BLOW-UP BEHAVIOR

CONSIDER
$$u_t = u_{xx} - u + u^p \qquad 0 < x < \pi$$
$$u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = u_0(x) \geq 0$$

WITH $p > 1$. WE SHOW THAT THE SOLUTION BLOWS UP (DIVERGES TO ∞) AT A FINITE $t = T$, WHEN $u_0(x)$ IS "LARGE ENOUGH".
IN OTHER WORDS THE ODE PART
$$u_t = -u + u^p \qquad u(0) = u_0 \qquad u_0 \text{ CONSTANT}$$
WHICH BLOWS UP IN FINITE TIME CONTROLS THE DYNAMICS.

WE LET
$$f(t) \equiv \int_0^\pi u(x,t) \sin x \, dx.$$

THEN
$$\int_0^\pi u_t \sin x \, dx = \int_0^\pi u_{xx} \sin x \, dx - \int_0^\pi u \sin x \, dx + \int_0^\pi u^p \sin x \, dx.$$

INTEGRATING BY PARTS AND USING THE BOUNDARY CONDITIONS, WE GET
$$(*) \qquad \frac{df}{dt} = -2f + \int_0^\pi u^p \sin x \, dx.$$

NOW RECALL HOLDERS INEQUALITY
$$\int_0^\pi a(x) b(x) \, dx \leq \left(\int_0^\pi [a(x)]^p \, dx \right)^{1/p} \left(\int_0^\pi [b(x)]^q \right)^{1/q} \qquad \frac{1}{p} + \frac{1}{q} = 1$$
$p > 0$, $q > 0$, $a > 0$, $b > 0$ $\forall x$.

CHOOSE $a(x) = u(\sin x)^{1/p}$ $\qquad b = [\sin(x)]^{1-1/p} = (\sin x)^{1/q}$

HENCE
$$\int_0^\pi u \sin x \, dx \leq \left(\int_0^\pi u^p \sin x \right)^{1/p} \cdot \left(\int_0^\pi \sin x \, dx \right)^{1-1/p}$$
$$\left(\int_0^\pi u^p \sin x \right)^{1/p} \geq \frac{f}{2^{1-1/p}} \quad \rightarrow \quad \int_0^\pi u^p \sin x \, dx \geq \frac{f^p}{2^{p-1}}.$$

THEREFORE, $(*)$ GIVES
$$\frac{df}{dt} \geq -2f + \frac{f^p}{2^{p-1}} \qquad f(0) = \int_0^\pi u_0(x) \sin x \, dx$$

NOW
$$\frac{df}{dt} = \frac{f^p}{2^{p-1}} \left[1 - \frac{f^{1-p}}{2^{-p}} \right] = \frac{f^p}{2^{p-1}} \left[1 - 2^p f^{1-p} \right]$$

THUS IF $1 - 2^p f_0^{1-p} > 0$ OR

$$f_0 > 2^{p/p-1}$$

WE GET BLOW-UP IN FINITE TIME. THIS IS BECAUSE THE ODE

$$\frac{dy}{dt} = \frac{y^p}{2^{p-1}}\left(1 - 2^p y^{1-p}\right)$$ FOR $p > 1$ HAS finite time blow-up.

IF $f \to \infty$ AT SOME TIME T, WE HAVE FROM $f(t) = \int_0^\pi u \sin x \, dx \Rightarrow \|u\|_{L_2} = \infty$ WHEN $t = T$.

WE WRITE $$\frac{dy}{y^p/2^{p-1} - 2y} = dt$$

$$I = \int_{y_0}^{y} \frac{d\lambda}{\lambda^p/2^{p-1} - 2\lambda} = \int_0^t ds = t, \quad \text{WITH} \quad y_0^p/2^{p-1} - 2y_0 > 0$$

NOW AS $y \to \infty$, $I \to I_0$. THUS $t \to T$ (where T IS FINITE).

HENCE THE BLOW-UP TIME IS $$T = \int_{y_0}^{\infty} \frac{d\lambda}{\lambda^p/2^{p-1} - 2\lambda}.$$

NOW THAT WE HAVE ESTABLISHED BLOW-UP, WE ASK WHAT DOES THE BLOW-UP PROFILE LOOK LIKE?

