PROBLEM 1. WE FIRST CONSIDER

\[ \frac{d}{dx} \left[ D(x, x/e) U_x \right] = f(x) \quad 0 < x < 1 \]

\[ U(0) = a, \quad U(1) = b \]

WE ASSUME THAT

\[ 0 < D_m(x) \leq D(x, y) \leq D_M(x) \quad 0 < x < 1, \quad 0 < y < \infty \]

THE GOAL IS TO DETERMINE THE LIMITING EQUATION FOR U ON THE MACROSCALE X.

WE LOOK FOR \( U = U(x, y) \) WITH \( y = \varepsilon^1 x \). THEN

\[ \frac{d}{dx} = \partial_x + \varepsilon^{-1} \partial_y \]

WE GET

\[ (\partial_y + \varepsilon \partial_x) \left[ D(x, y) (\partial_y + \varepsilon \partial_x) U \right] = \varepsilon^2 f(x) \]

WE THEN EXPAND

\[ U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \ldots \]

SUBSTITUTING INTO THE EQUATION WE OBTAIN,

\[ \partial_y \left[ D \partial_y U_0 \right] + \varepsilon \left[ \partial_x \left( D \partial_y U_0 \right) + \partial_y \left( D \partial_x U_0 \right) + \partial_y \left( D \partial_y U_1 \right) \right] \]

\[ + \varepsilon^2 \left[ \partial_y \left( D \partial_y U_1 \right) + \partial_y \left( D \partial_y U_1 \right) + \partial_x \left( D \partial_y U_1 \right) + \partial_x \left( D \partial_y U_0 \right) \right] \]

\[ = \varepsilon^2 f(x) \]

EQUIVATING COEFFICIENT IN \( \varepsilon \):

1. \[ \partial_y \left[ D \partial_y U_0 \right] = 0 \]

2. \[ \partial_y \left[ D \partial_y U_1 \right] = -\partial_x \left[ D \partial_y U_0 \right] - \partial_y \left[ D \partial_x U_0 \right] \]

3. \[ \partial_y \left[ D \partial_y U_2 \right] = -\partial_y \left[ D \partial_x U_1 \right] - \partial_x \left[ D \partial_y U_1 \right] - \partial_x \left[ D \partial_y U_0 \right] + f(x) \]
THE SOLVABILITY CONDITION IS THAT $u_0$ IS BOUNDED AS $y \to \infty$. WE SOLVE (1) TO GET

$$u_0 = c_0(x) \int_{y_0}^{y} \frac{1}{d(x,s)} ds + c_1(x).$$

**C₀, C₁ ARBITRARY.**

BUT SINCE $u_0$ IS BOUNDED AS $y \to \infty$ WE MUST SET $c_0(x) = 0$. HENCE

$$u_0 = u_0(x)$$

ONLY.

AT NEXT ORDER, (2) YIELD

$$dy \left[ D \frac{dy}{dy} u_1 \right] = - \left( \frac{dy}{dy} D \right) u_0 x.$$

NOTICE THAT $u_1 = -x \frac{dx}{dx} u_0$ SOLVE HOMOGENEOUS FORM OF (4). THE SOLUTION TO (4)

IS THEN,

$$u_1(x, y) = b_1(x) + b_0(x) \int_{y_0}^{y} \frac{ds}{d(x,s)} - y u_0 x.$$

NOW FOR $u_1$ TO BE BOUNDED AS $y \to \infty$ WE GET

$$\lim_{y \to \infty} \left[ u_0 x - \frac{1}{y} b_0(x) \int_{y_0}^{y} \frac{ds}{d(x,s)} \right] = 0.$$

THEREFORE,

$$u_0 x = <D^{-1} >_y b_0(x)$$

WHERE $<D^{-1}>_y \equiv \lim_{y \to \infty} \frac{1}{y} \int_{y_0}^{y} \frac{1}{d(x,s)} ds.$

AT NEXT ORDER, WE USE $u_{1y} = \frac{b_0}{d} - u_0 x$ IN (3) TO GET

$$dy \left[ D \frac{dy}{dy} u_2 \right] = - \frac{dy}{dy} \left[ D \frac{dx}{dx} u_1 \right] - \frac{dx}{dx} \left[ D(b_0/d - u_0 x) \right] - \frac{dx}{dx} \left[ D \frac{dx}{dx} u_0 \right] + F.$$

THIS GIVES

$$dy \left[ D \frac{dy}{dy} u_2 \right] = - \frac{dy}{dy} \left[ D \frac{dx}{dx} u_1 \right] - b_0' + F.$$

INTEGRATING WE GET

$$D \frac{dy}{dy} u_2 = - D \frac{dx}{dx} u_1 + (F - b_0') y.$$

HENCE, WE CALCULATE

$$u_2 = d_1(x) - \int_{y_0}^{y} d_2 x u_1 ds + b_0' \int_{y_0}^{y} \frac{ds}{d(x,s)} + (F - b_0') \int_{y_0}^{y} \frac{s}{d(x,s)} ds.$$

THE LAST INTEGRAL IS $O(y)$ AS $y \to \infty$ UNLESS $F = b_0'$. 
\[ b_0(x) = \frac{u_0(x)}{<D^{-1}>_0} \text{ from (5)}. \]

Hence, we have the homogenized equation,

\[ \partial_x [\bar{D}(x) \partial_x u_0] = f(x) \quad \text{bar harmonic mean} \]

\[ u_0(0) = a, \quad u_0(1) = b \quad \text{problem only depends on macroscale}. \]

Where

\[ \bar{D}(x) = \frac{1}{<D^{-1}>_0} \quad <D^{-1}>_0 = \lim_{Y \to \infty} \frac{1}{Y} \int_Y^Y \frac{ds}{D(x,s)}. \]

Remarks: The following properties hold

(i) If \( 0 < D_M(x) \leq D(x,y) \leq D_M(x) \quad \rightarrow \quad 0 < D_m(x) \leq \bar{D}(x) \leq D_M(x) \).

