232)

We also have either

\[ x = b \sin (2t) + \varepsilon \left( \frac{b}{24} \sin (4t) + O(\varepsilon^2) \right) \] (3.233)

or

\[ x = a \cos (2t) + \varepsilon \left( -\frac{a}{8} + \frac{a}{24} \cos (4t) \right) + O(\varepsilon^2), \] (3.234)

which both have period \( \pi \), as expected.

Case \( n = 1 \)

In the case \( n = 1 \), eliminating the secular terms in equation 3.211 tells us that

\[ 0 = -\delta_1 a - \frac{a}{2}, \] (3.235)

\[ 0 = -\delta_1 b + \frac{b}{2} \] (3.236)

and so we must either have

\[ \delta_1 = -\frac{1}{2}, \quad b = 0 \] (3.237)

or

\[ \delta_1 = \frac{1}{2}, \quad a = 0. \] (3.238)

In either of these cases, equation 3.211 becomes

\[ x_1'' + x_1 = -\frac{a}{2} \cos (3t) - \frac{b}{2} \sin (3t). \] (3.239)

As before, we let

\[ x_1 = \sum_{i=1}^{\infty} c_i \sin (it) + d_i \cos (it) \] (3.240)

and find that

\[ c_3 = \frac{b}{16}, \quad d_3 = \frac{a}{16}. \] (3.241)

As a result,

\[ x_1 = \frac{b}{16} \sin (3t) + \frac{a}{16} \cos (3t). \] (3.242)
Then equation 3.207 becomes
\[
x''_2 + x_2 = -\delta_1 \left( \frac{b}{16} \sin (3t) + \frac{a}{16} \cos (3t) \right) \\
- \delta_2 \left( a \cos (t) + b \sin (t) \right) \\
- \left( \frac{b}{16} \sin (3t) + \frac{a}{16} \cos (3t) \right) \cos (2t)
\]

\[
= -\delta_1 \left( \frac{b}{16} \sin (3t) + \frac{a}{16} \cos (3t) \right) \\
- \delta_2 a \cos (t) - \delta_2 b \sin (t) \\
- \frac{b}{32} \sin (t) - \frac{b}{32} \sin (5t) - \frac{a}{32} \cos (t) - \frac{a}{32} \cos (5t) .
\]

(3.243)

In order to eliminate the secular terms, we must have
\[
0 = -a \left( \delta_2 + \frac{1}{32} \right) 
\]

(3.245)

\[
0 = -b \left( \delta_2 + \frac{1}{32} \right) .
\]

(3.246)

As a result, \( \delta_2 = -1/32 \), so that either
\[
\delta = 1 - \epsilon \frac{1}{2} - \epsilon^2 \frac{1}{32} + O \left( \epsilon^3 \right) 
\]

(3.247)
or
\[
\delta = 1 + \epsilon \frac{1}{2} - \epsilon^2 \frac{1}{32} + O \left( \epsilon^3 \right) .
\]

(3.248)

We also have either
\[
x = a \cos (t) + \epsilon \frac{a}{16} \cos (3t) + O \left( \epsilon^2 \right) 
\]

(3.249)
or
\[
x = b \sin (t) + \epsilon \frac{b}{16} \sin (3t) + O \left( \epsilon^2 \right) ,
\]

(3.250)

which are periodic with period \( 2\pi \), as expected.

**Case n = 0**

In the case \( n = 0 \), we get
\[
x_0 = a + bt .
\]

(3.251)

Now we expect a periodic solution, so \( b = 0 \). As a result, equation 3.206 becomes
\[
x''_1 = -\delta_1 a - a \cos (2t) .
\]

(3.252)

In analogy with before, when we eliminated secular terms, we must have \( \delta_1 = 0 \). As a result, we have
\[
x''_1 = -a \cos (2t)
\]