WE CONSIDER

$$u_t = \Delta u + f(u) \qquad x \in \mathbb{R}^N$$
$$u(x,0) = u_0(x) \geq C$$

ASSUME $f(u) > 0$. AND $f'(u) > 0$.

THE ODE $$\frac{du}{dt} = f(u) \quad \text{WITH} \quad u(0) = u_0$$

CAN BE SOLVED TO GIVE

$$\int_{u_0}^{u} \frac{1}{f(\lambda)} d\lambda = t.$$

HENCE WE HAVE FINITE TIME BLOW-UP FOR THE ODE IF

$$\int_{u_0}^{\infty} \frac{1}{f(\lambda)} d\lambda \text{ IS FINITE.}$$

THEN THE BLOW-UP TIME IS $$T = \int_{u_0}^{\infty} \frac{1}{f(\lambda)} d\lambda$$

FOR INSTANCE IF $f(u) = u^p$ WE CALCULATE

$$\int_{u_0}^{u} \frac{1}{f(\lambda)}\, d\lambda = \int_{u_0}^{u} \lambda^{-p}\, d\lambda = \frac{1}{1-p}\left[u^{1-p} - u_0^{1-p} \right] = t \qquad p > 1$$

AS $u \to \infty$ WE HAVE $t \to T = \frac{1}{p-1} u_0^{1-p} > 0$.

RATHER THAN LOOKING FOR A FUNCTION THAT BLOW-UP AS $t \to T^-$ (i.e. THE u VARIABLE) WE WILL LOOK FOR A SOLUTION $V \to 0$ AS $t \to T^-$.

WE INTRODUCE

$$V = \int_{u}^{\infty} \frac{d\lambda}{f(\lambda)}$$

FOR THE ODE WE WOULD OBTAIN THAT $V = \int_{u_0}^{\infty} \frac{d\lambda}{f(\lambda)} - \int_{u_0}^{u} \frac{d\lambda}{f(\lambda)} = T - t$

AND SO V IS LINEAR NEAR THE BLOW-UP TIME.

NOW WE CALCULATE

$$V_t = \frac{-1}{f(u)} u_t \qquad \to \qquad u_t = -f(u)\, V_t$$

ALSO $\quad u_x = -f(u)\, V_x \qquad u_{xx} = -f'(u)\, u_x V_x - f(u)\, u_{xx}$

THUS $\quad u_{xx} = +f'(u)\, f(u)\, V_x^2 - f(u)\, u_{xx}$

$$\Delta u = f'(u)\, f(u)\, |\nabla V|^2 - f(u)\, \Delta V$$

WE SUBSTITUTE INTO $\quad u_t = \Delta u + f(u)$

$$\longrightarrow \left\{ \begin{array}{l} V_t = -1 + \Delta V - f'(u(V))\, |\nabla V|^2 \\ V(x,0) = \displaystyle\int_{u_0(x)}^{\infty} \frac{d\lambda}{f(\lambda)} \end{array} \right.$$

WE WILL ASSUME THAT BLOW-UP FIRST OCCURS AT $X = 0,\ t = T$ AND THAT NEAR THERE $V(x,t)$ HAS THE LOCAL EXPANSION

$$V(x,t) = V_0(t) + \frac{r^2}{2!} V_2(t) + \frac{r^4}{4!} V_4(t) + \cdots$$

NOW GIVEN RADIAL SYMMETRY, WE CALCULATE

$$\Delta W = W_{rr} + \frac{N-1}{r} W_r \qquad |\nabla W|^2 = W_r^2$$

NOW IF

$$V = V_0 + \frac{r^2}{2!} V_2 + \frac{r^4}{4!} V_4 + \dots \qquad \text{FOR} \quad r \ll 1$$

