Math 551: Two Strong Localized Perturbation Problems in 2-D

**Worked Example 1:** Consider the following eigenvalue problem in a 2-D domain with \( \Omega \) small holes:

\[
\begin{aligned}
\triangle u + \lambda u &= 0, \quad x \in \Omega \setminus \cup_{j=1}^{K} \Omega_{\varepsilon_j}, \quad (0.1a) \\
\partial_{\nu} u &= 0, \quad x \in \partial \Omega; \\
\int_{\Omega \setminus \cup_{j=1}^{K} \Omega_{\varepsilon_j}} u^2 \, dx &= 1 \quad (0.1b) \\
\end{aligned}
\]

We assume that each hole \( \Omega_{\varepsilon_j} \) is centered at \( x_j \in \Omega \). Assume further that the holes have a common logarithmic capacitance \( d \equiv d_1 = \ldots = d_K \).

1. Derive a two-term expansion for the lowest eigenvalue \( \lambda_0 \) of this problem in the form

\[
\lambda_0 \sim \lambda_{00} \nu + \lambda_{01} \nu^2 + O(\nu^3), \quad (0.2)
\]

where \( \lambda_{00} \) and \( \lambda_{01} \) are to be found, and \( \nu \equiv -1/\log(\varepsilon d) \). (Hint: the result for \( \lambda_{01} \) will involve a sum of the entries of a certain \( K \times K \) Green’s function matrix).

2. For the case of a concentric annular domain \( \varepsilon < r < 1 \) with \( r = |x| \), show that general your two-term asymptotic result above reduces to \( \lambda_0 \sim 2\nu + 3\nu^2/2 \) where \( \nu = -1/\log \varepsilon \). Verify that this two-term result agrees with the result obtained from an asymptotic approximation of the transcendental equation for the exact lowest eigenvalue. The exact transcendental equation for \( \lambda \) is obtained by making \( u(r) = J_0(\sqrt{\lambda} r) + a Y_0(\sqrt{\lambda} r) \) satisfy \( u(1) = u(\varepsilon) = 0 \). (Hint: in approximating the solution to the transcendental equation you will need the following behavior for \( J_0(z) \) and \( Y_0(z) \) for \( z \to 0 \):

\[
J_0(z) \sim 1 - z^2/4 + z^4/64 + \cdots; \quad Y_0(z) \sim \frac{2}{\pi} \left[ (\log(z/2) + \gamma ) \left( 1 - \frac{z^2}{4} \right) + \frac{z^2}{4} \right] + \cdots,
\]

where \( \gamma \) is Euler’s constant.

3. Recall that the first passage time \( w(x) \) for Brownian motion in a 2-D domain starting a point \( x \in \Omega \) in a domain with \( K \) traps, and with diffusivity \( D \), satisfies

\[
\begin{aligned}
\triangle w = -\frac{1}{D}, \quad x \in \Omega \setminus \cup_{j=1}^{K} \Omega_{\varepsilon_j}, \\
\partial_{\nu} w &= 0, \quad x \in \partial \Omega; \\
\int_{\Omega \setminus \cup_{j=1}^{K} \Omega_{\varepsilon_j}} w \, dx &= 1 \quad (0.3a) \\
\end{aligned}
\]

From your answer for \( \lambda_0 \) above, calculate a two-term asymptotic expansion for the average mean first passage time, defined by \( \bar{w} = |\Omega \setminus \cup_{j=1}^{K} \Omega_{\varepsilon_j}|^{-1} \int_{\Omega \setminus \cup_{j=1}^{K} \Omega_{\varepsilon_j}} w \, dx \). Here \( |\Omega \setminus \cup_{j=1}^{K} \Omega_{\varepsilon_j}| \sim |\Omega| + O(\varepsilon^2) \) denotes the area of the domain with the holes removed.