(ii) If \( \lim_{Y \to \infty} D(x,y) = D_\infty(x) \), then by L'Hôpital rule

\[ <D^{-1}>_\infty = D_\infty(x). \]

If \( D(x,y) \) is periodic in \( y \) so that \( D(x,y) = D(x,y + Y_p) \) \( \forall x, y \)

then we can retrace the derivation to get that \( u_0, u_1, u_2 \)

are periodic in \( y \). We conclude that,

\[ D \partial_y u_1 = -D u_0 x + b_0 \quad \rightarrow \quad u_1 = -\partial_x u_0 + b_1 + b_0 \int_0^Y \frac{ds}{D(x,s)} \]

We want \( u_1(x, y) = u_1(x, y + Y_p) \). Hence

\[ \partial_y u_1 = -u_0 x + \frac{b_0}{D(x,y)} \quad \rightarrow \quad \int_0^{Y_p} \partial_y u_1 \, dy = 0 \]

Given

\[ u_0 x = b_0(x) <D^{-1}>_p \quad <D^{-1}>_p = \frac{1}{Y_p} \int_0^{Y_p} \frac{ds}{D(x,s)} . \]

The equation for \( u_2 \) yields (7) with \( \bar{D}(x) = \frac{1}{<D^{-1}>_p} \).
(iii) For an arbitrary \( D(x, y) \) that is periodic in \( y \) with period \( y_0 \), the average \( D_{\text{avg}}(x) \) is

\[
D_{\text{avg}}(x) = \frac{1}{y_0} \int_0^{y_0} D(x, s) \, ds.
\]

The following inequality holds:

\[
\bar{D}(x) \leq D_{\text{avg}}(x).
\]

**Example 1** Suppose that \( D(x, y) = \frac{1}{1 + d x + B \cos y} \). We calculate that

\[
< D^{-1} >_{\infty} = \lim_{y \to \infty} \frac{1}{y} \int_0^y (1 + d x + B \cos s) \, ds = 1 + d x.
\]

Hence the homogenized equation for

\[
\frac{d}{dx} \left[ \frac{1}{1 + d x + B \cos y} \right] u_{0x} = f(x) \quad \text{is} \quad \frac{d}{dx} \left[ \bar{D}(x) u_{0x} \right] = f(x)
\]

with \( \bar{D}(x) = (1 + d x)^{-1} \).

Now when \( f = 0 \), the homogenized problem to leading order solves

\[
\frac{d}{dx} \left[ \bar{D} \frac{du_0}{dx} \right] = 0 \quad u_0(0) = 0 \quad u_0(1) = 1
\]

We calculate \( u_{0x} = \frac{C}{\bar{D}} = C (1 + d x) \rightarrow u_0 = C \int_0^x (1 + d \lambda) \, d\lambda \)

We want \( u_0(1) = 1 \) so that

\[
u_0(x) = \frac{1}{1 + d/2} \left( x + d x^2/2 \right)
\]

The full solution satisfies

\[
\frac{d}{dx} \left[ \frac{1}{1 + d x + B \cos(x/\epsilon)} \frac{du_0}{dx} \right] = 0, \quad u_0(0) = 0 \quad u(1) = 1
\]

We calculate that

\[
u = \frac{(x + d x^2/2 + \epsilon B \sin(x/\epsilon))}{1 + d/\epsilon + \epsilon B \sin(1/\epsilon)}
\]

so \( u - u_0 = O(\epsilon) \).
Now if we had incorrectly used the average $D_{\text{avg}}$ then

$$D_{\text{avg}}(X) = \frac{1}{2\pi} \int_0^{2\pi} \frac{dy}{1 + dX + B \cos(y)} = \frac{1}{\sqrt{(1 + dX)^2 - B^2}}$$

where $D = \frac{1}{1 + dX}$.

Now

$$\frac{d}{dx} \left( D_{\text{avg}} \frac{du}{dx} \right) = 0 \quad u(0) = 0, \quad u(1) = 1.$$  

We calculate

$$D_{\text{avg}} \frac{dU}{dx} = C \quad U = C \int_0^X \sqrt{(1 + d\lambda)^2 - B^2} \, d\lambda$$

with

$$C = \left( \int_0^1 \frac{d\lambda}{\sqrt{(1 + d\lambda)^2 - B^2}} \right)^{-1}.$$
MULTI-DIMENSIONAL PROBLEMS

CONSIDER THE 2-D PROBLEM

\[ \nabla \cdot \left[ D \left( x \varepsilon \right) \nabla u \right] = f \left( x \right) \quad x \in \Omega \]

\[ u \left( x \right) = u_0 \left( x \right) \text{ on } \partial \Omega \]

ASSUME THAT \( D \left( y \right) \) IS \( \gamma_p \) PERIODIC IN \( \mathbb{R}^d \) WITH BASIC CELL

\[ \gamma = \left\{ y : \left( y_1, \ldots, y_n \right) : 0 < y_i < \gamma_p; \quad i = 1, \ldots, N \right\} \]

\[ D \left( y + \gamma_p \right) = D \left( y \right) \]

FOR EXAMPLE IF \( D = 6 + \cos \left( 2 \pi y_2 - 3 \gamma_2 \right) \) THEN \( \gamma_p = \left( \pi, 2\pi/3 \right) \).