WE CALCULATE

$$\Delta V = N V_2 + V_4 \frac{(N+2)}{6} r^2$$

$$|\nabla V|^2 = V_r^2 = r^2 V_2^2 + \dots$$

$$W_t = V_{0_t} + \frac{r^2}{2} V_{2_t} + \dots$$

$$f'(u(V)) \sim f'(u(V_0))$$

NOW SUBSTITUTING WE GET

$$V_{0_t} + \frac{r^2}{2!} V_{2_t} + \dots = N V_2 + \frac{V_4 (N+2) r^2}{6} - 1 - f'(u(V_0)) r^2 V_2^2 + \dots$$

EQUATING POWERS OF r:

$$\left\{ \begin{aligned} V_{0_t} &= -1 + N V_2 \\ V_{2_t} &= -2 f'[u(V_0)] V_2^2 + \frac{(2+N)}{3} V_4 \end{aligned} \right.$$

WE WILL SOLVE THESE EQUATIONS APPROXIMATELY FOR $t \to T^-$. WE WANT $V_0 = 0$ AT $t = T$. WE ASSUME

$$V_2 \ll 1, \qquad V_4 \ll f'(u(V_0)) V_2^2 \qquad \text{AS} \quad t \to T^-.$$

TO A FIRST APPROXIMATION WE HAVE

$$V_0 \sim T - t \qquad \text{FOR} \qquad T - t \ll 1.$$

THE NEXT EQUATION GIVES

$$V_{2_t} = -2 f'[u(V_0)] V_2^2 \qquad \text{WITH} \qquad V_0 = T - t.$$

<u>REMARK</u>

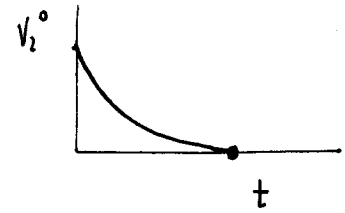
(i) CONSIDER THE ODE $\quad V_{2_t} = -g(t) V_2^2,$

$$V_2(0) = V_2^0 > 0$$

WITH $g(t) > 0$. THEN, $V_2 > 0$ FOR ALL t AND $V_2 \to 0$ IN FINITE TIME.

<u>PROOF</u> $\displaystyle\int_{V_2^0}^{V_2} \frac{ds}{s^2} = -\int_0^t g(\tau)\,dt \equiv -G(t)$ $\qquad G(t) = \int_0^t g(\tau)\,d\tau > 0$

$$\rightarrow \quad -\frac{1}{V_2} + \frac{1}{V_2^0} = -G(t).$$



THUS, $$\frac{1}{V_2} = \frac{1}{V_2^0} + G(t) > 0$$

IF $V_2 \to 0^+$ THEN $t \to \infty$ SINCE $G(\infty)$ IS INFINITE.

THEREFORE, SINCE WE HAVE ASSUMED $f'(u) > 0$ OUR PROBLEM

$$V_{2_t} = -g(t)\,V_2^2$$

WITH $g(t) = 2f'(u(V_0))$ IS OF THIS TYPE. WE WANT $V_0, V_2, V_4, \ldots > 0$ BEFORE THE BLOW-UP TIME WITH $V_0, V_2, V_4 \longrightarrow 0^+$ AS $t \to T^-$.

NOW WE CALCULATE

$$\frac{1}{V_2} = \frac{1}{V_2^0} + G(t)$$

WITH $\displaystyle G(t) = \int_0^t g(\tau)\,d\tau$ AND $g(\tau) = 2f'[u(T-\tau)]$.

NOW WE CALCULATE:

LET $s = T - \tau$. $\quad ds = -d\tau$.

$$G(t) = -\int_T^{T-t} 2f'[u(s)]\,ds = \int_{T-t}^{T} 2f'[u(s)]\,ds$$

NOW $\displaystyle s = \int_u^\infty \frac{1}{f(\lambda)}\,d\lambda$ IMPLIES $\displaystyle 1 = -\frac{1}{f(u)}\frac{du}{ds}$. THUS,

$$G(t) = \int_{s=T-t}^{s=T} 2f'[u(s)]\frac{ds}{du}\,du = -2\int_{s=T-t}^{s=T} \frac{f'[u(s)]}{f[u(s)]}\,du$$

$$G(t) = -2\log\Big[f(u(s))\Big]\Big|_{s=T-t}^{s=T} = 2\log\Big[f(u(T-t))\Big] - 2\log\Big[f(u(T$$