4. Show how your result for \( \lambda_0 \) above immediately applies to determining a critical value of the diffusivity \( D \) for the extinction threshold of a population satisfying the diffusion logistic model \( \dot{U}_t = D \triangle U + \mu U (1 - U/\beta) \) in a 2-D domain with reflecting outer boundary, and with localized regions where the population is extinct. Here \( \mu \) and \( \beta \) are positive constants. (I am looking for a simple explanation here)

**Solution:**

1. We look for a two-term expansion for the principal eigenvalue \( \lambda_0(\varepsilon) \) as

\[
\lambda_0(\varepsilon) = \lambda_1 \nu + \lambda_2 \nu^2 + \cdots, \quad \nu = -1/\log(\varepsilon d).
\]

In the outer region, away from \( O(\varepsilon) \) neighbourhoods of the holes, we expand the outer solution for \( u \) as

\[
u u = u_0 + \nu u_1 + \nu^2 u_2 + \cdots.
\]

The leading-order term is

\[
u u_0 = |\Omega|^{-1/2}.
\]
where $|\Omega|$ is the area of $\Omega$. Upon substituting (0.4) and (0.5) into (0.1 a) and (0.1 b), and collecting powers of $\nu$, we obtain that $u_1$ satisfies

$$\Delta u_1 = -\lambda_1 u_0, \quad x \in \Omega \setminus \{x_1, \ldots, x_K\}; \quad \int_{\Omega} u_1 \, dx = 0,$$

(0.7 a)

$$\partial_n u_1 = 0, \quad x \in \partial \Omega; \quad u_1 \text{ singular as } x \to x_j, \quad j = 1, \ldots, K,$$

(0.7 b)

while $u_2$ satisfies

$$\Delta u_2 = -\lambda_2 u_0 - \lambda_1 u_1, \quad x \in \Omega \setminus \{x_1, \ldots, x_K\}; \quad \int_{\Omega} (u_1^2 + 2u_0 u_2) \, dx = 0,$$

(0.8 a)

$$\partial_n u_2 = 0, \quad x \in \partial \Omega; \quad u_2 \text{ singular as } x \to x_j, \quad j = 1, \ldots, K.$$

(0.8 b)

Now in the $j^{th}$ inner region we introduce the new variables by

$$y = \epsilon^{-1}(x - x_j), \quad v(y) = u(x_j + \epsilon y).$$

(0.9)

We then expand the inner solution as

$$v(y) = \nu A_{0j} v_{cj}(y) + \nu^2 A_{1j} v_{cj}(y) + \cdots.$$  

(0.10)

Upon substituting (0.9) and (0.10) into (0.1 a) and (0.1 c), we obtain that $v_{cj}$ satisfies

$$\Delta_y v_{cj} = 0, \quad y \not\in \Omega_j; \quad v_{cj} = 0, \quad y \in \partial \Omega_j,$$

(0.11 a)

$$v_{cj}(y) \sim \log |y| - \log d + o(1), \quad \text{as } |y| \to \infty.$$  

(0.11 b)

Here $\Delta_y$ is the Laplacian in the $y$ variable, and $\Omega_j \equiv \epsilon^{-1} \Omega_{\epsilon_j}$. We consider the special case where $d$ is independent of $j$.

Upon using the far-field form (0.11 b) in (0.10), and writing the resulting expression in outer variables, we get

$$v = A_{0j} + \nu [A_{0j} \log |x - x_j| + A_{1j}] + \nu^2 [A_{1j} \log |x - x_j| + A_{2j}] + \cdots.$$  

(0.12)

The far-field behavior (0.12) must agree with the local behavior of the outer expansion (0.5). Therefore, we obtain that

$$A_{0j} = u_0 = |\Omega|^{-1/2}, \quad j = 1, \ldots, K,$$

(0.13 a)

$$u_1 \sim u_0 \log |x - x_j| + A_{1j}, \quad \text{as } x \to x_j, \quad j = 1, \ldots, K,$$

(0.13 b)

$$u_2 \sim A_{1j} \log |x - x_j| + A_{2j}, \quad \text{as } x \to x_j, \quad j = 1, \ldots, K.$$

(0.13 c)

Equations (0.13 b) and (0.13 c) give the required singularity structure for $u_1$ and $u_2$ in (0.7) and (0.8), respectively.