NOW WE WILL EXPAND

\[ u \varepsilon \left( x \right) = u_0 \left( x, y \right) + \varepsilon \: u_1 \left( x, y \right) + \varepsilon^2 \: u_2 \left( x, y \right) + \ldots \]

WITH \( \nabla = \nabla_x + \varepsilon^{-1} \nabla_y \). WE SUBSTITUTE TO GET,

\[ \left( \nabla_x + \varepsilon^{-1} \nabla_y \right) \cdot \left[ D \left( y \right) \left( \nabla_x + \varepsilon^{-1} \nabla_y \right) u \right] = f \left( x \right) \]

NOW WE CALCULATE,

\[ \varepsilon^{-2} \nabla_y \cdot \left[ D \left( y \right) \nabla_y u_0 \right] + \varepsilon^{-1} \left[ \nabla_y \cdot \left[ D \nabla_y u_1 \right] + \nabla_y \cdot \left[ D \nabla_x u_0 \right] + \nabla_x \cdot \left[ D \nabla_y u_0 \right] \right] \]

\[ + \left[ \nabla_y \cdot \left[ D \nabla_y u_2 \right] + \nabla_y \cdot \left[ D \nabla_x u_1 \right] + \nabla_x \cdot \left[ D \nabla_y u_1 \right] + \nabla_x \cdot \left[ D \nabla_x u_0 \right] \right] = f \left( x \right) \]

NOW TO LEADING ORDER

\[ \nabla_y \cdot \left[ D \left( y \right) \nabla_y u_0 \right] = 0 \quad \text{ FOR } \gamma \in \gamma \]

NOW THE PERIODICITY CONDITION IS THAT FOR ANY SYMMETRIC POINTS ON \( \partial \gamma \) LABELLED BY \( \gamma_L \) AND \( \gamma_R \) WE HAVE

\[ u_j \left( \gamma_L \right) = u_j \left( \gamma_R \right), \quad \nabla_y u_j \left( \gamma_L \right) = \nabla_y u_j \left( \gamma_R \right) \]

NOTICE IN 1-D CAN'T BE EXTENDED SMOOTHLY

WHEREAS CAN BE EXTENDED SMOOTHLY.
Now to leading order we get

\[ u_0 = u_0(x). \]

At the next order we get,

\[ \nabla_y \cdot [D \nabla_y u_j] = -\nabla_y \cdot [D \nabla_x u_0] \quad \text{for } y \in \Omega. \]

We write this as

\[ \nabla_y \cdot [D \nabla_y u_j] = -\nabla_y \cdot D \cdot \nabla_x u_0 = -\partial_{y_1} D \cdot \partial_{x_1} u_0 - \cdots - \partial_{y_N} D \cdot \partial_{x_N} u_0. \]

We now let \( w_j(y) \) be the solution of the unit cell problem

\[ \text{(Unit cell problem)} \quad \nabla_y \cdot [D \nabla_y w_j] = -\partial_{y_j} D \quad j = 1, \ldots, N \]

Then the solution is

\[ u_1(x,y) = \sum_{j=1}^{N} w_j(y) \partial_{x_j} u_0 + b_1(x) \]

Now we have at next order

\[ \nabla_y \cdot [D \nabla_y u_2] = -\nabla_y \cdot [D \nabla_x u_1] - \nabla_x \cdot [D \nabla_y u_1] - \nabla_x \cdot [D \nabla_x u_0] + f(x) \]

We write this as

\[ \nabla_y \cdot [D \nabla_y u_2 + D \nabla_x u_1] = -\nabla_x \cdot [D \nabla_y u_1] - \nabla_x \cdot [D \nabla_x u_0] + f(x) \]

Integrating over the unit cell

\[ \int_{\Omega} \nabla_y \cdot [D \nabla_y u_2 + D \nabla_x u_1] \, dy = \int_{\Omega} \nabla_x \cdot [D \nabla_y u_1] \, dy - \int_{\Omega} \nabla_x \cdot [D \nabla_x u_0] \, dy + \int_{\Omega} f \, dy \]

Using the divergence theorem

\[ \int_{\partial \Omega} \nabla \cdot [D \nabla_y u_2 + D \nabla_x u_1] \, ds = -\int_{\Omega} \nabla_x \cdot [D \nabla_y u_1] \, dy - \int_{\Omega} \nabla_x \cdot [D \nabla_x u_0] \, dy + f(x) |\Omega| \]

Now by periodicity of \( u_2 \) and \( u_1 \) we get using the formula for \( u_1 \) that

\[ \begin{align*}
\mathcal{O} &= -\int_{\Omega} \nabla_x \cdot [D \nabla_y u_1] \, dy - \left( \int_{\Omega} f(y) \, dy \right) \Delta_x u_0 + f(x) |\Omega| \quad \text{(Recall } |\Omega| = 1) \end{align*} \]
From (1) we get
\[ \nabla_y u_i = \sum_{j=1}^{N} \nabla_y w_{ij} \, d_{x_j} u_0 . \]

Then (4) becomes
\[ \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{\gamma} D_{ij} \, \nabla_y w_{ij} \, d_{x_i} x_j u_0 \, dy + \int_{\gamma} D(y) \, dy \, \Delta_x u_0 = f(x) \]

Now \[ \Delta_x u_0 = \sum_{i=1}^{N} d_{x_i} x_i u_0 . \]

We then define
\[ D_{ij} = \frac{1}{|\gamma|} \int_{\gamma} D(y) \left[ \delta_{ij} + \nabla_y w_{ij} \right] \, dy . \]

Equation (5) becomes
\[ \sum_{i=1}^{N} \sum_{j=1}^{N} D_{ij} \, \nabla_{x_i} x_j u_0 = f(x) \]

This can be written as the following statement:

The solution to \( \nabla \cdot [D(x/\varepsilon) \nabla u] = f(x) \), \( u = u_b \) on \( \partial \Omega \)

can be approximated by, at least in the interior of the domain,

\[ \nabla \cdot [D \nabla u_0] = f(x) \text{ in } \Omega \]

\[ u = u_b \text{ on } \partial \Omega . \]

where \( D \) is the matrix with entries \[ D_{ij} = \frac{1}{|\gamma|} \int_{\gamma} D(y) \left[ \delta_{ij} + \nabla_y w_{ij} \right] \, dy \]

where the periodic function \( w_j \) satisfies the cell problem

\[ \nabla_y \cdot [D \nabla_y w_j] = -\partial_{y_j} D \text{ in } \Omega \]

\[ w_j \text{ periodic} \]
(i) The original problem is isotropic since $D(x/e)$ is the diffusivity. The homogenized problem (6) is anisotropic since $D$ is not a diagonal matrix. Hence isotropy on microscale does not imply isotropy on macroscale.