THEREFORE, $$G(t) = 2\log[f(u(T-t))] - 2\log[f(u(T))].$$

RECALL FROM THE TRANSFORMATION $u = u(v)$ DEFINED BY

$$v = \int_u^\infty \frac{1}{f(\lambda)}\, d\lambda$$

THEN AS $v \to 0$ WE MUST HAVE $u \to +\infty$. HENCE $f[u(T-t)] \to +\infty$ AS $t \to T^-$. HENCE FROM

$$\frac{1}{V_2} = \frac{1}{V_2^0} + G(t)$$

WE GET

$$V_2(t) \to \frac{1}{2\log[f(u(T-t))]} \quad \text{AS} \quad t \to T^-.$$

NOW THE NEXT ITERATION FOR V_{0_t} IS

$$V_{0_t} = -1 + NV_2$$

WE WILL IMPOSE $V_0 = 0$ AT $t = T$ (OR $V_0 \to 0$ AS $t \to T^-$).

WE GET

$$V_0 = T - t - N\int_t^T V_2(\tau)\, d\tau$$

NOW AS $t \to T^-$ WE HAVE $V_2 \sim \dfrac{1}{2\log[f(u(T-t))]}$

SO THAT

$$V_0 \sim T - t - \frac{N}{2}\int_t^T \frac{d\tau}{\log[f(u(T-\tau))]}$$

AND

$$\frac{r^2 V_2}{2!} \sim \frac{r^2}{4\log[f(u(T-t))]}$$

THE TWO TERM EXPANSION READS (AS $t \to T^-$ AND $r \to 0$)

$$V \sim T - t - \frac{N}{2}\int_t^T \frac{d\tau}{\log[f(u(T-\tau))]} + \frac{r^2}{4\log[f(u(T-t))]} + \cdots$$

NOW FOR $T - t \ll 1$ WE CAN APPROXIMATE THE INTEGRAND AS $\int_a^b f(x)\,dx \cong (b-a)f(a) + \cdots$ AS $b - a \to 0$.

THIS GIVES,

$$V \sim T - t - \frac{N}{2}\frac{(T-t)}{\log[f(u(T-t))]} + \frac{r^2}{4\log[f(u(T-t))]}$$

BUT

$$V = \int_u^\infty \frac{1}{f(\lambda)}\,d\lambda$$

AND SO WE GET THE FINAL RESULT FOR THE BLOW-UP OF u

$$(*) \quad \int_u^\infty \frac{1}{f(\lambda)}\,d\lambda \sim T - t - \frac{N}{2}\frac{(T-t)}{\log[f(u(T-t))]} + \frac{r^2}{4\log[f(u(T-t))]}$$

<u>EXAMPLE 1</u> SUPPOSE $f(\lambda) = e^\lambda$. THEN,

$$V = \int_u^\infty \frac{1}{f(\lambda)}\,d\lambda = -e^{-\lambda}\Big|_u^\infty = e^{-u}.$$

THUS WE HAVE $u = -\log V > 0$ AND SO $f(u) = e^{-\log V} = \frac{1}{|V|}$.

THUS $f[u(V)] = \frac{1}{|V|}$ AND $f[u(T-t)] = \frac{1}{|T-t|}$ AS $t \to T^-$.

HENCE $\log[f(u(T-t))] = -\log|T-t| = |\log(T-t)|$ PROVIDED $t - T < 0$.

SUBSTITUTING INTO $(*)$ WE GET

$$e^{-u} \sim T - t - \frac{N}{2}\frac{(T-t)}{|\log(T-t)|} + \frac{r^2}{4|\log(T-t)|}$$

$$e^{-u} \sim (T-t)\left[1 - \frac{N}{2|\log(T-t)|} + \frac{r^2}{4(T-t)|\log(T-t)|}\right]$$

WRITING THIS AS

$$u \sim -\log\left[(T-t)(1+y)\right] \quad \text{WITH } y \ll 1 \text{ WHEN } t \to T^-$$

AND $r^2 \ll (T-t)|\log(T-t)|$

WE GET WITH $\log(1+y) \sim y + \dots$ AS $y \to 0$ THAT

$$u \sim -\log(T-t) - \frac{N}{2|\log(T-t)|} + \frac{r^2}{4(T-t)|\log(T-t)|}$$ is the blow-up profile.