(3.253)
so that
\[ x_1 = \frac{a}{4} \cos (2t) \]  
(3.254)
and equation 3.207 becomes
\[ x_2'' = -\delta_2 a - \frac{a}{4} \cos (2t) \cos (2t) \]  
(3.255)
\[ = -\delta_2 a - \frac{a}{8} - \frac{a}{8} \cos (4t) \]  
(3.256)
so that we must have
\[ 0 = -a \left( \delta_2 + \frac{1}{8} \right) . \]  
(3.257)
so that $\delta_2 = -1/8$ and
\[ \delta = 0 - \epsilon^2 \frac{1}{8} \]  
(3.258)
with
\[ x = a + \epsilon \frac{a}{4} \cos (2t) + O \left( \epsilon^2 \right) , \]  
(3.259)
which is again periodic with period $\pi$, as expected.

These approximations to $\delta (\epsilon)$ for $\epsilon$ small are compared to the approximation in the previous section (which is valid for both small and large $\epsilon$) in figure 3.4.
3.3.3 Damped Case

We follow Richards [24]. Our equation is

$$x'' + kx' + (\delta + \epsilon \cos (2t)) x = 0.$$  \hfill (3.260)

If we let

$$y(t) = e^{\frac{bt}{2}} x(t),$$  \hfill (3.261)

we obtain that

$$y'' + (a + \epsilon \cos (2t)) y = 0$$  \hfill (3.262)

where

$$a = \delta - \frac{k^2}{4}. \quad (3.263)$$

Now equation 3.260 isn’t of the form of equation 3.124 (§3.2.3), but equation 3.262 is. As a result, we know that the solution to equation 3.262 is of the form

$$y(t) = e^{\mu_1 t} p_1(t) + e^{\mu_2 t} p_2(t) \quad (3.264)$$

where $\mu_1$ and $\mu_2$ satisfy

$$e^{\mu \pi} = \rho = \phi \pm \sqrt{\phi^2 - 1} \quad (3.265)$$

where $\phi$ is half of the trace of $B$ for $y(t)$ above when we use the initial conditions $X(0) = I$. As a result, the largest $\mu$ (the one most likely to cause instability) satisfies

$$e^{\mu \pi} = \rho = \phi + \sqrt{\phi^2 - 1} \quad (3.266)$$

so that

$$\mu \pi = \ln \left( \phi + \sqrt{\phi^2 - 1} \right) \quad (3.267)$$
$$\mu = \cosh^{-1} (\phi) \quad (3.268)$$
$$\mu = \frac{\cosh^{-1} (\phi)}{\pi}. \quad (3.269)$$

Now in order for $x(t)$ to be stable, we must have

$$0 \geq \Re \left( \mu - \frac{k}{2} \right) \quad (3.270)$$
$$\frac{k}{2} \geq \Re (\mu) \quad (3.271)$$

with $\mu$ as above. This can be used to numerically determine the stability of the damped equation. The result for $k = 0.2$ is shown in figure 3.5.

3.3.4 Damped Case with $\epsilon$ small

Consider the damped Mathieu equation

$$x'' + kx' + (\delta + \epsilon \cos (2t)) x = 0.$$  \hfill (3.272)
Figure 3.5: The border of the region of stability of the Mathieu equation, in the damped case.

Near $\delta = 1$

Suppose that $k$ is of order $\epsilon$. Then we can write $k = \epsilon k_1$ and expand near $\delta = 1$,

\[
\begin{align*}
\delta &= 1 + \epsilon \delta_1 + \ldots \\
x &= x_0 + \epsilon x_1 + \ldots
\end{align*}
\]

Plugging this in and equating terms of equal order, we obtain

\[
\begin{align*}
x_0'' + x_0 &= 0 \\
x_1'' + x_1 &= -k_1 x_0' - \cos(2t) x_0 - \delta x_0.
\end{align*}
\]