The problem for $u_1$ with singular behavior (0.13 b) can be written in terms of the delta function as

$$\Delta u_1 = -\lambda_1 u_0 + 2\pi A_0 \sum_{j=1}^{K} \delta(x - x_j), \quad x \in \Omega; \quad \int_{\Omega} u_1 \, dx = 0,$$

(0.14 a)

$$\partial_n u_1 = 0, \quad x \in \partial \Omega.$$  

(0.14 b)

Upon using the divergence theorem we obtain that $-\lambda_1 u_0 \int_{\Omega} 1 \, dx + 2\pi A_0 K = 0$, so that with $u_0 = A_0$ from (0.13 a), we get

$$\lambda_1 = \frac{2\pi K}{|\Omega|}.$$  

(0.15)
The solution to (0.14) can be written in terms of the Neumann Green’s function as
\[ u_1 = -2\pi u_0 \sum_{i=1}^{K} G_N(x; x_i), \]  
where the Neumann Green’s function \( G_N(x; \xi) \) satisfies
\[ \triangle G_N = \frac{1}{|\Omega|} - \delta(x - \xi), \quad \xi \in \Omega; \quad \partial_n G_N = 0, \quad x \in \partial \Omega, \]  
(0.17 a)
\[ G_N(x; \xi) \sim -\frac{1}{2\pi} \log |x - \xi| + R_N(\xi; \xi) + o(1), \quad \text{as} \quad x \to \xi; \quad \int_{\Omega} G_N(x; \xi) \, dx = 0. \]  
(0.17 b)
The constant \( R_N(\xi; \xi) \) is the regular part of \( G_N \) at the singularity. Since \( G_N \) has a zero spatial average, it follows from (0.16) that \( \int_{\Omega} u_1 \, dx = 0 \), as required in (0.14 a).

Next, we expand \( u_1 \) as \( x \to x_j \). We use the local behavior for \( G_N \), given in (0.17 b), to obtain from (0.16) that
\[ u_1 \sim u_0 \log |x - x_j| - 2\pi u_0 \left[ R_{Njj} + \sum_{i=1, i \neq j}^{K} G_{Ni j} \right], \quad x \to x_j, \]  
(0.18)
where \( G_{Ni j} = G_N(x_j; x_i) \) and \( R_{Njj} = R_N(x_j; x_j) \). Comparing (0.18) and the required singularity behavior (0.13 b), we obtain that
\[ A_{1j} = -2\pi u_0 \left[ R_{Njj} + \sum_{i=1}^{K} G_{Ni j} \right], \quad j = 1, \ldots, N. \]  
(0.19)
Next, we write the problem (0.8) in \( \Omega \) as
\[ \triangle u_2 = -\lambda_2 u_0 - \lambda_1 u_1 + 2\pi \sum_{j=1}^{K} A_{1j} \delta(x - x_j), \quad x \in \Omega; \quad \partial_n u_2 = 0, \quad x \in \partial \Omega. \]  
(0.20)
Since \( \int_{\Omega} u_1 \, dx = 0 \) and \( u_0 = |\Omega|^{-1/2} \), the divergence theorem applied to (0.20) determines \( \lambda_2 \) as \( \lambda_2 u_0 |\Omega| = 2\pi \sum_{j=1}^{K} A_{1j} \). Finally, we use (0.19) for \( A_{1j} \), we get
\[ \lambda_2 = \frac{4\pi^2}{|\Omega|} p(x_1, \ldots, x_K), \quad p(x_1, \ldots, x_K) \equiv \sum_{j=1}^{N} \left( R_{Njj} + \sum_{i=1, i \neq j}^{K} G_{Ni j} \right). \]  
(0.21)
Combining (0.4) with (0.15) and (0.21) we get the two-term expansion given in equations (5.27) and (5.28) of the Corollary in §5 of the workshop notes given by
\[ \lambda_0(\varepsilon) \sim \frac{2\pi \nu K}{|\Omega|} - \frac{4\pi^2 \nu^2}{|\Omega|} p(x_1, \ldots, x_K) + \cdots, \quad \nu = -1/\log(\varepsilon d). \]  
(0.22)
For the case of one circular hole of radius \( \varepsilon \) (for which \( d = 1 \)) in a circle of area \( |\Omega| = \pi \), the result above reduces to
\[ \lambda \sim 2\nu - 4\pi \nu^2 R_{N11}, \quad \nu \equiv -1/\log \varepsilon. \]  
(0.23)
Here \( R_{N11} \) is the regular part of the Neumann Green’s function at the center of the hole. For the unit disk and a source point at the origin, so that \( x_1 = 0 \), the Neumann Green’s function \( G_N(r; 0) \), satisfying (0.17), is radially symmetric and has the form
\[ G_N(r; 0) = \frac{r^2}{4\pi} - \frac{1}{2\pi} \log r + A \sim -\frac{1}{2\pi} \log r + A + o(1), \quad r \to 0, \]  
(0.24)
where \( A \equiv R_{N11} \) is a constant to be found from the constraint \( \int_{\Omega} G_N \, dx = 0 \). Notice that \( \partial_r G_N = 0 \) on \( r = 1 \).
The exact eigenvalue relation for the lowest eigenvalue is

