BLOW-UP BEHAVIOR

Consider
\[ u_t = u_{xx} - u + u^p \quad 0 < x < \pi \]
\[ u(0, t) = u(\pi, t) = 0, \quad u(x, 0) = u_0(x) > 0 \]

with \( p > 1 \). We show that the solution blow up (diverges to \( \infty \)) at a finite \( t = T \), when \( u_0(x) \) is "large enough". In other words, the ODE part

\[ u_t = -u + u^p \quad u(0) = u_0 \quad u_0 \text{ constant} \]

which blow up in finite time control the dynamics.

We let
\[ F(t) = \int_0^\pi u(x, t) \sin x \, dx. \]

Then
\[ \int_0^\pi u_t \sin x \, dx = \int_0^\pi u_{xx} \sin x \, dx - \int_0^\pi u \sin x \, dx + \int_0^\pi u^p \sin x \, dx. \]

Integrating by parts and using the boundary condition, we get
\[ (x) \frac{dF}{dt} = -2F + \int_0^\pi u^p \sin x \, dx. \]

Now recall Holder's inequality
\[ \int_0^\pi a(x) b(x) \, dx \leq \left( \int_0^\pi [a(x)]^p \, dx \right)^{1/p} \left( \int_0^\pi [b(x)]^q \, dx \right)^{1/q} \quad \frac{1}{p} + \frac{1}{q} = 1 \]
\[ p > 0, q > 0, a > 0, b > 0, \quad \forall x. \]

Choose \( a(x) = u(\sin x) \)
\[ b = [\sin x]^p \]

Hence
\[ \int_0^\pi u \sin x \, dx \leq \left( \int_0^\pi u^p \sin x \, dx \right)^{1/p} \left( \int_0^\pi \sin x \, dx \right)^{1-1/p} \]
\[ \left( \int_0^\pi u^p \sin x \, dx \right)^{1/p} \geq \frac{\pi}{2^{1-1/p}} \quad \rightarrow \quad \int_0^\pi u^p \sin x \, dx \geq \frac{\pi^p}{2^{p-1}}. \]

Therefore, (9) gives
\[ \frac{dF}{dt} \geq -2F + \frac{F^p}{2^{p-1}} \quad F(0) = \int_0^\pi u_0(1 \sin x \, dx \]

Now
\[ \frac{dF}{dt} = \frac{F^p}{2^{p-1}} \left[ 1 - \frac{F^{1-p}}{2^{-p}} \right] = \frac{F^p}{2^{p-1}} \left[ 1 - 2^{-p} F^{1-p} \right] \]
Thus if \( 1 - 2^p f_0^{1-p} > 0 \) or
\[
\frac{f_0}{2^p/p - 1} > 2^p/p - 1
\]

we get blow-up in finite time. Thus if because the ODE
\[
\frac{dy}{dt} = \frac{y^p}{2^p/p - 1} (1 - 2^p y^{1-p}) \quad \text{for } p > 1 \quad \text{has finite time blow-up.}
\]

If \( F \to \phi \) at some time \( T \), we have from \( F(t) = \int_0^t u \sin x \, dx \Rightarrow \| u \|_{L_2} \to \infty \) when \( t \to T \).

\[
I = \int_{y_o}^y \frac{d\lambda}{\lambda^{p/2^p - 1} - 2\lambda} = \int_0^t ds = t, \quad \text{with} \quad \frac{y_o^{p/2^p - 1} - 2y_o}{2^p/p - 1} > 0
\]

now as \( y \to \infty \), \( I \to I_o \). Thus \( t \to T \) where \( T \) is finite.

Hence the blow-up time is
\[
T = \int_{y_o}^\phi \frac{d\lambda}{\lambda^{p/2^p - 1} - 2\lambda}.
\]

Now that we have established blow-up, we ask what does the blow-up profile look like?

We consider
\[
\frac{dU}{dt} = AU + F(U) \quad \text{for } X \in \mathbb{R}^N \quad \text{assume} \quad F(U) > 0 \quad \text{and} \quad F'(U) > 0.
\]

The ODE
\[
\frac{dU}{dt} = F(U) \quad \text{with} \quad U(0) = U_0
\]

can be solved to give
\[
\int_{U_0}^U \frac{1}{F(\lambda)} \, d\lambda = t.
\]

Hence we have finite time blow-up for the ODE if
\[
\int_{U_0}^\phi \frac{1}{F(\lambda)} \, d\lambda \text{ is finite.}
\]

Then the blow-up time is
\[
T = \int_{U_0}^\phi \frac{1}{F(\lambda)} \, d\lambda.
\]
For instance if \( f(u) = u^p \) we calculate

\[
\int_0^u \frac{1}{f(u)} \, \text{d}u = \int_0^u \lambda^{-p} \, \text{d}\lambda = \frac{1}{-p} \left[ u^{-p} - u_0^{-p} \right] = t \quad \text{for } p > 1
\]

As \( u \to 0 \) we have \( t \to T = \frac{1}{p-1} u_0^{-p} > 0 \).