This tells us that

\[
x_0 = a \cos t + b \sin t
\]

so that

\[
\begin{align*}
x_1'' + x_1 &= -k_1(-a \sin t + b \cos t) - \cos(2t)(a \cos t + b \sin t) \\
&\quad - \delta_1(a \cos t + b \sin t) \\
&= k_1 a \sin t - k_1 b \cos t - \frac{a}{2} (\cos t + \cos(3t)) \\
&\quad - b \frac{1}{2} (-\sin t + \sin(3t)) - \delta_1(a \cos t + b \sin t)
\end{align*}
\]
In order to eliminate secular terms, we must have
\begin{align}
    k_1 a + \frac{b}{2} - \delta_1 b &= 0 \quad (3.280) \\
    -k_1 b - \frac{a}{2} - \delta_1 a &= 0 \quad (3.281)
\end{align}
which can be written as
\[
\begin{bmatrix}
    k_1 & \frac{1}{2} - \delta_1 \\
    -\frac{1}{2} - \delta_1 & -k_1
\end{bmatrix}
\begin{bmatrix}
    a \\
    b
\end{bmatrix} = \begin{bmatrix}
    0 \\
    0
\end{bmatrix}. \quad (3.282)
\]
In order for this to have a nonzero solution, the determinant of the matrix must be zero, so we must have
\begin{align}
    0 &= -k_1^2 + \left( \frac{1}{2} + \delta_1 \right) \left( \frac{1}{2} - \delta_1 \right) \quad (3.283) \\
    0 &= k_1^2 + \delta_1^2 - \frac{1}{4} \quad (3.284) \\
    \delta_1 &= \pm \sqrt{\frac{1}{4} - k_1^2} \quad (3.285)
\end{align}
so that
\begin{align}
    \delta &= 1 + \epsilon \delta_1 + O(\epsilon^2) \quad (3.286) \\
    &= 1 \pm \sqrt{\frac{\epsilon^2}{4} - k_1^2} + O(\epsilon^2). \quad (3.287)
\end{align}

**Near \( \delta = 4 \)**

For larger values of \( \delta \), in order \( \epsilon \) to still be small at the edge of stability, we must have \( k \) quite a bit smaller. As a result, near \( \delta = 4 \), we choose \( k \) to be of order \( \epsilon^2 \). Then we can write \( k = \epsilon^2 k_1 \) and expand near \( \delta = 4 \),
\begin{align}
    \delta &= 4 + \epsilon \delta_1 + \epsilon^2 \delta_2 + \ldots \quad (3.288) \\
    x &= x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots \quad (3.289)
\end{align}
We need to expand these to order \( \epsilon^2 \) because it will turn out that \( \delta_1 = 0 \).

Plugging this in and equating terms of equal order, we obtain
\begin{align}
    x_0'' + 4x_0 &= 0 \quad (3.290) \\
    x_1'' + 4x_1 &= -\delta_1 x_0 - \cos(2t) x_0 \quad (3.291) \\
    x_2'' + 4x_2 &= -k_1 x_0' - \delta_1 x_1 - \delta_2 x_0 - \cos(2t) x_1. \quad (3.292)
\end{align}
This tells us that
\[ x_0 = a \cos(2t) + b \sin(2t) \quad (3.293) \]
so that
\[ x''_1 + 4x_1 = -\delta_1 x_0 - \cos (2t) x_0 \]
\[ = -\delta_1 (a \cos (2t) + b \sin (2t)) \]
\[ - \cos (2t) (a \cos (2t) + b \sin (2t)) \]  \hspace{1cm} (3.294)
\[ = -\delta_1 a \cos (2t) - \delta_1 b \sin (2t) \]
\[ - \frac{a}{2} \cos (4t) - \frac{a}{2} \sin (4t) - \frac{b}{2} \cdot 0 \]  \hspace{1cm} (3.295)

In order to eliminate secular terms, we must have
\[ \delta_1 a = 0, \quad \delta_1 b = 0 \]  \hspace{1cm} (3.297)
so we must have \( \delta_1 = 0 \). As a result, we have
\[ x''_1 + 4x_1 = -\frac{a}{2} - \frac{a}{2} \cos (4t) - \frac{b}{2} \sin (4t). \]  \hspace{1cm} (3.298)