(ii) End up with same homogenized problem if $D = D(x, x/e)$.

(iii) Now it is easy to show that $D$ is a symmetric positive definite matrix when $D(y) > d > 0$ for $y \in \mathcal{Y}$.

**Special Case #1:** Periodic in $\mathcal{Y}_N$.

Now let $D(y_1, \ldots, y_N) = \tilde{D}(y_N)$. Then from the cell problem we have

$W_j = 0$ for $j \neq N$

$W_N = W_N(y_N)$

$\tilde{D} W_N = -\tilde{D} + C$

$W_N$ periodic on $0 < y_N < 1$

We solve to get

$W_N = -y_N + \int_0^{y_N} \frac{1}{\tilde{D}(\lambda)} d\lambda + \text{arbitrary constant}$

$\int_0^1 \frac{1}{\tilde{D}(\lambda)} d\lambda$

Now $D_{y_N} W_N = -1 + \frac{1}{\tilde{D}} \int_0^1 \frac{1}{\tilde{D}(\lambda)} d\lambda$

Thus we have

$D_{ij} = 0$ if $i \neq j$

$D_{ii} = \int_0^{y_N} D(y) dY_N$

$D_{NN} = \frac{1}{\int_0^1 \tilde{D}(y_N)} dY_N$

**Effective Conductivity**

Effective conductivity $D_{ij}$ of a layered medium is given by the arithmetic mean in directions parallel to the layers but by the geometric mean in the normal direction to the layer.
Now we have

\[ D_{ij} = \int_Y D(y) \left[ \delta_{ij} + \delta_y w_j \right] \, dy = \int_Y D(y) \left[ \nabla w_j + e_j \right] \cdot e_i \, dy \quad e_j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \]

Now for the cell problem we multiply by any function \( \phi \) which is periodic in \( y \). Then

\[ \phi \nabla_y \left[ D \nabla_y w_j \right] = -\phi \partial_y D \]

Using \( \nabla_y \left[ \phi F \right] = \nabla_y \phi \cdot F + \phi \nabla_y F \) we get with \( F = D \nabla_y w_j \)

\[ \int_Y \nabla_y \left[ \phi D \nabla_y w_j \right] \, dy - \int_Y \phi \partial_y \nabla_y w_j \, dy = -\int_Y \phi \partial_y D \, dy \]

0 by periodicity. So \( \int_Y D \nabla_y \phi \cdot \nabla_y w_j = -\int_Y \phi \partial_y D \, dy \)

Using integration by parts and \( \phi \partial_y D = \partial_y \phi (D) - \partial_y (\phi D) \) we get

\[ \int_Y D \nabla_y \phi \cdot \nabla_y w_j \, dy + \int_Y D \partial_y \phi \, dy = 0 \]

This can be written as

\[ (x) \quad \int_Y D \left[ \nabla w_j + e_j \right] \cdot \nabla \phi \, dy = 0 \]

\( (x) \) hold for any \( \phi \) that is periodic in \( y \).

Let \( \phi = w_i \). Then

\[ \int_Y D e_j \cdot \nabla w_i \, dy = -\int_Y D \nabla w_i \cdot \nabla w_j \, dy \]

So

\[ \int_Y D \partial_y w_i \, dy = -\int_Y D \nabla w_i \cdot \nabla w_j \, dy. \]

This means that for \( i \neq j \) we have can also show \( D \) positive definite.

\[ D_{ij} = -\int_Y D \nabla w_i \cdot \nabla w_j \, dy \rightarrow D_{ij} \text{ is symmetric } D_{ji} = D_{ij}. \]

We can write

\[ D_{ij} = \int_Y D(y) \left[ \delta_{ij} - \nabla w_i \cdot \nabla w_j \right] \, dy \quad \text{for } i \neq j. \]
For the next problem, consider

\[ \nabla \cdot \left[ D(x/\epsilon) \nabla u \right] + f(x) = 0 \quad \text{in} \quad B^\epsilon \]

\[ \hat{n} \cdot D(x/\epsilon) \nabla u = 0 \quad \text{on} \quad \Gamma^\epsilon \]

\[ u = u_d \quad \text{on} \quad \Gamma \]

\( B^\epsilon \): region without the holes, \( \Gamma^\epsilon \): boundary of the holes, \( \hat{n} \): unit normal out of \( B^\epsilon \).

