EXAMPLE (DRIFT-DIFFUSION SIGNALLING PROBLEM)

\[ u_t + c u_x = \varepsilon u_{xx}, \quad 0 < x < \infty, \quad t > 0 \]
\[ u(x,0) = 0, \quad u(0,t) = f(t) \]

Here, \( c > 0 \) is constant.

For \( \varepsilon = 0 \) the first order PDE \( u_t + cu_x = 0 \) can be solved by method of characteristics. The solution is

\[ u_0 = \begin{cases} 
  f(t-x/c), & 0 < x < ct \\
  0, & x > ct 
\end{cases} \]

However, if \( f(0) \neq 0 \) then \( u_0 \) has a jump discontinuity across \( t = x/c \). This must be smoothed out by the diffusive term.

Near \( x = ct \), we let \( y = \varepsilon^{-p}(x-ct) \) and \( v(y,t) = u(ct+\varepsilon^p y, t) \)

for some \( p > 0 \) to be found. We substitute into \( u_t + cu_x = \varepsilon u_{xx} \)

to obtain \( u_t = v_y (-c \varepsilon^p) + v_t, \quad u_x = \varepsilon^{-p} v_y, \quad u_{xx} = \varepsilon^{-1-2p} v_{yy} \).

This gives

\[-c \varepsilon^p v_y + v_t + c \varepsilon^{-1-2p} [\varepsilon^p v_y] = \varepsilon^{1-2p} v_{yy} \]

Choosing \( p = \frac{1}{2} \) we get

\[ v_{yy} = v_t, \quad t > 0, \quad -\infty < y < \infty \]

\[ v \to \begin{cases} 
  f(0^+), & y \to -\infty \\
  0, & y \to +\infty 
\end{cases} \] (1)

As the matching condition (\( y \to -\infty \) implies \( x-ct \to 0^- \), and \( y \to +\infty \) implies \( x-ct \to 0^+ \)). We give the initial condition

\[ v(y,0) = \begin{cases} 
  f(0^+), & y < 0 \\
  0, & y > 0 
\end{cases} \] (2)

consistent with (1) and (2).

To solve (1) and (2) we take a Laplace transform in time,

\[ \tilde{v}(y,s) = \int_0^{\infty} e^{-st} v(y,t) \, dt. \]
we obtain
\[ S \bar{V} - \bar{V}(y,0) = \bar{V}_{yy} \]
so that
\[ \bar{V}_{yy} - S \bar{V} = \begin{cases} -\frac{f(0^+)}{2} & \text{if } y < 0 \\ 0 & \text{if } y > 0 \end{cases} \]
\[ \bar{V}(y,s) = \begin{cases} \frac{f(0^+)}{s} + Ae^{\sqrt{s}y} & \text{if } y < 0 \\ Be^{-\sqrt{s}y} & \text{if } y > 0 \end{cases} \]

Imposing \( \bar{V} \) bounded at \( |y| \to \infty \) for \( \text{Re} (\sqrt{s}) > 0 \) gives
\[ \bar{V}(y,s) = \begin{cases} \frac{f(0^+)}{s} + Ae^{\sqrt{s}y} & \text{if } y < 0 \\ Be^{-\sqrt{s}y} & \text{if } y > 0 \end{cases} \]

Now imposing \( \bar{V}, \bar{V} \) continuous across \( y = 0 \) gives \( A = -B \) and \( B = \frac{1}{2} \frac{f(0^+)}{2s} \)
we have
\[ \bar{V}(y,s) = \begin{cases} \frac{f(0^+)}{s} + \frac{f(0^+)}{2s} e^{\sqrt{s}y} & \text{if } y < 0 \\ \frac{f(0^+)}{2s} e^{-\sqrt{s}y} & \text{if } y > 0 \end{cases} \]