Rather than looking for a function that blow-up as \( t \to T^- \) (i.e. the \( u \) variable) we will look for a solution \( v \to 0 \) as \( t \to T^- \).

We introduce

\[
v = \int_0^\infty \frac{\text{d}\lambda}{f(\lambda)}
\]

For the ODE we would obtain that

\[
v = \int_0^\infty \frac{\text{d}\lambda}{f(\lambda)} - \int_0^u \frac{\text{d}\lambda}{f(\lambda)} = T - t
\]

and so \( v \) is linear near the blow-up time.

Now we calculate

\[
v_t = -\frac{1}{f(u)} u_t \quad \Rightarrow \quad u_t = -f(u) v_t
\]

Also

\[
u_x = -f(u) v_x \quad u_{xx} = -f'(u) u_x v_x - f(u) u_{xx}
\]

Thus

\[
u_{xx} = f'(u) f(u) v_x^2 - f(u) u_{xx}
\]

\[
\Delta u = f'(u) f(u) |v|^2 - f(u) \Delta v
\]

We substitute into

\[
u_t = \Delta u + f(u)
\]

\[
vt = -1 + \Delta v - f'(u(v)) |v|^2
\]

\[
v(x,0) = \int_0^\infty \frac{\text{d}\lambda}{f(\lambda)}
\]

We will assume that blow-up first occurs at \( x = 0, t = T \) and that near there \( v(x,t) \) has the local expansion

\[
v(x,t) = v_0(t) + \frac{1}{2} v_2(t) + \frac{1}{4} v_4(t) + \ldots
\]
Given radial symmetry, we calculate

\[ \Delta W = W_{rr} + \frac{N-1}{r} W_r \quad |\nabla W|^2 = W_r^2 \]

Now, if

\[ V = V_0 + \frac{r^2}{k_0} V_2 + \frac{r^4}{4 k_0} V_4 + \ldots \quad \text{for} \quad r \ll 1 \]

we calculate

\[ \Delta V = N V_2 + V_4 \left( \frac{N+2}{6} \right) r^2 \]

\[ |\nabla V|^2 = V_r^2 + r^2 V_2^2 + \ldots \]

\[ V_t = V_{0t} + \frac{r^2}{2} V_{2t} + \ldots \]

\[ f'(u(V_0)) \sim f'(u(V_0)) \]

Now substituting we get

\[ V_{0t} + \frac{r^2}{2} V_{2t} + \ldots = N V_2 + V_4 \left( \frac{N+2}{6} \right) r^2 - 1 - f'(u(V_0)) \frac{r^2}{2} V_2^2 + \ldots \]

Equating powers of \( r \):

\[ \begin{cases} V_{0t} = -1 + N V_2 \\ V_{2t} = -2 f'(u(V_0)) V_2^2 + \frac{(2+N) V_4}{3} \end{cases} \]

We will solve these equations approximately for \( t \to T^- \). We want \( V_0 = 0 \) at \( t = T \). We assume

\[ V_1 \ll 1, \quad V_4 \ll f'(u(V_0)) V_2^2 \quad \text{as} \quad t \to T^- \]

To a first approximation, we have

\[ V_0 \sim T - t \quad \text{for} \quad T - t \ll 1 \]

The next equation gives

\[ V_{2t} = -2 f'(u(V_0)) V_2^2 \quad \text{with} \quad V_0 = T - t \]

Remark

(i) Consider the ODE

\[ V_{2t} = -g(t) V_2^2 \]

\[ V_2(0) = V_2^0 > 0 \]

with \( g(t) > 0 \). Then, \( V_1 > 0 \) for all \( t \) and \( V_2 \to 0 \) in finite time.
PROOF

\[
\int_{V_t}^{V_0} \frac{ds}{S^2} = - \int_0^t g(t) \, dt = -G(t)
\]

\[
\implies - \frac{1}{V_2} + \frac{1}{V_2^0} = -G(t).
\]

Thus,

\[
\frac{1}{V_2} = \frac{1}{V_2^0} + G(t) > 0
\]

If \( V_1 \to 0^+ \) then \( t \to \infty \) since \( G(\infty) \) is infinite.