\[ \lambda_0 \sim 2\nu + 3\nu^2/2 + \cdots. \]  

The integral constraint reduces to \( \int_0^1 G_{N\tau} \, dr = 0 \), which yields

\[ \int_0^1 \frac{r^3}{4\pi} \, dr - \frac{1}{2\pi} \int_0^1 r \log r \, dr + A \int_0^1 r \, dr = \frac{1}{16\pi} + \frac{1}{8\pi} + \frac{A}{2} = 0, \]

so that \( A = -3/(8\pi) \). Thus, \( A = R_{N11} = -3/(8\pi) \) is the regular part of the Neumann Green’s function at the origin. The two-term expansion (0.23) then becomes

\[ \lambda_0 \sim 2\nu + 3\nu^2/2 + \cdots. \]  

We substitute

\[ \nu \sim \frac{3}{2 \pi}, \quad J_0(\sqrt{\lambda}) = \frac{J_0(\sqrt{\lambda}c)}{Y_0(\sqrt{\lambda}c)} Y''_0(\sqrt{\lambda}). \]  

Since \( \lambda \to 0 \) as \( \varepsilon \to 0 \), we use the large argument expansion for each term above, and neglect algebraically small terms in \( \varepsilon \). We substitute

\[ J_0'(z) \sim -z/2 + z^3/16 + \cdots; \quad J_0(z) \approx 1, \]

\[ Y_0(z) \sim \frac{2}{\pi} \left( \log (z/2) + \gamma_e \right) + \cdots, \quad Y''_0(z) \sim \frac{2}{\pi z} \left[ 1 + \frac{z^2}{4} - \frac{z^2}{2} \left( \log (z/2) + \gamma_e \right) \right], \]

into (0.26), and perform a little algebra to get

\[ \frac{\lambda}{2} - \frac{\lambda^2}{16} \sim \frac{\nu}{1 - \nu \left( \log (\sqrt{\lambda}/2) + \gamma_e \right)} \left[ 1 + \frac{\lambda}{4} - \frac{\lambda}{2} \left( \log (\sqrt{\lambda}/2) + \gamma_e \right) \right], \]

where \( \nu = -1/\log \varepsilon \). We then expand \( \lambda = \lambda_0 + \nu^2 \), so that the equation above, upon using the leading term of the Binomial series, becomes

\[ \nu + \lambda_1 \nu^2/2 - \nu^2/4 \sim \nu (1 + \nu) (1 + \frac{\nu}{2} - \nu^2), \quad \chi \equiv \log \left( \sqrt{\lambda}/2 \right) + \gamma_e. \]

Expanding this out, the \( \nu \chi \) term cancels and from the \( O(\nu^2) \) terms we get \( \lambda_1/4 - 1/4 = 1/2 \). This gives \( \lambda_1 = 3/2 \), and so \( \lambda \sim 2\nu + 3\nu^2/2 \), which agrees with (0.25).