<u>EXAMPLE 2</u> SUPPOSE $f(u) = u^p$. THEN

$$v = \int_u^\infty \Lambda^{-p}\, d\Lambda = \frac{1}{p-1} u^{1-p}$$

OR $$u^{1-p} = (p-1)\, v \longrightarrow u = (p-1)^{1/1-p}\, v^{1/1-p}$$

NOW $$u^p = a_p\, v^{p/1-p} \qquad a_p = (p-1)^{p/1-p}$$

THUS AS $v \to 0$ WE GET

$$\log f(u) = p \log u \sim p\left[\log\left(v^{p/1-p}\right) + \log a_p^{1/p}\right]$$

TO A FIRST APPROXIMATION,

$$\log f(u(v)) \sim \frac{p}{1-p} \log v \qquad \text{AS} \quad v \to 0.$$

WE CAN WRITE THIS AS

$$\log f(u(v)) \sim \frac{p}{p-1} |\log v| \qquad \text{SINCE} \quad p > 1 \quad \text{AND} \quad v \to 0.$$

SUBSTITUTING INTO (*) ON page 18 we get

$$\frac{1}{p-1} u^{1-p} \sim T-t - \frac{N(p-1)(T-t)}{2p|\log(T-t)|} + \frac{r^2(p-1)}{4p|\log(T-t)|}$$

$$u^{1-p} \sim (p-1)[T-t]\left(1 - \frac{N(p-1)}{2p|\log(T-t)|} + \frac{r^2(p-1)}{4p(T-t)|\log(T-t)|}\right)$$

$$(T-t)^{1/p-1} u \sim K\left(1 - \frac{N(p-1)}{2p|\log(T-t)|} + \frac{r^2(p-1)}{4p(T-t)|\log(T-t)|}\right)^{1/1-p} \quad \text{where} \quad K = (p-1)^{1/1-p}$$

USING TAYLOR SERIES $(1+y)^{1/1-p} \sim y/_{1-p}$ FOR $p \ll 1$ WE GET

$$(T-t)^{1/p-1} u \sim K\left(1 + \frac{N}{2p|\log(T-t)|} - \frac{r^2}{4p(T-t)|\log(T-t)|}\right)$$

BLOW-UP IN ONE-DIMENSION

WE NOW CONSIDER BLOW-UP IN DIMENSION 1 FOR

$$(1) \qquad u_t = u_{xx} + u^p \qquad -\infty < x < \infty, \quad p > 1$$

LET a, T BE THE BLOW-UP POINT AND BLOW-UP TIME, RESPECTIVELY. THEN

<u>THEOREM</u> (FILIPAS, KOHN, CPAM 1992).

$$(2) \qquad (T-t)^{1/p-1} u(x,t) \sim K + \frac{K}{2p|\log(T-t)|}\left(1 - \frac{|x-a|^2}{2(T-t)}\right).$$

WHERE $\quad (3) \qquad K = (p-1)^{-1/p-1}.$

THIS IS PRECISELY THE RESULT WE OBTAINED ON p. 19.

WE NOW SKETCH HOW THIS RESULT IS OBTAINED. WE INTRODUCE NEW VARIABLES

$$(4) \quad \begin{cases} u(x,t) = (T-t)^{-1/p-1} w(y,s) \\ y = (x-a)/\sqrt{T-t} \quad \text{AND} \quad s = -\log(T-t). \end{cases}$$

WITH

CHANGING VARIABLES (1) BECOMES

$$(5) \quad \begin{cases} w_s - \dfrac{1}{\rho}\nabla\cdot(\rho\nabla w) + \dfrac{1}{p-1} w = w^p \\ \rho = e^{-y^2/4} \end{cases} \qquad \nabla \equiv \frac{d}{dy}$$

WHERE

$s \to$ time derivative

NOTICE $\quad s \to +\infty \Rightarrow t \to T^-$.