We can repeat the same analysis as for the previous problem noting that

\[ \hat{n} \cdot D(x/\epsilon) \nabla u = \hat{n} \cdot D(x/\epsilon) \left[ \nabla_y u_y + \nabla_x u_0 \right] + \epsilon \hat{n} \cdot D(x/\epsilon) \left[ \nabla_y u_y + \nabla_x u_1 \right] + \ldots = 0 \]

on \( \Gamma^\epsilon \). Hence we obtain the homogenized problem

\[ \nabla \cdot \left[ \mathcal{D} \nabla u_0 \right] = f(x) \]

\[ \mathcal{D} = \int \mathcal{D}(y) \left[ \delta_{ij} + \phi_{ij} \right] W_j \, dy \]

with

\[ u_j = \sum_j W_j(y) \, dx_j \, u_0 + b_i(x) \]

We get

\[ \hat{n} \cdot D(x/\epsilon) \left[ \sum_j \nabla_y W_j \, dx_j \, u_0 + \delta_{ij} \, dx_j \, u_0 \right] = 0 \quad \text{on} \quad \Gamma^\epsilon. \]

This boundary condition can be written as

\[ \hat{n} \cdot D(x/\epsilon) \sum_j \left( \nabla_y W_j + e_j \right) \, dx_j \, u_0 = 0 \quad \text{on} \quad \Gamma^\epsilon. \]

Hence the unit cell problem is

\[ \nabla_y \cdot \left[ \mathcal{D}(y) \nabla W_j \right] = -\delta_{ij} \, D \quad \text{in} \quad B \]

\( W_j \): periodic boundary condition on \( dy \) \( j \)th position.

\[ \hat{n} \cdot \left[ \nabla_y W_j + e_j \right] = 0 \quad \text{on} \quad \Gamma \]

\( e_j = (0,0,\ldots,1,\ldots,0,0) \)
Homogenization of Rough Interfaces

The boundary has the form

\[ z = h(y/\varepsilon) \]

\[ h(y) = h(y + 1) \text{ period}. \]

We must solve the problem

\[ \nabla \cdot [\sigma \nabla u] + \alpha u = f(x, z) \quad z \neq h(y/\varepsilon) \]

\[ \hat{n} = (\varepsilon^{-1}h_y, -1) \]

\[ \sigma(x, z) = \begin{cases} \sigma^+ & z > h(y/\varepsilon) \\ \sigma^- & z < h(y/\varepsilon) \end{cases} \]

With \( \int u = [\sigma \nabla u \cdot \hat{n}] = 0 \) across the interface.

We look for a solution in the form

\[ u(x, y, z, \varepsilon) = u_0(x, y, z) + \varepsilon u_1(x, y, z) + \varepsilon^2 u_2(x, y, z) + \ldots \]

With \( y = x/\varepsilon \). We have

\[ \varepsilon^2 (\sigma u_y)_y + \varepsilon^3 (\sigma u_x)_y + \varepsilon^2 (\sigma u_y)_x + (\sigma u_x)_x + (\sigma u_z)_z + \alpha u = f \]

We want \( u_0 \) periodic in \( y \).

Now

\[ \nabla u \cdot \hat{n} = (\partial_x + \varepsilon\partial_y, \partial_x, \partial_z) u \cdot (\varepsilon^{-1}h_y, -1) = \varepsilon^{-2} u_y h_y + \varepsilon^{-1} u_x h_y - u_z. \]

Substituting and equating powers of \( \varepsilon \)

\[ \begin{cases} \sigma u_{0y} = 0 \quad z \neq h(y) \\ [u_0] = [\sigma u_{0y}] = 0 \end{cases} \]

\[ \begin{cases} (\sigma u_{1y})_y = - (\sigma u_{0x})_y - (\sigma u_{0y})_x \quad z \neq h(y) \\ [u_1] = 0, \quad [\sigma (u_{1y} + u_{0x})] = 0 \end{cases} \]

\[ \begin{cases} (\sigma u_{2y})_y + (\sigma u_{1x})_y = - (\sigma u_{1y})_x - (\sigma u_{0x})_x - (\sigma u_{0z})_z - \alpha u_0 + f \quad z \neq h(y) \\ [u_2] = 0, \quad [\sigma (u_{2y} h_y + u_{1x} h_y - u_{0z})] = 0 \end{cases} \]
This implies that

\[ \forall (x, y, z) + 1 = \forall (x, y + 1, z) = \forall (x, y, z) + \frac{\sigma_{\text{eff}}}{\sigma^+} (y_2 - y_1) + \frac{\sigma_{\text{eff}}}{\sigma^-} (y_2 - y_1) \]

Equivalently,

\[ \sigma_{\text{eff}} = \left( \frac{y_2 - y_1}{\sigma^+} + \frac{1 - (y_2 - y_1)}{\sigma^-} \right)^{-1} \rightarrow \sigma_{\text{eff}} = \sigma_{\text{eff}} (z). \]

We then integrate the equation at next order over 0 ≤ y ≤ 1 to get

\[ \int_0^1 \sigma (u_{y^2} + u_{x^1})_y \, dy = - \int_0^1 (\sigma u_{y^2}) x \, dy - \int_0^1 (\sigma u_{y^2}) x \, dy - \int_0^1 (\sigma u_{y^2}) z \, dy - \int_0^1 \sigma u_0 \, dy + \int_0^1 f \, dy \]

but

\[ u_{y^2} + u_{x^2} = \forall_y u_{x^2}. \]

\[ \int_0^1 \sigma (u_{y^2} + u_{x^2})_y \, dy = - \int_0^1 (\sigma u_{y^2}) u_{x^2} \, dy - \int_0^1 \sigma dy - \int_0^1 (u_{y^2} + u_{x^2} )_y \, dy = - \int_0^1 (u_{y^2} + u_{x^2} )_y \, dy \]

Using \( \sigma y = \sigma_{\text{eff}} \) and \( \sigma = \int_0^1 \sigma \, dy \), we get

\[ \int_0^1 \sigma (u_{y^2} + u_{x^2})_y \, dy = - \sigma_{\text{eff}} u_{yo_{x^2}} - \sigma u_{y^2} + \int_0^1 \sigma (u_{y^2} + u_{x^2} )_y \, dy. \]