(3) Let \( \phi_j, \lambda_j \) be the eigenpairs of (0.1) for \( j = 0, 1, 2 \ldots \) ordered by \( \lambda_0 < \lambda_1 < \lambda_2 \ldots \). We calculated an asymptotic expansion for the lowest eigenpair \( \lambda_0 \) and \( \phi_0 \) above. We will normalize the eigenpairs by \( \int_{\Omega_p} \phi_j^2 \, dx = 1 \), and we know that the eigenfunctions are orthogonal in the sense that \( \int_{\Omega_p} \phi_j \phi_k \, dx = 0 \) for \( j \neq k \). We then expand the solution \( w \) of (0.3) in terms of \( \phi_j \) as \( w = \sum_{j=0}^\infty c_j \phi_j \). By orthogonality, we obtain that

\[ c_j = \int_{\Omega_p} w \phi_j \, dx \]  

Next, we multiply the equation in (0.3) by \( \phi_j \) and use Green’s second identity to obtain

\[ \int_{\Omega_p} \phi_j \Delta w + dx - \frac{1}{D} \int_{\Omega_p} \phi_j \, dx + \lambda_j \int_{\Omega_p} \phi_j w \, dx = 0. \]

Thus, \( c_j = (D\lambda_j)^{-1} \int_{\Omega_p} \phi_j \, dx \), so that from (0.29) we get

\[ w = \frac{1}{D} \sum_{j=0}^\infty \frac{\phi_j}{\lambda_j} \int_{\Omega_p} \phi_j \, dx. \]

Now we calculate \( \bar{w} \) to get

\[ \bar{w} = \frac{1}{|\Omega_p|} \int_{[\Omega_p]} w \, dx = \frac{1}{D} \sum_{j=0}^\infty \frac{1}{\lambda_j} \left( \int_{\Omega_p} \phi_j \, dx \right)^2. \]
Finally, we notice that $\lambda_0 \to 0$ as $\varepsilon \to 0$ and that $\int_{\Omega \setminus \Omega_p} \phi_j \, dx \to 0$ as $\varepsilon \to 0$ for $j \geq 1$. This follows since for $\varepsilon \to 0$ the first eigenfunction satisfies $\phi_0 \sim |\Omega|^{-1/2}$, and the orthogonality of eigenfunction property holds. Thus only the $j = 0$ term above is retained, and with $\phi_0 \sim |\Omega|^{-1/2}$, we calculate
\[
\bar{w} \sim \frac{1}{\lambda_0 D |\Omega|} \left( \int_{\Omega} |\Omega|^{-1/2} \, dx \right)^2 = \frac{1}{D \lambda_0}.
\]
Finally, we use our two-term estimate for $\lambda_0$ as given above in (0.22) to get the two-term expansion for the average mean first time
\[
\bar{w} \sim \frac{|\Omega|}{2\pi \nu K D} + \frac{|\Omega| p(x_1, \ldots, x_K)}{K^2 D} + \cdots, \quad \nu = -1/\log \varepsilon. \tag{0.30}
\]
If we want to minimize $\bar{w}$ we must choose the trap locations to minimize $p(x_1, \ldots, x_L)$.

(4) Suppose that the traps have radius $\sigma$ and that the length scale of the domain is $L$. If we assume that $\sigma \ll L$, and define $\varepsilon = \sigma/L$ and scale $U$ by the saturation constant $u = \beta U$, we obtain under steady-state conditions the nonlinear eigenvalue problem
\[
\Delta u + \lambda u (1 - u) = 0, \quad x \in \Omega \setminus \Omega_p; \quad \Omega_p \equiv \bigcup_{j=1}^K \Omega_{\varepsilon_j}, \tag{0.31 a}
\]
\[
\partial_n u = 0, \quad x \in \partial \Omega; \quad u = 0, \quad x \in \partial \Omega_{\varepsilon_j}, \quad j = 1, \ldots, K. \tag{0.31 b}
\]
Here $\lambda \equiv L^2 \mu/D$ is a dimensionless parameter. Notice that $u = 0$ is a solution for all values of $\lambda$. This is the extinct fish solution. We want to know at what minimum value of $\lambda$ will a branch of nontrivial solutions bifurcate from the $u = 0$ solution. Linearizing around $u = 0$, the local bifurcating branch is at the first eigenvalue $\lambda = \lambda_0$ of the Laplacian problem (0.1). Thus
\[
\frac{L^2 \mu}{D} = \lambda_0(\varepsilon) \sim \frac{2\pi \nu K}{|\Omega|} - \frac{4\pi^2 \nu^2}{|\Omega|^2} p(x_1, \ldots, x_K) + \cdots, \quad \nu = -1/\log(\varepsilon d). \tag{0.32}
\]
would give a threshold value of $D$ for a bifurcating solution branch.
**Worked Example 2:** Consider the following problem in the 2-D circular disk \( \Omega = \{ x \mid |x| \leq 2 \} \) that contains three small holes