<u>REMARK</u> (i) CONSIDER $\quad u_t = u_{xx} + u^p$

IF WE SET $\quad x = \lambda\tilde{x}, \; t = \lambda^2\tilde{t}, \; u = \lambda^q\tilde{u}$

THEN WE OBTAIN $\quad \tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}\tilde{x}} + \tilde{u}^p \quad$ WHEN $\quad q = -2/p-1.$

THIS SUGGESTS THAT IF WE SUBSTITUTE

$$u = (T-t)^{-1/p-1} F\left(\tilde{x}/\sqrt{T-t}\right)$$

WE WILL GET AN ODE FOR $F(\Lambda)$. NOTICE THAT (4) HAS A SLIGHTLY DIFFERENT FORM.

NOW FOR $|y|$ BOUNDED, WE IMPOSE THE STEADY-STATE CONDITION

$$W(y,s) \to K \qquad \text{AS} \quad s \to \infty,$$

WHERE $\qquad K^p = \dfrac{K}{p-1} \qquad$ OR $\qquad K = (p-1)^{-1/p-1}.$

NOW WE LINEARIZE BY LETTING $W = K + V$ WHERE $V \ll 1$.

WE DEFINE

$$g(w) = w^p - \frac{1}{p-1} w \qquad\qquad g'(K) = 1$$

SO THAT

$$g(K+V) = g(K) + g'(K)V + \frac{g''(K)}{2} V^2 + \dots \qquad g''(K) = p/K$$

THUS FROM (5) WE OBTAIN

$$(6) \qquad V_s - \frac{1}{\alpha} \nabla \cdot [\alpha \nabla V] - V = \frac{p}{2K} V^2 + O(V^3)$$

WE ARE INTERESTED IN THE BEHAVIOR OF (6) AS $s \to +\infty$ ON BOUNDED SETS IN $|y|$. IN PARTICULAR, WE WILL SHOW THAT

$$(7) \qquad V(y,s) \sim \frac{K}{2ps}\left(1 - y^2/2\right) \qquad \text{IN 1-D}$$

WITH A SIMILAR RESULT IN 2-D BEING

$$V(y,s) \sim \frac{K}{ps}\left(1 - \frac{|y|^2}{4}\right) \qquad \text{IN 2-D.}$$

AFTER WE SHOW (7) THEN WE USE

$$W = K + V + \dots$$

TO GET

$$W = K + \frac{K}{2ps}\left(1 - y^2/2\right) \qquad \text{AS} \quad s \to \infty.$$

RECALLING THE DEFINITION OF y AND s FROM P. 20, WE OBTAIN

$$u(x,t) \sim (T-t)^{-1/p-1}\left[K + \frac{K}{+2p|\log(T-t)|}\left(1 - \frac{(x-a)^2}{2(T-t)}\right)\right] \qquad t \to T^-$$

WHICH IS THE MAIN RESULT. (WE ALSO REQUIRE $(x-a)^2/(T-t)$ BOUNDED).

<u>REMARK</u> (i) SIMILARLY WE GET THE CORRECT RESULT IN 2-D.

WE DEFINE $\mathcal{L} v \equiv \frac{1}{\mathcal{R}} \nabla \cdot (\mathcal{R} \nabla v) + v$

SO THAT THE EQUATION READS

$$v_s - \mathcal{L} v = \frac{P}{2K} v^2$$

WE HAVE SOME PROPERTIES OF $\mathcal{L}$:

<u>LEMMA</u> CONSIDER THE EIGENVALUE PROBLEM

$$\mathcal{L} \phi = \lambda \phi$$

WHICH WE WRITE AS

$$(*) \quad \frac{d}{dy}\left(\mathcal{R} \frac{d\phi}{dy}\right) + \mathcal{R} \phi = \lambda \mathcal{R} \phi \qquad \mathcal{R} = e^{-y^2/4}$$