Therefore, since we have assumed \( f'(u) > 0 \) our problem

\[
V_{2t} = -g(t) V_2^3
\]

with \( g(t) = 2 f'(u(V_0)) \) is of this type. We want \( V_0, V_1, V_A, \ldots > 0 \)

before the blow-up time with \( V_0, V_1, V_A \to 0^+ \) as \( t \to T^- \).

Now we calculate

\[
\frac{1}{V_2} = \frac{1}{V_2^0} + G(t)
\]

with

\[
G(t) = \int_0^t g(t) \, dt \quad \text{and} \quad g(t) = 2 f'[u(T-t)]
\]

Now we calculate:

Let \( s = T-t \). \( ds = -dt \).

\[
G(t) = - \int_{T-t}^{T} 2 f'[u(s)] \, ds = \int_{T-t}^{T} 2 f'[u(s)] \, ds
\]

Now

\[
S = \int_{u}^{\infty} \frac{1}{f(u)} \, du \quad \text{implies} \quad 1 = - \frac{1}{f(u)} \frac{du}{ds}.
\]

Thus,

\[
G(t) = \int_{S-T-t}^{S-T} 2 f'[u(s)] \, ds \frac{du}{ds} = -2 \int_{S=T-t}^{S=T} \frac{f'[u(s)]}{f[u(s)]]} \, du
\]

\[
G(t) = -2 \log \left[ f(u(t)) \right] \bigg|_{S=T-t}^{S=T} = 2 \log \left[ f(u(T-t)) \right] - 2 \log \left[ f(u(T)) \right]
\]

Therefore,

\[
G(t) = 2 \log \left[ f(u(T-t)) \right] - 2 \log \left[ f(u(T)) \right]
\]
Recall from the transformation \( U: \Omega(v) \) defined by
\[
U = \int_{v}^{\infty} \frac{1}{f(\lambda)} \, d\lambda
\]
then as \( v \to 0 \) we must have \( U \to +\infty \). Hence \( F[U(t-t)] \to +\infty \) as \( t \to T^- \). Hence from
\[
\frac{1}{V_2} = \frac{1}{V_2^0} + G(t)
\]
we get
\[
V_2(t) \to \frac{1}{2 \log[F(U(t-t))]}
\]
as \( t \to T^- \).

Now the next iteration for \( V_n \) is
\[
V_n(t) = 1 + N V_2
\]
We will impose \( V_0 \to 0 \) at \( t = T \) (or \( V_0 \to 0 \) as \( t \to T^- \)).

We get
\[
V_0 = T - t - \int_{t}^{T} V_2(\tau) \, d\tau
\]
Now as \( t \to T^- \), we have \( V_2 \sim \frac{1}{2 \log[F(U(t-t))]}) \)
so that
\[
V_0 \sim T - t - \frac{N}{2} \int_{t}^{T} \frac{d\tau}{\log[F(U(t-t))]}.
\]

And
\[
\frac{\Gamma^2}{2^0} \sim \frac{\Gamma^2}{4 \log[F(U(t-t))]}.
\]
The two term expansion reads (as \( t \to T^- \) and \( \Gamma \to 0 \))
\[
V \sim T - t - \frac{N}{2} \int_{t}^{T} \frac{d\tau}{\log[F(U(t-t))]} + \frac{\Gamma^2}{4 \log[F(U(t-t))]} + \ldots
\]
Now for \( T - t \ll 1 \) we can approximate the integrand as \( \int_{a}^{b} f(x) \, dx \approx (b-a) f(a) \) as \( b-a \to 0 \).
\[ \nu \sim T - t - \frac{N}{2} \frac{(T-t)}{\log[\nu(t-t)]} + \frac{r^2}{4 \log[\nu(t-t)]} \]

But
\[ \nu = \int u \frac{1}{f(u)} \, du \]

And so we get the final result for the blow-up of \( U \)
\[ (x) \int u \frac{1}{f(u)} \, du \sim T - t - \frac{N}{2} \frac{(T-t)}{\log[\nu(t-t)]} + \frac{r^2}{4 \log[\nu(t-t)]} \]
\[ u \sim \log (T-t) - \frac{N}{2|\log (T-t)|} + \frac{r^2}{4(T-t)|\log (T-t)|} \quad u \text{ the blow-up profile.} \]