Expanding \( x_1 \) in terms of sines and cosines and equating coefficients, we find that
\[ x_1 = \frac{a}{8} + \frac{a}{24} \cos (4t) + \frac{b}{24} \sin (4t). \]  \hspace{1cm} (3.299)

As a result, we have that
\[ x''_2 + 4x_2 = -k_1 x'_0 - \delta_1 x_1 - \delta_2 x_0 - \cos (2t) x_1 \]
\[ = -k_1 (-2a \sin (2t) + 2b \cos (2t)) - 0 \]
\[ - \delta_2 (a \cos (2t) + b \sin (2t)) \]
\[ - \cos (2t) \left( -\frac{a}{8} + \frac{a}{24} \cos (4t) + \frac{b}{24} \sin (4t) \right) \]  \hspace{1cm} (3.300)
\[ = \left( 2k_1 a - \delta_2 b - \frac{b}{48} \right) \sin (2t) \]
\[ + \left( -2k_1 b - \delta_2 a + \frac{a}{8} - \frac{a}{48} \right) \cos (2t) \]
\[ - \frac{a}{48} \cos (6t) - \frac{b}{48} \sin (6t). \]  \hspace{1cm} (3.301)

In order to eliminate secular terms, we must have
\[ 0 = 2k_1 a - \delta_2 b - \frac{b}{48} \]  \hspace{1cm} (3.302)
\[ 0 = -2k_1 b - \delta_2 a + \frac{5b}{48} \]  \hspace{1cm} (3.303)

which can be written as
\[ \begin{bmatrix} 2k_1 & -\delta_2 - \frac{1}{48} \\ -\delta_2 + \frac{5}{48} & -2k_1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]  \hspace{1cm} (3.305)
In order to have a nonzero solution to this, we must have that the determinant of the matrix is zero. As a result,

$$0 = -4k^2 - \left(\delta_2 + \frac{1}{48}\right)\left(\delta_2 - \frac{5}{48}\right)$$  \hspace{1cm} (3.306)

$$\delta_2 = \frac{1}{12} \pm \frac{1}{144} - \frac{4}{2} \left(\frac{-5}{48} + 4k^2\right)$$  \hspace{1cm} (3.307)

where

$$\delta = 1 + \epsilon^2 \delta_2 + O(\epsilon^3).$$  \hspace{1cm} (3.308)

These approximations are compared to the result from §3.3.3 in figure 3.6.

### 3.3.5 Hill’s Equation

Consider Hill’s equation, which is a generalised version of the Mathieu equation

$$x'' + (\delta + \epsilon b(t))x = 0$$  \hspace{1cm} (3.309)

where $b$ is periodic with period $\pi$. Let us assume that

$$\int_0^\pi b(t) \, dt = 0$$  \hspace{1cm} (3.310)
and that we can expand $b(t)$ as
\[
b(t) = \sum_{n=1}^{\infty} c_n \cos (2nt) + d_n \sin (2nt). \tag{3.311}
\]

We wish to determine an expansion for the solution where $\epsilon$ is small. Now we know this occurs near $\delta = m^2$ for positive integers $m$, so we expand
\[
\delta = m^2 + \epsilon \delta_1 + \ldots \tag{3.312}
\]
\[
x = x_0 + \epsilon x_1 + \ldots \tag{3.313}
\]

Then we obtain that
\[
x''_0 + m^2 x_0 = 0 \tag{3.314}
\]
\[
x''_1 + m^2 x_1 = -\delta_1 x_0 - b(t)x_0 \tag{3.315}
\]
so that
\[
x_0 = a \cos (mt) + b \sin (mt) \tag{3.316}
\]
and
\[
x''_1 + m^2 x_1 = -\delta_1 x_0 - b(t)x_0 \tag{3.317}
\]
\[
= -\delta_1 (a \cos (mt) + b \sin (mt)) - (a \cos (mt) + b \sin (mt))
\]
\[
\times \sum_{n=1}^{\infty} (c_n \cos (2nt) + d_n \sin (2nt)) \tag{3.318}
\]