Recalling \( f[1] = \frac{1}{s} \) and \( f^{-1} \left[ \frac{e^{-a\sqrt{s}}}{s} \right] = \text{erfc} \left( \frac{a}{2\sqrt{t}} \right) \) for \( a > 0 \)
where \( \text{erfc} (z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt \) (with \( \text{erfc} (0) = 1 \), \( \text{erfc} (\infty) = 0 \))
our solution is
\[ \bar{V}(y,t) = \begin{cases} \frac{f(0^+)}{s} + \frac{f(0^+)}{2s} \text{erfc} \left( -\frac{y}{2\sqrt{t}} \right) & \text{if } y < 0 \\ \frac{f(0^+)}{2s} \text{erfc} \left( \frac{y}{2\sqrt{t}} \right) & \text{if } y > 0 \end{cases} \]

Now since \( y = \epsilon^{-\frac{1}{2}} (x-ct) \) we conclude that for \( x-ct = O(\epsilon^{\frac{1}{4}}) \) that the diffusive smoothing across \( x = ct \) is given by
\[ u \sim \begin{cases} \frac{f(0^+)}{s} + \frac{f(0^+)}{2s} \text{erfc} \left( -\frac{(x-ct)}{2\sqrt{\epsilon t}} \right) & \text{if } x-ct = O(\epsilon^{\frac{1}{4}}) < 0 \\ \frac{f(0^+)}{2s} \text{erfc} \left( \frac{x-ct}{2\sqrt{\epsilon t}} \right) & \text{if } x-ct = O(\epsilon^{\frac{1}{4}}) > 0 \end{cases} \]

Notice that the effective width of this layer is \( O(\sqrt{\epsilon t}) \) and so \( \alpha \uparrow A_j \uparrow \)
SIMILARITY SOLUTIONS

We consider \( U_t = \Delta U \). Let \( X = \sqrt{t} \), \( T = \frac{t}{L^2} \) and label \( \tilde{U}(x, t) := U[\sqrt{t}, \frac{t}{L^2}] \).

Then from the PDE:
\[
\frac{1}{\sqrt{t}} \tilde{U}_t = \Delta \tilde{U} \quad \text{so PDE is invariant if} \quad T = \frac{L^2}{t}, \quad \text{or} \quad L = \sqrt{t}.
\]

We have \( \frac{\sqrt{t}}{\sqrt{t}} = \frac{X}{\sqrt{t}} \) independent of scaling \( L \).

Now for \( U_t = \Delta U \) thus suggest we look for a solution of the form \( U(x, t) = t^p g(X/\sqrt{t}) \) with \( \Lambda = X/\sqrt{t} \).

We calculate
\[
U_t = p t^{p-1} g + t^p \frac{\partial}{\partial t} \left( X t^{-1/2} \right) g' = p t^{p-1} g + t^p \left( -\frac{1}{2} x t^{-1/2} \right) g'.
\]
\[
\Delta U = t^p \frac{\partial^2}{\partial t^2} \left( X t^{-1/2} \right) g' = t^p \frac{1}{2} \Lambda g''.
\]

We substitute into \( U_t = \Delta U \) to obtain
\[
p t^{p-1} g + \frac{1}{2} \Lambda g'' = t^p \frac{1}{2} \Lambda g' \cdot
\]

As such we obtain for \( \Lambda = X/\sqrt{t} \) that
\[
g'' + \frac{1}{2} \Lambda g' = p g \cdot
\]

where \( p \) is a parameter.

EXAMPLE

Solve \( U_t = \Delta U \) on \( t > 0, \ 0 < x < \infty \) with \( U(x, 0) = 0, \ U(0, t) = 1 \).

Solution: Take \( p = 0 \) and put \( U = g(\Lambda) \) with \( \Lambda = X/\sqrt{t} \). Now (X) gives
\[
g'' + \frac{1}{2} \Lambda g' = 0 \quad \text{on} \quad \Lambda > 0 \quad \text{with} \quad g(0) = 1, \quad [ \text{so that} \ U(0, t) = 1 ]
\]
\[
\quad \quad \quad \quad \quad \text{and} \quad g(\infty) = 0 \quad [ \text{so that} \ U(x, \infty) = 1 ].
\]

Then \( g = \text{span} \{ 1, \int_0^\Lambda e^{-s^2/4} \, ds \} \). We conclude by inspection that
\[
g(\Lambda) = \int_0^\Lambda e^{-s^2/4} \, ds / \int_0^\infty e^{-s^2/4} \, ds.
\]