Now, since \( \sigma h_y (u_{y^2} + u_{x^2}) - \sigma u_{y^2} = 0 \), \( \sigma (u_{y^2} + u_{x^2}) \) is not continuous. This gives,

\[ \sigma (u_{y^2} + u_{x^2}) \bigg|_{y = y_1} - \sigma (u_{y^2} + u_{x^2}) \bigg|_{y = 0} + \sigma (u_{y^2} + u_{x^2}) \bigg|_{y = y_2} - \sigma (u_{y^2} + u_{x^2}) \bigg|_{y = y_1} + \sigma (u_{y^2} + u_{x^2}) \bigg|_{y = y_2} \]

By periodicity, \( \sigma (u_{y^2} + u_{x^2}) \big|_{y = 0} = \sigma (u_{y^2} + u_{x^2}) \big|_{y = 1} \). Hence,

\[ - \left[ \sigma (u_{y^2} + u_{x^2}) \right]_{y = 1} - \left[ \sigma (u_{y^2} + u_{x^2}) \right]_{y = y_1} = - \sigma_{\text{eff}} u_{yo_{x^2}} - \sigma u_{y^2} + \int_0^1 \sigma (u_{y^2} + u_{x^2} )_y \, dy. \]

This gives,

\[ \frac{\sigma u_{yo_{x^2}}}{h_y(y_1)} - \frac{\sigma u_{yo_{x^2}}}{h_y(y_2)} = - \sigma_{\text{eff}} u_{yo_{x^2}} - \sigma u_{y^2} + \int_0^1 \sigma (u_{y^2} + u_{x^2} )_y \, dy. \]

But \( u_{yo_{x^2}} \) is independent of \( y \), hence.
TO LEADING ORDER

\[ \sigma u_0 = a \quad \rightarrow \quad u_0 = a \int_0^y \frac{1}{\sigma} ds + b \quad a, b \text{ depend on } x, z. \]

NOW THIS SHOWS \( u_0 = u_0(x, z) \). AT NEXT ORDER WE GET

\( (\sigma u_1 + \sigma u_0)_x = 0 \)

\[ [u_1] = 0 \quad [\sigma(u_1 + u_0 x)] = 0 \]

INTRODUCE A FUNCTION \( V \) so that \( V u_0 x = u_1 + y u_0 x \) AND \( V_y u_0 x = u_1 y + u_0 x \)

\[ u_1 = V u_0 x - y u_0 x \quad \rightarrow \quad u_{1 y} + u_0 x = V_y u_0 x. \]

THIS YIELDS,

\( (\sigma V_y)_y = 0 \quad z \neq h(y) \)

\[ [V] = 0 \quad [\sigma V_y] = 0 \]

THIS IMPLIES THAT \( \sigma V_y \) IS INDEPENDENT OF \( y \).

WE PLOT THE LOCAL GEOMETRY,

WE LABEL \( \sigma V_y = \sigma_{\text{eff}} \). WE CALCULATE

\[ V = V(x, y, z) + \sigma_{\text{eff}} (y - y_i) \quad y_i \leq y \leq y_2 \quad h(y) < z < 0 \]

\[ V = V(x, y, z) + \sigma_{\text{eff}} (y_2 - y_i) + \sigma_{\text{eff}} (y - y_2) \quad y_2 < y \leq y_1 + 1 \]

NOW WE WANT \( u_1(x, y, z) = u_1(x, y + 1, z) \). THUS

\[ V(x, y, z) = V(x, y + 1, z) - 1 \]
Now by continuity of $U_0$

\[ w(0) = 1 + R \]

\[ w(-A) = T \]

Also by the continuity of $[\sigma > U_z]_{z=0}$ we get

\[ \langle \sigma^z \rangle (-i P_z + R i P_z) = \langle \sigma > w'(0) \]

Thus

\[ w'(0) = -i P_z (1 - R) = -i P_z (2 - w(0)) \]

Now similarly the condition $[\langle \sigma > U_z]_{z=-A}$ we get,

\[ w'(-A) = -i \hat{P}_z T = -i \hat{P}_z w(-A) \]

Therefore in the layer we must solve

\[ \frac{d}{dz} \left[ \langle \sigma > \frac{dw}{dz} \right] - \sigma_{eff} P_z^2 w + K^2 w = 0 \]

Or equivalently,

\[ \frac{d}{dz} \left[ \langle \sigma > \frac{dw}{dz} \right] + (K^2 - \sigma_{eff} P_z^2) w = 0 \quad -A \leq z \leq 0 \]

\[ w'(0) = -i P_z (2 - w(0)) \]

\[ w'(-A) = -i \hat{P}_z w(-A) \]

Then the reflection and transmission coefficients are defined in terms of the solution by

\[ R = w(0) - 1 \]

\[ T = w(-A) \]
Consider wave propagation and reflection off of an interface.

We have
\[ \frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla \cdot [\sigma \nabla \phi] \]

We let \( \phi = e^{i \omega t} u \) to get
\[ \nabla \cdot [\sigma \nabla u] + k^2 u = 0 \quad k = \omega c. \]

Now suppose that we have an incident wave
\[ u_x = e^{i (p_x x - p_z z)} \quad \sigma_+ (p_x^2 + p_z^2) + k^2. \]

Then the wave field has the form
\[ u = \begin{cases} 
   e^{i p_x x} (e^{-i p_z z} + R e^{i p_z z}) & z > 0 \\
   e^{i p_x x} (e^{-i \tilde{p}_z z} T) & z < 0 
\end{cases} \]

where \( R, T \) are reflection, transmission coefficients. Note, \( \sigma_- (p_x^2 + \tilde{p}_z^2) = k^2. \)

Now by the continuity of \( u \rightarrow 1 + R = T \)
\[ \sigma_- u_x \rightarrow \sigma_+ (-i p_z + R i p_z) = -i \tilde{p}_z \sigma_- T \]

Therefore
\[ p_z \sigma_+ (R - 1) = -\sigma_- \tilde{p}_z (R + 1) \]