\[
\begin{align*}
\Delta u &= 0, \quad x \in \Omega \setminus \bigcup_{j=1}^{3} \Omega_{\varepsilon_j} , \\
u &= 4 \cos(2\theta), \quad |x| = 2. \\
u &= \alpha_j, \quad x \in \partial \Omega_{\varepsilon_j}, \quad j = 1, 2, 3.
\end{align*}
\]

Suppose that each of the holes has an elliptical shape with semi-axes \( \varepsilon \) and \( 2\varepsilon \). Apply the theory for summing infinite logarithmic expansions to first derive and then numerically solve a linear system for the source strengths. In your implementation assume that the holes are centered at cartesian coordinate locations \( \mathbf{x}_1 = (1/2, 1/2), \mathbf{x}_2 = (1/2, 0) \) and \( \mathbf{x}_3 = (-1/4, 0) \). Take the boundary values on the holes to be \( \alpha_1 = 1, \alpha_2 = 0 \) and \( \alpha_3 = 2 \). Plot (on a computer) the source strengths versus \( \varepsilon \). (Hint: You will need to recall the method of images for calculating the required Green’s function in a circular disk)

**Solution:**

We let the holes be centered at \( x_1, \ldots, x_N \). In the outer region, defined away from \( \Omega_{\varepsilon_j} \) for \( j = 1, \ldots, N \), we expand

\[ u(x; \varepsilon) \sim U_{OH}(x) + U_0(x; \nu) + \sigma(\varepsilon)U_1(x; \nu) + \cdots, \]

where we assume that \( \sigma \ll \nu^m \) for any integer \( m > 0 \). Since the holes have a common shape, we have that \( \nu = -1/\log(\varepsilon d) \) where \( d \) is the common logarithmic capacitance of the holes. In (0.34), \( U_{OH}(x) \) is the smooth function satisfying the unperturbed problem in the unperturbed domain \( \Omega \)

\[ \Delta U_{OH} = 0, \quad x \in \Omega; \quad U_{OH} = f, \quad x \in \partial \Omega. \]

Substituting (0.34) into (0.33 a) and (0.33 b), and letting \( \Omega_{\varepsilon_j} \rightarrow \mathbf{x}_j \) as \( \varepsilon \rightarrow 0 \), we get that \( U_0 \) satisfies

\[ \begin{align*}
\Delta U_0 &= 0, \quad x \in \Omega \setminus \{ x_1, \ldots, x_N \}, \\
U_0 &= 0, \quad x \in \partial \Omega, \\
U_0 & \text{ is singular as } x \rightarrow x_j, \quad j = 1, \ldots, N.
\end{align*} \]

The singularity behavior for \( U_0 \) as \( x \rightarrow x_j \) will be found below by matching the outer solution to the far-field behavior of the inner solution to be constructed near each \( \Omega_{\varepsilon_j} \).

In the \( j^{th} \) inner region near \( \Omega_{\varepsilon_j} \) we introduce the inner variables \( y \) and \( v(y; \varepsilon) \) by

\[ y = \varepsilon^{-1}(x - x_j), \quad v(y; \varepsilon) = u(x_j + \varepsilon y; \varepsilon). \]

We then expand \( v(y; \varepsilon) \) as

\[ v(y; \varepsilon) = \alpha_j + \nu \gamma_j v_{c_j}(y) + \mu_0(\varepsilon) V_{1j}(y) + \cdots, \]

where \( \gamma_j = \gamma_j(\nu) \) is a constant to be determined. Here \( \mu_0 \ll \nu^k \) as \( \varepsilon \rightarrow 0 \) for any \( k > 0 \). In (0.38), the logarithmic gauge function \( \nu \) is defined by

\[ \nu = -1/\log(\varepsilon d), \]

where \( d \) is specified below. By substituting (0.37) and (0.38) into (0.33 a) and (0.33 c), we conclude that \( v_{c_j}(y) \) is the unique solution to

\[ \begin{align*}
\Delta_y v_{c_j} &= 0, \quad y \notin \Omega_j; \\
v_{c_j} &= 0, \quad y \in \partial \Omega_j, \\
v_{c_j}(y) & \sim \log |y| - \log d + o(1), \quad \text{as } |y| \rightarrow \infty.
\end{align*} \]