**Example 2** Suppose \( F(u) = u^p \). Then

\[ v = \int_u^\infty \lambda^{-p} d\lambda = \frac{1}{p-1} u^{1-p} \]

Or \( u^{1-p} = (p-1)v \rightarrow u = (p-1)^{1/p-1} \sqrt{v} \)

Now \( u^p = a_p \sqrt{v}^{p/(1-p)} \quad a_p = (p-1)^{p/(1-p)} \)

Thus as \( v \to 0 \) we get

\[ \log F(u) = p \log u \sim p \log \left( v^{1/(1-p)} \right) + \log a_p^{1/p} \]

To a first approximation,

\[ \log F(u|v) \sim \frac{p}{1-p} \log v \quad \text{as} \quad v \to 0. \]

We can write this as

\[ \log F(u|v) \sim \frac{p}{p-1} |\log v| \quad \text{since} \quad p > 1 \text{ and } v \to 0. \]

Substituting into (4) on page 18 we get

\[ I \quad \frac{u^{1-p}}{p-1} \sim T-t - \frac{N(p-1)(T-t)}{2p|\log (T-t)|} + \frac{r^2}{4p|\log (T-t)|} \]

\[ u^{1-p} \sim (p-1) \left( T-t \right) \left( 1 - \frac{N(p-1)}{2p|\log (T-t)|} + \frac{r^2}{4p(T-t)|\log (T-t)|} \right) \]

\[ (T-t)^{-p} u \sim \kappa \left( 1 - \frac{N(p-1)}{2p|\log (T-t)|} + \frac{r^2}{4p(T-t)|\log (T-t)|} \right)^{1/p-1} \quad \text{where} \quad \kappa = (p-1)^{1/p-1} \]

Using Taylor series \((1+y)^{1/p} \sim y^{1/p}\) for \( p < 1 \) we get

\[ (T-t)^{-p} u \sim \kappa \left( 1 + \frac{N}{2p|\log (T-t)|} - \frac{r^2}{4p(T-t)|\log (T-t)|} \right) \]
We now consider blow-up in dimension 1 for

\[ u_t = u_{xx} + u^p \quad -\infty < x < \infty, \quad p > 1 \]

Let \( q, T \) be the blow-up point and blow-up time, respectively. Then

**Theorem** (Filișan, Kohn, CPAM 1992).

\[ (T-t)^{1/p-1} u(x,t) \sim K + \frac{K}{2p \log(1/T-t)} \left( 1 - \frac{|x-a|^2}{2(1/T-t)} \right) \]

where \( K = (p-1) \).

This is precisely the result we obtained on p. 19.

We now sketch how this result is obtained. We introduce new variables

\[ u(x,t) = (T-t)^{-1/p-1} w(y,s) \]

with \( y = (x-a)/\sqrt{T-t} \) and \( s = -\log(T-t) \).

Changing variable, (1) becomes

\[ w_t - \frac{1}{\varrho} \nabla \cdot (\varrho \nabla w) + \frac{1}{p-1} w = w^p \]

where \( \varrho = e^{-y^2/4} \)

\[ s \rightarrow \text{time derivative} \]

Notice \( s \rightarrow +\infty \Rightarrow t \rightarrow T^- \).

**Remark** (i) Consider \( u_t = u_{xx} + u^p \)

If we set \( x = \lambda \tilde{x}, \quad t = \lambda^2 \tilde{t}, \quad u = \lambda^b \tilde{u} \)

then we obtain \( \tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}x} + \tilde{u}^p \) when \( b = -2/p-1 \).

This suggests that if we substitute

\[ u = (T-t)^{-1/p-1} f \left( \frac{\tilde{x}}{\sqrt{T-t}} \right) \]

we will get an ODE for \( f(A) \). Notice that (4) has a slightly different form.
Now for $|y|$ bounded, we impose the steady-state condition

$$W(y, s) \rightarrow K \quad \text{as} \quad s \rightarrow \infty,$$

where

$$K^p = \frac{K}{p^*} \quad \text{or} \quad K = (p-1)^{-1/p-1}.$$

Now we linearize by letting $W = K + V$ where $V \ll 1$.