We calculate
\[
\int_0^\Lambda e^{-s^2/4} \, ds = 2 \int_0^{\Lambda/2} e^{-y^2} \, dy = \sqrt{\pi} \quad \text{since} \quad \int_0^\infty e^{-y^2} \, dy = \sqrt{\pi}/2.
\]

Hence
\[
g(\Lambda) = \frac{1}{\sqrt{\pi}} \int_0^{\Lambda/2} e^{-y^2} \, dy = \text{erfc} \left( \frac{\Lambda}{2} \right).
\]

We conclude that
\[
U(x, t) = \text{erfc} \left( \frac{X}{\sqrt{t}} \right) \quad \text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-y^2} \, dy.
\]
Example 2

Consider \( u_t = u_{xx}, \ -\infty < x < \infty \)

with \( u(x, 0) = \delta(x) \).

Solution

We must have \( \int_{-\infty}^{\infty} u(x, t) \, dx = \int_{-\infty}^{\infty} u(x, 0) \, dx = 1 \).

We put \( u = t^p g(\sqrt{t}) \) and integrate:

\[
\int_{-\infty}^{\infty} t^p g(\sqrt{t}) \, dx = \int_{-\infty}^{\infty} t^p g(\sqrt{t}) \, dt = 1.
\]

Thus take \( p = -\frac{1}{2} \) and \( \int_{-\infty}^{0} g(\lambda) \, d\lambda = 1 \). Setting \( p = -\frac{1}{2} \) in (1) gives

\[
g'' + \frac{1}{2} (1 g + g) = 0 \rightarrow g'' + \frac{1}{2} (1 g) = 0. \text{ We want } g \rightarrow 0 \text{ at } \lambda \rightarrow \pm \infty.
\]

and \( \int_{-\infty}^{\infty} g(\lambda) \, d\lambda = 1 \). Thus \( g' + \frac{1}{2} \lambda g = 0 \) and solving \( g = A e^{-\lambda^2/4} \).

Now find \( A \) by

\[
\int_{-\infty}^{\infty} A e^{-\lambda^2/4} \, d\lambda = 1 \rightarrow 2A \int_{-\infty}^{\infty} e^{-y^2/4} \, dy = 1 \rightarrow 2A \sqrt{\pi} = 1 \rightarrow A = \frac{1}{2\sqrt{\pi}}.
\]

We conclude that \( g(\lambda) = \frac{1}{2\sqrt{\pi}} e^{-\lambda^2/4} \) so that \( u(x, t) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} \).
PIPE FLOW

\[ \begin{align*}
& \text{in } 0 < x < \omega : \\
& \varepsilon \left( U' + \frac{1}{r} U \right)' + U' + U'' = (1 - \varepsilon^2) U \\
& U(x, 1) = \begin{cases} 
0, & x < 0 \\
1, & x > 0 
\end{cases} \quad \text{as } x \to \infty
\end{align*} \]

SIDE VIEW

STATIONARY OF

\[ U_t' + (1 - \varepsilon^2) U_x = \varepsilon A U \to U = \int_0^x (x - (1 - \varepsilon^2) t) \to \text{RIGHT} \]

UPSTREAM \( U_0 \equiv 0 \).

NOW PUT IN A BL NEAR \( \varepsilon = 1 \) FOR \( x > 0 \).

DEFINING \( \varepsilon = \frac{1 - \varepsilon}{\varepsilon} \), \( V(x, r) = U[\sqrt{x}, 1 - \varepsilon^2 r] \).

\[ 1 - \varepsilon^2 = 1 - (1 - \varepsilon^2) \varepsilon \approx 2 \varepsilon^3 \rho. \]

\[ \varepsilon^{1-2\beta} V_{r r} - \frac{\varepsilon^{1-\beta}}{1 - \varepsilon^2 \rho} V_r + \varepsilon V_{x x} = 2 \varepsilon^2 \rho V_x + \cdots \]

MUST BALANCE UNDERLINED TERMS: \( \beta = 1 - 2 \beta = \beta \to \beta = \frac{1}{3} \).