THIS EIGENVALUE PROBLEM IS SELF-ADJOINT WITH RESPECT TO THE INNER PRODUCT $(f, g)_{\mathcal{R}} \equiv \int_{-\infty}^{\infty} f g \mathcal{R} \, dy$. THE EIGENVALUES HAVE THE PROPERTY:

(1) THERE ARE TWO POSITIVE EIGENVALUES e_1^+, e_2^+ GIVEN BY

$$e_1^+ = 1, \quad e_2^+ = \frac{1}{2} \qquad \phi_1^+ = c_0^+ \qquad \phi_2^+ = c_1^+ y$$

(2) THERE IS ONE ZERO EIGENVALUE GIVEN BY

$$e_1^0 = 0 \qquad \phi_1^0 = c_0^0 \left(\frac{1}{2} y^2 - 1\right)$$

(3) THE NEGATIVE EIGENVALUES ARE

$$e_j^- = 1 - \frac{(j+2)}{2} \qquad j = 1, 2, 3, \ldots$$

WITH EIGENFUNCTIONS $\phi_j^-(y) = c_j^- H_{j+2}(y/2)$

where $H_K(x) = (-1)^K e^{x^2} \frac{d^K}{dx^K}(e^{-x^2})$ $K = 0, 1, 2, \ldots$ ARE HERMITE POLYNOMIALS.

PROOF IN ONE-SPACE DIMENSION WE HAVE FROM (*)

$$\phi_{yy} - \frac{y}{2}\phi_y + \phi = \lambda\phi$$

NOW let $x = y/2$ AND $\psi(x) = \phi(y)$. THIS GIVES

$$\psi_{xx} - 2x\psi_x = \mu\psi \qquad \mu = 4(\lambda - 1)$$

WHICH IS THE WELL-KNOWN HERMITE EQUATION WITH EIGENVALUES

$$\mu_k = -2k \qquad k = 0, 1, 2, \ldots \qquad \psi = H_k(x).$$

HENCE

$$\lambda_k = -\frac{k}{2} + 1, \quad k = 0, 1, 2, \ldots$$

$$\phi_k(y) = c_k H_k(y/2).$$

DECOMPOSING THE EIGENSPACE INTO POSITIVE, ZERO, NEGATIVE EIGENVALUES GIVES THE RESULT. ∎

WE USE THE NORMALIZATION CONDITION $(\phi_k, \phi_k)_{\mathcal{R}} \equiv 1$ TO CALCULATE THE CONSTANT C_k. IN THE NOTATION OF p. 22 WE GET

$$c_0^+ = \frac{1}{\sqrt{2}}\pi^{-1/4}, \quad c_1^+ = \frac{1}{2}\pi^{-1/4}, \quad c_1^0 = \frac{1}{2}\pi^{-1/4}.$$

WE WILL ALSO USE THE IDENTITY

$$\phi_k'(y) = \left(\frac{k}{2}\right)^{1/2}\phi_{k-1}(y) \qquad k = 1, 2, 3, \ldots$$

WHICH IS A CONSEQUENCE OF THE WELL-KNOWN RECURSION FORMULA

$$H_k'(y) = 2k\,H_{k-1}(y),$$

FOR HERMITE POLYNOMIALS.

REMARK (i) IN TWO-DIMENSIONS THE EIGENVALUES ARE THE SAME BUT THE DIMENSION OF THE UNSTABLE EIGENSPACE IS 3 AND THE DIMENSION OF THE NULLSPACE IS 3.

(ii) THE ZERO EIGENSPACE IS SPANNED BY

$$c_0 c_2\left(\frac{1}{2}y_1^2 - 1\right), \quad c_0 c_2\left(\frac{1}{2}y_2^2 - 1\right), \quad c_1^2\, y_1 y_2.$$

NOW WE RETURN TO

$$V_s - \mathcal{L} V = \frac{P}{2K} V^2 .$$

WE DECOMPOSE V IN AN EIGENFUNCTION EXPANSION

$$V = \left(B_1(s)\, \phi_1^+(y) + B_2(s)\, \phi_2^+(y) \right) + \alpha_1(s)\, \phi_1^0(y) + \left(\gamma_1(s)\, \phi_1^-(y) + \gamma_2(s)\, \phi_2^-(y) + \right.$$

SINCE WE WANT $V \longrightarrow 0$ AS $s \longrightarrow \infty$ WE MUST ELIMINATE THE GROWING MODES $B_1(s)$ AND $B_2(s)$.