We define

$$g(w) = w^p - \frac{1}{p^*} w$$

so that

$$g(K + V) = g(K) + g'(K) V + \frac{g''(K)}{2} V^2 + \cdots$$

Thus from (5) we obtain

$$V = \frac{P}{2K} V^2 + O(V^3)$$

(6)

We are interested in the behavior of (6) as $s \rightarrow \infty$ on bounded sets in $|y|$. In particular, we will show that

$$V(y, s) \sim \frac{K}{2ps} \left(1 - \frac{y^2}{2}\right) \quad \text{in 1-D}$$

With a similar result in 2-D being

$$V(y, s) \sim \frac{K}{p^*} \left(1 - \frac{|y|^2}{4}\right) \quad \text{in 2-D}.$$  

After we show (7), then we use

$$W = K + V + \cdots$$

to get

$$W = K + \frac{K}{2ps} \left(1 - \frac{y^2}{2}\right) \quad \text{as} \quad s \rightarrow \infty.$$

Recalling the definition of $y$ and $s$ from p. 20, we obtain

$$u(x, t) \sim (T-t)^{-1/p^*} \left[ K + \frac{K}{+2p\log(T-t)} \left(1 - \frac{(x-a)^2}{2(T-t)}\right) \right] \quad t \rightarrow T^-$$

Which is the main result. (We also require $(x-a)^2/(T-t)$ bounded).
REMARK (i) Similarly we get the correct result in 2-D.

We define
\[ dV = \frac{1}{\rho} \nabla \cdot (\rho \nabla V) + V \]

so that the equation reads
\[ V_s - dV = \frac{\rho}{2\kappa} V^2 \]

We have some properties of \( d \):

**Lemma** Consider the eigenvalue problem
\[ d \phi = \lambda \phi \]

which we write as
\[ d \left( \lambda \frac{d\phi}{dy} \right) + \lambda \phi = \lambda \lambda \phi \quad \lambda = e^{-y/4} \]

This eigenvalue problem is self-adjoint with respect to the inner product
\[ (f, g) = \int_0^\infty f \overline{g} \, dy \].

The eigenvalues have the property:

1. There are two positive eigenvalues \( \phi_1^+, \phi_2^+ \) given by
   \[ \phi_1^+ = 1, \quad \phi_2^+ = \frac{1}{2} \quad \phi_1^+ = c_0^+, \quad \phi_2^+ = c_1^+ \]

2. There is one zero eigenvalue given by
   \[ \phi_1^0 = 0 \quad \phi_1^0 = c_0^0 \left( \frac{1}{2} y^2 - 1 \right) \]

3. The negative eigenvalues are
   \[ \phi_j^- = 1 - \frac{j(j+2)}{2} \quad j = 1, 2, 3, \ldots \]

   with eigenfunctions
   \[ \phi_j^-(y) = c_j^- H_{j+2}(\sqrt{\frac{y}{2}}) \]

   where
   \[ H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} (e^{-x^2}) \quad k = 0, 1, 2, \ldots \]

   are Hermite polynomials.
Proof: In one-space dimension we have from (16)
\[ \phi_{yy} - \frac{y}{2} \phi_y + \phi = \Lambda \phi \]

Now let \( x = \frac{y}{2} \) and \( \psi(x) = \phi(y) \). This gives
\[ \psi_{xx} - 2x \psi_x = \mu \psi \quad \mu = 4(\Lambda - 1) \]

which is the well-known Hermite equation with eigenvalues
\[ \mu = -2K, \quad K = 0, 1, 2, \ldots \]
\[ \psi = H_K(x) \]

Hence
\[ \Lambda = -\frac{K+1}{2}, \quad K = 0, 1, 2, \ldots \]
\[ \phi_K(y) = c_K H_K \left( \frac{y}{2} \right) \]

Decomposing the eigenspace into positive, zero, negative eigenvalues gives the result. \( \blacksquare \)

We use the normalization condition \( (\phi_K, \phi_K) = 1 \) to calculate the constant \( c_K \). In the notation of p. 22 we get
\[ c_0^+ = \frac{1}{\sqrt{2}} \pi^{-1/4}, \quad c_1^+ = \frac{1}{2} \pi^{-1/4}, \quad c_1^- = \frac{1}{2} \pi^{-1/4} \]

We will also use the identity
\[ \phi_K(y) = \left( \frac{y}{2} \right)^{K/2} \phi_{K-1}(y) \quad K = 1, 2, 3, \ldots \]

which is a consequence of the well-known recursion formula
\[ H_K(y) = 2y H_{K-1}(y) \]

for Hermite polynomials.