To eliminate secular terms, if $m = 0$, we must have $\delta_1 a = \delta_1 b = 0$, and so $\delta_1 = 0$. As a result for $m = 0$, we must expand everything to second order. We will return to this later. For $m \neq 0$, we must have
\[
0 = -\delta_1 a - \frac{ac_m}{2} - \frac{bd_m}{2} \tag{3.320}
\]
\[
0 = -\delta_1 b - \frac{ad_m}{2} + \frac{bc_m}{2} \tag{3.321}
\]
which we can rewrite as
\[
\begin{bmatrix}
-\delta_1 - \frac{c_m}{2} & -\frac{d_m}{2} \\
-\frac{d_m}{2} & -\delta_1 + \frac{c_m}{2}
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}. \tag{3.322}
\]
As a result, we must have

\[ \delta_1^2 = \frac{1}{4} (c_m^2 + d_m^2) \]  

and so

\[ \delta = m^2 \pm \frac{\epsilon}{2} \sqrt{c_m^2 + d_m^2}. \]  

**Case \( m = 0 \)**

Recall that we determined that in the \( m = 0 \) case, we must expand everything to second order. As a result, we expand

\[ \delta = \epsilon \delta_1 + \epsilon^2 \delta_2 + \ldots \]  
\[ x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots \]  

From before, plugging in \( m = 0 \), we have \( x_0 = a \) and \( \delta_1 = 0 \), so that

\[ \begin{align*}
    x''_1 &= -\delta_2 a - \sum_{n=1}^{\infty} c_n a \cos (2nt) + d_n a \sin (2nt) \\
    x''_1 &= -\sum_{n=1}^{\infty} c_n a \cos (2nt) + d_n a \sin (2nt) \\
    x_1 &= \sum_{n=1}^{\infty} \frac{c_n a}{4n^2} \cos (2nt) + \frac{d_n a}{4n^2} \sin (2nt). 
\end{align*} \]  

The second-order equation gives us

\[ \begin{align*}
    x''_2 &= -\delta_2 x_2 - \delta_2 x_0 - b(t)x_1 \\
    x''_2 &= -\delta_2 a - \left( \sum_{i=1}^{\infty} c_i \cos (2it) + d_i \sin (2it) \right) \\
    &\quad \times \left( \sum_{j=1}^{\infty} \frac{c_j a}{4j^2} \cos (2jt) + \frac{d_j a}{4j^2} \sin (2jt) \right) 
\end{align*} \]  

In order to eliminate the secular-like terms, we must have

\[ \begin{align*}
    0 &= -\delta_2 a - \sum_{i=1}^{\infty} \frac{c_i^2 a}{8i^2} + \frac{d_i^2 a}{8i^2} \\
    \delta_2 &= -\frac{1}{8} \sum_{i=1}^{\infty} \frac{c_i^2 + d_i^2}{i^2} \\
    \delta &= -\epsilon^2 \frac{1}{8} \sum_{i=1}^{\infty} \frac{c_i^2 + d_i^2}{i^2}. 
\end{align*} \]
3.4 Applications of Mathieu’s Equation

3.4.1 Pendulum with Oscillating Pivot

Suppose we have a mass \( m \) attached at the end of a massless pendulum of length \( L \). Suppose the pivot point \( P \) oscillates in the vertical direction according to some function \( p(t) \). Then the angle \( \theta \) from the vertical to the pendulum obeys

\[
\theta'' + \left( \frac{g + p''(t)}{L} \right) \sin(\theta) = 0. \tag{3.335}
\]

We choose to measure the angle \( \theta \) such that when the pendulum is vertical, pointed upward (at what is usually the unstable stationary solution), \( \theta = \pi \). When the pendulum is near the top, \( \theta \approx \pi \). Let \( x = \theta - \pi \) so that \(|x| \ll 1\). Then our model is approximately

\[
x'' + \left( \frac{g + p''(t)}{L} \right) (-x) = 0. \tag{3.336}
\]