Solving for \( R, T \) we get
\[ R = \frac{p_z \sigma_+ - \tilde{p}_z \sigma_-}{p_z \sigma_+ + \tilde{p}_z \sigma_-} \quad T = \frac{2 p_z \sigma_+}{p_z \sigma_+ + \tilde{p}_z \sigma_-} \]

Now consider the rough interface problem
\[ \nabla \cdot [\sigma \nabla u] + k^2 u = 0 \]
\[ u_0 = \begin{cases} 
   e^{i p_x x} (e^{-i p_z z} + R e^{i p_z z}) & z > 0 \\
   e^{i p_x x} w(z) & -A \leq z < 0 \\
   e^{i p_x x} (e^{-i \tilde{p}_z (z + A) T}) & z \leq -A 
\end{cases} \]

where \( w \) satisfies the homogenized equation in the layer. (See HW)
\begin{align*}
(\sigma^+ - \sigma^-) \left[ \frac{1}{h_1(y)} - \frac{1}{h_2(y)} \right] \omega_{oZ} &= -\sigma_{\text{eff}} \omega_{oXX} - <\sigma> \omega_{oZZ} - a \omega_o + f \\
\text{NOW} \quad <\sigma> &= \sigma^+ (1-y_1) + \sigma^- (1-y_2 + y_1), \text{ WE CALCULATE,} \\
<\sigma>_Z &= \sigma^+ (y_2' - y_1') + \sigma^- (y_1' - y_2') \\
\text{BUT SINCE} \quad z = h(y), \quad 1 = h'(y) y' \rightarrow y_1' = \frac{1}{h'(y)}, \quad y_2' = \frac{1}{h_2'(y)}, \text{ SUBSTITUTING} \\
\text{INTO THE ABOVE EQUATION WE OBTAIN,} \\
<\sigma>_Z \omega_{oZ} + <\sigma> \omega_{oZZ} + \sigma_{\text{eff}} \omega_{oXX} + a \omega_o &= f \\
\text{WE WRITE THIS COMPACTLY AS} \\
\frac{\partial}{\partial z} \left[ <\sigma> \omega_{oZ} \right] + \sigma_{\text{eff}} \omega_{oXX} + a \omega_o &= f \\
\text{WHERE} \\
\sigma_{\text{eff}} &= \left( \frac{y_2 - y_1}{\sigma^+} + \frac{1 - (y_2 - y_1)}{\sigma^-} \right)^{-1} = \sigma_{\text{eff}}(z) \\
<\sigma> &= \sigma^+ (y_2 - y_1) + \sigma^- (1 - y_2 + y_1) \\
\text{NOW IN THE LAYER, WE CAN WRITE} \\
\nabla \cdot \left( \tilde{\sigma}(z) \nabla \omega_o \right) + a \omega_o &= f \quad -A < z < 0 \\
\text{homogenized} \\
\text{equation} \\
\text{with} \\
\tilde{\sigma}(z) = \begin{pmatrix} \sigma_{\text{eff}}(z) & 0 \\ 0 & <\sigma> \end{pmatrix}
\end{align*}

\text{IN SUMMARY, WE HAVE WITH } h_1(y+1) = h(y), \text{ MIN } h(y) = -A, \text{ MAX } h(y) = 0, \quad y_2 = \frac{y}{\varepsilon}

\begin{align*}
\nabla \cdot \left[ \sigma^+ \nabla \omega_o \right] + a \omega_o &= f \quad \text{in } z > 0 \\
\nabla \cdot \left( \tilde{\sigma}(z) \nabla \omega_o \right) + a \omega_o &= f \quad \text{in } -A < z < 0 \\
\nabla \cdot \left[ \sigma^- \nabla \omega_o \right] + a \omega_o &= f \quad \text{in } z < -A
\end{align*}
Darcy's law is a phenomenologically derived constitutive law that describes the slow flow of a fluid through a porous medium, such as a sand and water mixture.

We consider a periodic array of sand particles in either $\mathbb{R}^2$ or $\mathbb{R}^3$ shown as:

\[
\begin{array}{c}
\text{fluid region} \\
\text{solid particle} \\
\text{size of particle} \\
\text{macroscopic over which pressure varies} \\
\text{then } \varepsilon = \frac{l}{L} \ll 1.
\end{array}
\]

Now we must solve Stokes equations

\[
\begin{cases}
\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{p}, & x \in \Omega \setminus \{\text{solid particles}\} \\
\n\mathbf{v} = 0, & x \in \Omega \setminus \{\text{solid particles}\} \\
\mathbf{v} = 0 & \text{on each solid particle boundary}
\end{cases}
\]

And there is some externally imposed pressure gradient across macroscopic boundary $\partial \Omega$. The unit cell $B$ is as shown in $\mathbb{R}^2$ as

\[
\begin{array}{c}
\Omega \\
\{\text{boundary of solid particle removed}\}
\end{array}
\]

We first need to re-scale $\mathbf{v}$. We must balance external pressure gradient on macroscopic length scale with viscous forces on microscopic $l$, i.e.

\[
\frac{\mu}{l^2} \left[ \mathbf{v} \right] = \frac{\mathbf{p}}{L} \quad \text{so} \quad \left[ \mathbf{v} \right] = \frac{\mathbf{p}}{L} \frac{l^2}{\mu}
\]
THUS MEANING THAT 
\[ \mathcal{V} = O(\varepsilon^3). \]

AS SUCH WE RESCALE \((x)\) BY REPLACING 
\[ v \mapsto \varepsilon^2 v \] TO GET

\[
\begin{cases}
\mu \varepsilon^2 \Delta \varepsilon^2 v = \nabla p & x \in \Omega \setminus \{\text{solid particles}\} \\
\nabla \cdot \varepsilon^2 v = 0 & x \in \Omega \setminus \{\text{solid particles}\} \\
\varepsilon^2 v = 0 & \text{on each particle boundary}
\end{cases}
\]