THEN TO LEADING ORDER

\[ \begin{cases} 
V_{r r} = 2 \rho V_{r x}, & x > 0, \quad 0 < \rho < \infty \\
V_r = 0 \text{ on } \rho = 0, & V_x \to 0 \text{ at } \rho \to \infty \\
V_{x x} = 0 \text{ at } x = 0 \text{ (intake)} 
\end{cases} \]

NOTICE \( x \) IS "TIME-LIKE" VARIABLE AND \( r \) IS WELL-POSED IF WE INTEGRATE THROUGH \( x \) IN CREASING VALUES. IT HAS VARIABLE COEFFICIENTS Owing TO "\( \rho \)" COEFFICIENT.

WE THEN LOOK FOR A SOLUTION IN FORM \( V_0 = f(\rho / x^3) \).

IF WE SUBSTITUTE WE OBTAIN

\[ f''(z) + 2z^2 \frac{\partial}{\partial z} \left[ z^3 \frac{\partial f}{\partial z} \right] = 0 \]

WHERE \( z = \rho / x^3 \).
Choose \( \beta = \frac{1}{3} \):

\[
\begin{align*}
z = 0 & \rightarrow \rho = 0 \rightarrow f(0) = 1 \\
x = 0 & \rightarrow z = \phi \rightarrow f(\phi) = 0.
\end{align*}
\]

We must solve

\[
f''(z) + \frac{2}{3} z^2 f'(z) = 0, \quad \text{with} \quad f(0) = 1 \quad \text{and} \quad f(\phi) = 0.
\]

\[
f(z) = \int_0^\phi e^{-2\lambda^3/9} d\lambda = \int_0^\phi e^{-2\lambda^3/9} d\lambda.
\]

Now

\[
\int_0^\phi e^{-2\lambda^3/9} d\lambda = \left( \frac{3}{2} \right) \left( \frac{2}{9} \right)^{2/3} \Gamma \left( \frac{1}{3} \right).
\]

Hence in the boundary layer we have \( z = \frac{\rho}{x^{1/3}} = \frac{(1 - \Gamma)}{(\varepsilon x)^{1/3}} \)

\[
\mathrm{U}(x, \Gamma) \approx \int_0^\phi e^{-2\lambda^3/9} d\lambda / \int_0^\phi e^{-2\lambda^3/9} d\lambda.
\]

We conclude that level curves of \( \mathrm{U} \) are when

\[
\frac{1 - \Gamma}{(\varepsilon x)^{1/3}} = \text{constant}
\]

i.e.

\[
\Gamma = 1 - \varepsilon \left( \frac{x}{\varepsilon} \right)^{1/3} \quad \text{for different } \varepsilon > 0.
\]

\[\]

\[
\text{Boundary layer theory is only valid when } \varepsilon \ll 1
\]

i.e. \( X \ll O(1/\varepsilon) \).
EXAMPLE (DIFFUSION FOR SHORT TIME)

Consider, in a disk, the time-dependent diffusion equation \( u = u(r, t) \) satisfying

\[
\begin{align*}
\frac{\partial u}{\partial t} & = D \left( \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial r} \right) & \text{in} & \quad 0 < r < a, \quad t > 0 \\
u & = 1 \quad \text{on} \quad r = a; \quad u \text{ bounded as} \quad r \to 0; \quad u(r, 0) = 0 \quad \text{at IC.}
\end{align*}
\]

Find an approximation for the solution for short time:

**Solution**

Use Laplace transform in time:

\[
\bar{u}(r, s) = \int_0^\infty e^{-st} u(r, t) \, dt.
\]

We obtain

\[
D \left( \frac{\partial \bar{u}}{\partial r} + \frac{1}{r} \frac{\partial \bar{u}}{\partial r} \right) = s \bar{u} - \bar{u}(r, 0)
\]

Thus

\[
\begin{align*}
\frac{\partial \bar{u}}{\partial r} + \frac{1}{r} \frac{\partial \bar{u}}{\partial r} - s \bar{u} & = 0 \\
\bar{u}(a, s) & = \frac{1}{s}; \quad \bar{u} \text{ bounded as} \quad r \to 0.
\end{align*}
\]

We recall that large \( s \) in L.T. \( \iff \) small \( t \).