Here \( \Omega_j \equiv \varepsilon^{-1}\Omega_{\varepsilon_j} \), and the logarithmic capacitance, \( d \), is determined by the shape of \( \Omega_j \). Since the holes were assumed to have the same shape then \( d \) is independent of \( j \).

Writing (0.40 b) in outer variables and substituting the result into (0.38), we get that the far-field expansion of \( v \) away from each \( \Omega_j \) is

\[ v \sim \alpha_j + \gamma_j + \nu \gamma_j \log |x - x_j|, \quad j = 1, \ldots, N. \]
Then, by expanding the outer solution (0.34) as \( x \to x_j \), we obtain the following matching condition between the inner and outer solutions:

\[
U_{0H}(x_j) + U_0 \sim \alpha_j + \gamma_j + \nu \gamma_j \log |x - x_j|, \quad \text{as} \quad x \to x_j, \quad j = 1, \ldots, N. \tag{0.42}
\]

In this way, we obtain that \( U_0 \) satisfies (0.36) subject to the singularity structure

\[
U_0 \sim \alpha_j - U_{0H}(x_j) + \gamma_j + \nu \gamma_j \log |x - x_j| + o(1), \quad \text{as} \quad x \to x_j, \quad j = 1, \ldots, N. \tag{0.43}
\]

Observe that in (0.43) both the singular and regular parts of the singularity structure are specified. Therefore, (0.43) will effectively lead to a linear system of algebraic equations for \( \gamma_j \) for \( j = 1, \ldots, N \).

The solution to (0.36 \(a\) and (0.36 \(b\), with \( U_0 \sim \nu \gamma_j \log |x - x_j| \) as \( x \to x_j \), can be written as

\[
U_0(x; \nu) = -2\pi \nu \sum_{i=1}^{N} \gamma_i G(x; x_i), \tag{0.44}
\]

where \( G(x; x_j) \) is the Green’s function satisfying

\[
\Delta G = -\delta(x - x_j), \quad x \in \Omega; \quad G = 0, \quad x \in \partial \Omega, \tag{0.45 \(a\)}
\]

\[
G(x; x_j) \sim -\frac{1}{2\pi} \log |x - x_j| + R(x_j; x_j) + o(1), \quad \text{as} \quad x \to x_j. \tag{0.45 \(b\)}
\]

Here \( R_{ij} \equiv R(x_j; x_j) \) is the regular part of \( G \).

Finally, we expand (0.44) as \( x \to x_j \) and equate the resulting expression with the required singularity behavior (0.43) to get

\[
\nu \gamma_j \log |x - x_j| - 2\pi \nu \gamma_j R_{ij} - 2\pi \nu \sum_{i=1}^{N} \gamma_i G(x_j; x_i) = \alpha_j - U_{0H}(x_j) + \gamma_j + \nu \gamma_j \log |x - x_j|, \quad j = 1, \ldots, N. \tag{0.46}
\]

In this way, we get the following linear algebraic system for \( \gamma_j \) for \( j = 1, \ldots, N \):

\[
-\gamma_j (1 + 2\nu \rho_{ij}) - 2\pi \nu \sum_{i=1}^{N} \gamma_i G_{ij} = \alpha_j - U_{0H}(x_j), \quad j = 1, \ldots, N. \tag{0.47}
\]

Here \( G_{ij} \equiv G(x_j; x_i) \) and \( \nu_j = -1/\log(\varepsilon d_j) \). We summarize the asymptotic construction as follows:

For \( \varepsilon \ll 1 \), the outer expansion from (0.34) is

\[
u \sim U_{0H}(x) - 2\pi \nu \sum_{i=1}^{N} \gamma_i G(x; x_i), \quad \text{for} \quad |x - x_j| = \mathcal{O}(1). \tag{0.48 \(a\)}
\]