Let \( p(t) = A \cos(\omega t) \) to obtain

\[
x'' + \left( -g + A\omega^2 \cos(\omega t) \right) x = 0. \tag{3.337}
\]

Now let \( 2\tau = \omega t \) so that

\[
\ddot{x} + \left( -\frac{4g}{\omega^2 L} + \frac{4A}{L} \cos(2\tau) \right) x = 0. \tag{3.338}
\]

We can finally let

\[
\delta = -\frac{4g}{\omega^2 L}, \quad \epsilon = \frac{4A}{L} \tag{3.339}
\]

to obtain

\[
\ddot{x} + (\delta + \epsilon \cos(2\tau)) x = 0 \tag{3.340}
\]

where \( \epsilon \) will be small if the amplitude of oscillations of the pivot is small compared to the length of the pendulum.

We wish to determine an \( \epsilon \) and \( \delta \), and hence an \( A \) and \( \omega \), such that the solution to the above equation (Mathieu’s equation) is stable for \( x \) small. Notice that the usual problem (\( A = 0 \) so \( \epsilon = 0 \)) is unstable; near \( x = 0 \) the solution grows exponentially in time.

3.4.2 Variable Length Pendulum

Consider now a pendulum with an oscillatory length. This time, the pendulum is pointed downward.
Derivation of Model

Suppose that there is some force $F$ on the mass along the pendulum. Then the forces on the mass at the end of the pendulum are given by

$$\begin{align*}
m x'' &= -F \sin (\theta) \quad (3.341) \\
m y'' &= F \cos (\theta) - mg. \quad (3.342)
\end{align*}$$

where

$$\begin{align*}
x &= L \sin (\theta) \quad (3.343) \\
y &= -L \cos (\theta). \quad (3.344)
\end{align*}$$

By letting $z = x + iy = -iLe^{i\theta}$, we obtain

$$z'' = (2L'\theta' + L\theta'' - iL'' + iL\theta^2) e^{i\theta} \quad (3.345)$$

so that

$$m (2L'\theta' + L\theta'' - iL'' + iL\theta^2) = iF - imge^{i\theta}. \quad (3.346)$$

By equating real parts, we then obtain

$$2L'\theta' + L\theta'' + g \sin (\theta) = 0. \quad (3.347)$$

Letting $\phi = L\theta$, this becomes

$$\phi'' - \phi \frac{L''}{L} + g \sin \left( \frac{\phi}{L} \right) = 0. \quad (3.348)$$

For $\theta \ll 1$, this is approximately

$$\phi'' + \left( \frac{g - L''}{L} \right) \phi = 0. \quad (3.349)$$

Transformation to Mathieu’s Equation

Let

$$L = L_0 (1 + \Delta \cos (\omega t)) \quad (3.350)$$

for $\Delta \ll 1$. Then we obtain

$$\begin{align*}
0 &= \phi'' + \left( \frac{g - L''}{L} \right) \phi \quad (3.351) \\
0 &= \phi'' + \left( \frac{g}{L_0 (1 + \Delta \cos (\omega t))} - \frac{-L_0 \Delta \omega^2 \cos (\omega t)}{L_0 (1 + \Delta \cos (\omega t))} \right) \phi \quad (3.352) \\
0 &= \phi'' + \left( \frac{g}{L_0 (1 - \Delta \cos (\omega t))} + \Delta \omega^2 \cos (\omega t) \right) \phi \quad (3.353) \\
0 &= \phi'' + \left( \frac{g}{L_0} + \Delta \left( \omega^2 - \frac{g}{L_0} \cos (\omega t) \right) \right) \phi. \quad (3.354)
\end{align*}$$
Letting $\Omega^2 = g/L_0$, this becomes
\[\phi'' + \left(\Omega^2 + \Delta \left(\omega^2 - \Delta^2\right) \cos(\omega t)\right) \phi = 0.\] (3.355)