Remark: (i) In two-dimension the eigenvalues are the same but the dimension of the unstable eigenspace is 3, and the dimension of the nullspace is 3.

(ii) The zero eigenspace is spanned by
\[ c_0 c_2 \left( \frac{1}{2} y_1^2 - 1 \right), \quad c_0 c_3 \left( \frac{1}{2} y_1^3 - 1 \right), \quad c_1^2 y_1, y_2 \]
NOW WE RETURN TO
\[ V_s - \frac{dV}{ds} = \frac{P}{2K} \varepsilon^2. \]

WE DECOMPOSE \( V \) IN AN EIGENFUNCTION EXPANSION

\[ V = (B_1(s) \phi_1^+(y) + B_2(s) \phi_2^+(y)) + \phi_1(s) \phi_1^0(y) + (g_1(s) \phi_1^-(y) + B_2(s) \phi_2^-(y)). \]

SINCE WE WANT \( V \to 0 \) AS \( s \to \infty \) WE MUST ELIMINATE THE GROWING MODES \( B_1(s) \) AND \( B_2(s) \).

THEREFORE, WE HAVE

\[ V = \phi_1(s) \phi_1^0(y) + (g_1(s) \phi_1^-(y) + B_2(s) \phi_2^-(y) + \ldots) \]

WE CALCULATE FROM OUR EIGENVALUE PROBLEM

\[ V_s - \frac{dV}{ds} = \dot{\phi}_1 \phi_1^0 + \left[ (\dot{x}_1 - e_1 \phi_1) \phi_1^- + (\dot{x}_2 - e_2 \phi_2) \phi_2^- + \ldots \right] \]

WITH \( e_j < 0 \) (NEGATIVE EIGENVALUES). NOW WE CALCULATE \( \frac{P}{2K} \varepsilon^2 \).

\[ \frac{P}{2K} \varepsilon^2 = \frac{P d_1^2 \phi_1^0}{2K} + \text{other terms that decay as } s \to \infty \]

GIVING

\[ \dot{d}_1 \phi_1^0 + \sum_{j=1}^\infty (\dot{x}_j - e_j \phi_j) \phi_j^- = \frac{P d_1^2 \phi_1^0}{2K} + \text{other terms that decay}. \]

Since eigenfunctions are orthogonal \( \int_0^\infty \phi_1^0(y) \phi_j^- dy = 0 \) we get

\[ \dot{d}_1 \int_0^\infty \phi_1^0(y)^2 dy = \frac{P}{2K} \left( \int_0^\infty \phi_1^0(y)^3 dy \right) d_1^2. \]
Now since \( \int_{-\infty}^{\infty} (\phi_1^0)^2 \, dy = 1 \) by the normalization condition,

we get

\[
\dot{d}_1 = \frac{p}{2K} d_1^2 \int_{-\infty}^{\infty} \alpha (c_1^0)^3 \left( \frac{1}{2} y^2 - 1 \right)^3 \, dy \quad \text{with} \quad \alpha = e^{-y/4}
\]

\[c_1^0 = \frac{1}{2\pi^{1/4}}.
\]

We can calculate this integral to get

\[
\int_{-\infty}^{\infty} \alpha (c_1^0)^3 \left( \frac{1}{2} y^2 - 1 \right)^3 \, dy = 4 c_1^0.
\]

Hence,

\[
d_1 = \frac{2p c_1^0}{K} d_1^2
\]

We are interested in the solution for large \( s \).

Let \( d_1 \sim A/s \) for large \( s \). We get

\[
-\frac{A}{s^2} = \frac{2p c_1^0}{K} \frac{A^2}{s^3} \quad \rightarrow \quad A = -\frac{K}{2pc_1^0}
\]

Thus

\[
d_1 \sim -\frac{K}{2pc_1^0 s}
\]

Next we have \( s \),

\[
V \sim d_1(s) \phi_1^0(y) = -\frac{K}{2pc_1^0} c_1^0 \left( \frac{1}{2} y^2 - 1 \right)
\]

Therefore,

\[
V(y,s) \sim \frac{K}{2ps} \left( 1 - \frac{1}{2} y^2 \right) \quad \text{as} \quad s \to \infty.
\]

Hence from page 21, we have

\[
W = K + \frac{K}{2ps} \left( 1 - y^2/2 \right) + \ldots \quad \text{as} \quad s \to \infty
\]

which gives the blow-up profile for \( u \) on page 21.