The inner expansion near \( \Omega_{ij} \) with \( y = \varepsilon^{-1}(x - x_j) \), is

\[
u \sim \alpha_j + \rho \gamma_j v_{ij}(y), \quad \text{for} \quad |x - x_j| = \mathcal{O}(\varepsilon). \tag{0.48 \(b\)}
\]

Here \( \nu = -1/\log(\varepsilon d) \), \( d \) is defined in (0.40 \(b\), \( v_{ij}(y) \) satisfies (0.40), \( U_{0H} \) satisfies the unperturbed problem (0.35), while \( G(x; x_j) \) and \( R(x_j; x_j) \) satisfy (0.45). Finally, the constants \( \gamma_j \) for \( j = 1, \ldots, N \) are obtained from the \( N \) dimensional linear algebraic system (0.47).

For the problem under consideration we have \( f = 4 \cos(2\theta) = 4(\cos^2 \theta - \sin^2 \theta) = x^2 - y^2 \) on \( (x^2 + y^2)^{1/2} = 4 \). Thus, the solution to the unperturbed problem (0.35) is simply

\[
U_{0H}(x, y) = x^2 - y^2. \tag{0.49}
\]

Next, the Green’s function satisfying (0.45) and its regular part are calculated from the method of images as

\[
G(x; x_j) = -\frac{1}{2\pi} \log \left(\frac{2|x - x_j|}{|x - x_j||x_j|}\right), \quad R_{jj} \equiv R(x_j; x_j) = -\frac{1}{2\pi} \log \left[\frac{2}{|x_j - x_j||x_j|}\right]. \tag{0.50}
\]
Here \( x'_j \) is the image point of \( x_j \) in the unit disk of radius two.

Next, we note that since each of the holes has an elliptic shape with semi-axes \( \varepsilon \) and \( 2\varepsilon \), then from the Table of the class notes their common logarithmic capacitance is \( d = 3/2 \). The holes are assumed to be centered at

\[ x_1 = (1/2, 1/2), \quad x_2 = (1/2, 0) \quad \text{and} \quad x_3 = (-1/4, 0), \]

and have the constant boundary values \( \alpha_1 = 1 \), \( \alpha_2 = 0 \) and \( \alpha_3 = 2 \).

Therefore, upon defining \( \nu = -1/\log(3\varepsilon/2) \) we obtain from (0.47) that \( \gamma_j \) for \( j = 1, \ldots, 3 \) is the solution of the linear system

\[
\begin{align*}
-\gamma_1 \left[ 1 + 2\pi \nu R_{11} \right] - 2\pi \nu \left[ \gamma_2 G(x_1; x_2) + \gamma_3 G(x_1; x_3) \right] &= 1, \\
-\gamma_2 \left[ 1 + 2\pi \nu R_{22} \right] - 2\pi \nu \left[ \gamma_1 G(x_2; x_1) + \gamma_3 G(x_2; x_3) \right] &= -1/4, \\
-\gamma_3 \left[ 1 + 2\pi \nu R_{33} \right] - 2\pi \nu \left[ \gamma_1 G(x_3; x_1) + \gamma_2 G(x_3; x_2) \right] &= 31/16.
\end{align*}
\]

Here \( R_{jj} \) and \( G(x_j; x_i) \) are to be evaluated from (0.50).

We solve this linear system numerically for \( \gamma_j \) as a function of \( \varepsilon \). The curves \( \gamma_j(\varepsilon) \) as a function of \( \varepsilon \) are plotted in Fig. 1. We observe that the leading-order approximation to (0.51), valid for \( \nu \ll 1 \), is simply \( \gamma_1 = -1 \), \( \gamma_2 = 1/4 \) and \( \gamma_3 = -31/16 \). From Fig. 1 we observe that this approximation, which neglects interaction effects between the holes, is rather inaccurate unless \( \varepsilon \) is very small.

![Figure 1. Plot of \( \gamma_j = \gamma_j(\varepsilon) \) for \( j = 1, 2, 3 \) obtained from the numerical solution to (0.51).](image-url)