Letting
\[\tau = \frac{\omega}{2} t, \quad \delta = \frac{4\Omega^2}{\omega^2}, \quad \epsilon = 4\Delta \left(1 - \frac{\Omega^2}{\omega^2}\right)\]
this becomes Mathieu’s equation:
\[\ddot{\phi} + \left(\delta + \epsilon \cos(2\tau)\right) \phi = 0.\] (3.356)

### 3.4.3 Ion Traps

As in the honours thesis by Fischer [8], we consider an ion trap as shown in figure 3.7. The side walls are described by
\[r^2 = 2z^2 + r_0^2\] (3.358)
where $r_0$ is the radius at the narrowest point. The end caps are described by
\[z^2 = \frac{r^2}{2} + z_0^2\] (3.359)
where $2z_0$ is the shortest distance between the two end caps.

Now if we apply a potential difference $A$ between the side walls and the end caps, taking the end caps to be ground, we obtain a potential of

$$V(z, r) = A \frac{r^2 - 2(z^2 - z_0^2)}{r_0^2 + 2z_0^2}$$

and hence an electric field of

$$E = -\nabla V = \frac{A}{r_0^2 + 2z_0^2} (-2r\hat{r} + 4z\hat{z}) .$$

As a result, in the $z$-direction, we have

$$mz'' = \frac{4QA}{d_0^2} z$$

where prime denotes differentiation with respect to $t$ and we have let $d_0^2 = r_0^2 + 2z_0^2$. If

$$A = U_0 - V_0 \cos (\omega t) ,$$

as in the thesis of King [15], our problem then becomes

$$z'' - \frac{4Q}{md_0^2} (U_0 - V_0 \cos (\omega t)) z .$$

Following King [15], we can then make the substitutions

$$\tau = \frac{\omega}{2} t, \quad \delta = \frac{16QU_0}{md_0^2 \omega^2}, \quad \epsilon = \frac{16QV_0}{md_0^2 \omega^2}$$

so that our equation once more takes the familiar form of Mathieu’s equation:

$$\ddot{z} + (\delta + \epsilon \cos (2\tau)) z = 0 .$$

**Stability for $U_0 = 0$**

In the case that $U_0 = 0$, our equation becomes

$$mz'' = -\frac{4QV_0}{d_0^2} \cos (\omega t) z .$$

We follow King [15]. We assume that the solution is composed of two parts: one which has large amplitude and small acceleration, the other which has small amplitude but large acceleration (something small but quickly oscillating). We approximate $z = z_M + z_\mu$ so that we can approximate our equation by

$$mz''_\mu = -\frac{4QV_0}{d_0^2} \cos (\omega t) z_M$$
so that
\[ z_\mu \approx \frac{4QV_0}{md_0^2\omega^2} \cos(\omega t) z_M. \] (3.369)

As a result, we obtain
\[ mz'' = -\frac{4QV_0}{d_0^2} \cos(\omega t) z \] (3.370)
\[ z''_M + z''_\mu = -\frac{4QV_0}{md_0^2} \cos(\omega t) (z_M + z_\mu) \] (3.371)
\[ z''_M = \frac{4QV_0}{md_0^2} \cos(\omega t) z_M = -\frac{4QV_0}{md_0^2} \cos(\omega t) z_M - \frac{16Q^2V_0^2}{m^2d_0^4\omega^2} \cos^2(\omega t) z_M \] (3.372)

Averaging over one period, this becomes
\[ z''_M = -\frac{8Q^2V_0^2}{m^2d_0^4\omega^2} z_M, \] (3.373)
which is a harmonic oscillator with frequency
\[ \frac{2\sqrt{2}QV_0}{md_0^2\omega}. \] (3.374)

As a result, for \( U_0 = 0 \), the ion trap acts like a harmonic oscillator, trapping the ion at its centre.


A physical analogy to the trap is shown in figure 3.8. If one constantly rotates the base at the correct frequency, the ball will be not roll down the base [25, 27].
Figure 3.8: A physical analogy to the ion trap.