THEREFORE, WE HAVE

$$V = \alpha_1(s)\, \phi_1^0(y) + \left(\gamma_1(s)\, \phi_1^-(y) + \gamma_2(s)\, \phi_2^-(y) + \dots \right)$$

WE CALCULATE FROM OUR EIGENVALUE PROBLEM

$$V_s - \mathcal{L} V = \dot{\alpha}_1 \phi_1^0 + \left[(\dot{\gamma}_1 - e_1^- \gamma_1)\, \phi_1^- + (\dot{\gamma}_2 - e_2^- \gamma_2)\, \phi_2^- + \dots \right]$$

WITH $e_j^- < 0$ (negative eigenvalues). NOW WE CALCULATE $\frac{P}{2K} V^2$.

$$\frac{P}{2K} V^2 = \frac{P \alpha_1^2}{2K} (\phi_1^0)^2 + \text{other terms that decay as } s \to \infty$$

(arising from negative spectra)

THUS $$V_s - \mathcal{L} V = \frac{P}{2K} V^2$$

GIVES $$\dot{\alpha}_1 \phi_1^0 + \sum_{j=1}^{\infty} (\dot{\gamma}_j - e_j^- \gamma_j)\, \phi_j^- = \frac{P \alpha_1^2}{2K} (\phi_1^0)^2 + \text{other terms that decay.}$$

SINCE eigenfunctions are orthogonal $\int_{-\infty}^{\infty} \mathcal{L} \phi_1^0 \phi_j^- \, dy = 0$ WE GET

$$\dot{\alpha}_1 \int_{-\infty}^{\infty} (\phi_1^0)^2 \mathcal{L} \, dy = \frac{P}{2K} \left(\int_{-\infty}^{\infty} \mathcal{L} (\phi_1^0)^3 \, dy \right) \alpha_1^2 .$$

NOW SINCE $\int_{-\infty}^{\infty} (\phi_1^0)^2 \mathcal{L}\, dy = 1$ BY THE NORMALIZATION CONDITION,

WE GET

$$\dot{\alpha}_1 = \frac{p}{2K} \alpha_1^2 \int_{-\infty}^{\infty} \mathcal{L} (c_1^0)^3 \left[\frac{1}{2} y^2 - 1 \right]^3 dy \qquad \text{WITH} \quad \mathcal{L} = e^{-y^2/4}, \quad c_1^0 = \frac{1}{2\pi^{1/4}}.$$

WE CAN CALCULATE THIS INTEGRAL TO GET

$$\int_{-\infty}^{\infty} \mathcal{L} (c_1^0)^3 \left(\frac{1}{2} y^2 - 1 \right)^3 dy = 4 c_1^0.$$

HENCE,

$$\dot{\alpha}_1 = \frac{2 p c_1^0}{K} \alpha_1^2$$

WE ARE INTERESTED IN THE SOLUTION FOR Large s.

LET $\alpha_1 \sim A/s$ FOR large s. we get

$$-\frac{A}{s^2} = \frac{2 p c_1^0}{K} \frac{A^2}{s^2} \longrightarrow A = -\frac{K}{2 p c_1^0}$$

THUS $\alpha_1 \sim -\dfrac{K}{2 p c_1^0 s}$.

NEXT WE LARGE s,

$$V \sim \alpha_1(s)\, \phi_1^0(y) = -\frac{K}{2 p c_1^0 s} c_1^0 \left(\frac{1}{2} y^2 - 1 \right)$$

THEREFORE,

$$V(y,s) \sim \frac{K}{2ps} \left(1 - \frac{1}{2} y^2 \right) \qquad \text{AS} \quad s \to \infty.$$

HENCE FROM page 21, we have

$$W = K + \frac{K}{2ps} \left(1 - y^2/2 \right) + \cdots \qquad \text{AS} \quad s \to \infty$$

which gives the blow-up profile for u on page 21.