WE LOOK FOR A SOLUTION WITH

\[
\begin{cases}
\varepsilon^2 v = v_0(x,y) + \varepsilon v_1(x,y) + \varepsilon^2 v_2(x,y) + \ldots & \text{with } y = x/\varepsilon \\
\rho = \rho_0(x,y) + \varepsilon \rho_1(x,y) + \ldots
\end{cases}
\]

WE HAVE

\[
\nabla \equiv \left( \frac{1}{\varepsilon} \nabla_y + \nabla_x \right), \text{ so}
\]

\[
\mu \varepsilon^2 \left( \frac{1}{\varepsilon} \nabla_y + \nabla_x \right) \cdot \left( \frac{1}{\varepsilon} \nabla_y + \nabla_x \right) \nabla = \left( \frac{1}{\varepsilon} \nabla_y + \nabla_x \right) \rho.
\]

THIS GIVES

\[
\mu \Delta_y v_0 + \varepsilon \mu \left[ \Delta_y v_1 + \nabla_y \cdot (\nabla_x v_0) + \nabla_x \cdot (\nabla_y v_0) \right] = \frac{1}{\varepsilon} \nabla_y \rho_0 + (\nabla_y \rho_1 + \nabla_x \rho_0)
\]

\[
\varepsilon^2 \nabla_y \cdot v_0 + (\nabla_y \cdot v_1 + \nabla_x \cdot v_0) + \ldots = 0.
\]

WE CONCLUDE THAT

\[
\nabla_y \rho_0 = 0 \quad \text{for } y \in B
\]

AND

\[
\nabla_y \rho_1 + \nabla_x \rho_0 = \mu \Delta_y v_0 \quad \text{for } y \in B
\]

\[
\nabla_y \cdot v_0 = 0
\]

\[
\nabla_x \cdot v_0 = 0 \quad \text{on } \Gamma
\]

WE CONCLUDE THAT \(\rho_0 = \rho_0(x)\), AT NEXT ORDER

\[
\nabla_x \cdot v_1 = -\nabla_x \cdot v_0 \quad \text{in } B.
\]
Now in (4) we write \( \nabla_x p_0 = \sum_j e_j \delta_{x_j} p_0 \) at \( \text{1st position} \)

\[
\mu \Lambda_y v_0 = \nabla_y p_1 + \sum_{j=1}^N e_j \delta_{x_j} p_0 \quad \text{in } B, \quad e_j = (0, \ldots, 1, 0)
\]

\( \nabla_y \cdot v_0 = 0 \) \( \text{in } B \)

\( v_0 = 0 \) \( \text{on } \Gamma \)

We look for a solution \((v_0, p_1)\) in the form

\[
v_0 = \sum_{j=1}^N (-W_j/p) \delta_{x_j} p_0 \quad \text{with } W_j = W_j(y)
\]

\[
p_1 = \sum_{j=1}^N \hat{\eta}_j(y) \delta_{x_j} p_0 \quad \text{with } \hat{\eta}_j = \hat{\eta}_j(y)
\]

Now imposing that the divergence condition is satisfied \( \nabla \cdot \delta_{x_j} p_0 \)

we obtain the unit cell problem

\[
\mathcal{L}_y W_j = \nabla_y \hat{\eta}_j - e_j \quad \text{in } B
\]

\[
\nabla_y \cdot W_j = 0 \quad \text{in } B
\]

\( W_j = 0 \) \( \text{on } \Gamma \)

Now notice that \( v_0 \) in (7) depends on both the slow and the fast scales. As such we define the average of \( v_0 \) as

\[
\mu(x) = \frac{1}{|B|} \int_B v_0(x, y) \, dy \quad \text{with } |B|: \text{volume of } B.
\]

Now if we substitute for \( v_0 \) from (7) we get

\[
\mu(x) = \left( \frac{1}{\mu |B|} \right) \int_B \sum_{j=1}^N W_j(y) \delta_{x_j} p_0 \, dy = \left( \frac{-1}{\mu} \right) \sum_{j=1}^N \hat{\eta}_j \delta_{x_j} p_0
\]
we now define
\[ K_j = \left( K_{nj}, \ldots, K_{nj} \right)^T \]

and
\[ \Omega = \left( K_{n1}, \ldots, K_{nn} \right) \] is the permeability matrix.

Then
\[ u(x) = -\frac{1}{\mu} \Omega \nabla p_0 \] is the relation between averaged velocity and pressure gradient.

Remark
(i) (10) is called Darcy's law.
(ii) We now show that \( \nabla_x \cdot u = 0 \).

We recall (5) that \( \nabla_x \cdot v_1 = -\nabla_x \cdot v_0 \) so that
\[ \nabla_x \cdot u = \frac{1}{B} \int_B \nabla_x \cdot v_0 \, dy = \frac{1}{B} \int_B \nabla_y \cdot v_1 \, dy \]
\[ = -\frac{1}{B} \int_B v_1 \cdot \hat{n} \, ds = -\frac{1}{B} \int_B v_1 \cdot \hat{n} \, ds = -\frac{1}{B} \int_B v_1 \cdot \hat{n} \, ds \]
\[ = 0 \text{ by the periodicity condition on } \Omega \]

Thus, we have \( \nabla_x \cdot u_0 = 0 \) so that the homogenized pressure satisfies
\[ \nabla \cdot (\Omega \nabla p_0) = 0 \text{ in } \Omega \]
with some condition on \( p_0 \) on \( \partial \Omega \).

(iii) It can be shown that \( \Omega \) is symmetric and positive definite.

(iv) The permeability matrix \( \Omega \) is found only by solving the cell problem (8) for \( \Omega_j \) and \( \Pi_j \) for \( j = 1, \ldots, N \).