Define \( \varepsilon = \frac{D}{s} \ll 1 \). \( \varepsilon \left( \frac{\partial \bar{u}}{\partial r} + \frac{1}{r} \frac{\partial \bar{u}}{\partial r} \right) = \bar{u} \)

\[
\bar{u}(a, s) = \frac{1}{s}.
\]

For \( \varepsilon \to 0^+ \), the outer solution is \( \bar{u} \equiv 0 \).

Putting a bl near \( r = a \) with \( \rho = (a-r)/\varepsilon^{1/2} \) gives us a first approximation \( \bar{u}_{pp} + 0(\varepsilon^{1/2}) - \bar{u} = 0 \); \( \bar{u} \to 0 \) as \( \rho \to \infty \), \( \frac{\bar{u}}{s} \) at \( \rho = 0 \).

The solution is \( \bar{u} = \frac{1}{s} e^{-\rho} \).

Thus

\[
\bar{u}(r, s) \sim \frac{1}{s} e^{-\frac{(a-r)}{\varepsilon^{1/2}}} \approx \frac{1}{\sqrt{D}} e^{-\frac{(a-r)}{\sqrt{D}}} \quad \varepsilon \ll 1
\]

Recall that

\[
\int e^{-\frac{A}{s^{1/2}}} \, ds = \text{erfc} \left( \frac{A}{2\sqrt{s}} \right) \quad \text{for} \quad A > 0.
\]

Thus

\[
u(r, t) \approx \text{erfc} \left[ \frac{a-r}{2\sqrt{D}t} \right] \quad \text{for} \quad t \ll 1.
\]

This is diffusion smoothing from the boundary of the disk.
BOUNDARY LAYER ANALYSIS

We will first consider an elliptic problem of the form

\[ \varepsilon \left( A_{11} u_{xx} + 2A_{12} u_{xy} + A_{22} u_{yy} \right) = u_y \quad \text{in } \Omega \subset \mathbb{R}^2 \quad \left( A_{11}, A_{22}, A_{12} \text{ constants} \right) \]

and \( \varepsilon \to 0^+ \).

We assume ellipticity so that \( A_{11} > 0, A_{22} > 0 \) and \( A_{12}^2 - A_{11} A_{22} < 0 \).

We take a domain \( \Omega \) and BC as shown.

The PDE (1) is steady-state for parabolic problem \( u_t + u_y = \varepsilon \left( A_{11} u_{xx} + 2A_{12} u_{xy} + A_{22} u_{yy} \right) \).

Thus for \( \varepsilon = 0 \), \( u = f(y-z) \). Thus flow is from bottom to top.

As such, since outer limit is \( u_y = 0 \), i.e., \( u = \text{constant} \) on lines \( x = \text{constant} \) we take outer limit to satisfy the BC on lower boundary.

Thus the leading order outer solution is \( u = \overline{u}_0 = \overline{u}_B(x) \).

Since \( \overline{u}_B(x) \) is assumed continuous, we have \( u \sim \overline{u}_B(x) \) except in a thin region near upper boundary.

Now near upper boundary we let

\[ z = f(y_B(x) - y) \quad v(x, z) = u(x, y_B(x) = e^p z) \].

We now calculate:

\[ u_{yy} = \frac{u_{zz}}{e^{2p}} \quad u_{xy} = -\frac{e^p}{e^{2p}} v_{zz} + o(e^{-p}) \]

\[ u_x = v_x + \frac{e^p}{e^{2p}} v_z \quad u_y = \frac{1}{e^p} v_z \]

\[ u_{xx} = (\frac{e^p}{e^{2p}})^2 v_{zz} + o(e^{-p}) \]

We substitute into \( \varepsilon \left( A_{11} u_{xx} + 2A_{12} u_{xy} + A_{22} u_{yy} \right) = u_y \) to obtain

\[ \varepsilon \left( A_{11} (\frac{e^p}{e^{2p}})^2 v_{zz} + \ldots + 2A_{12} \left( -\frac{e^p}{e^{2p}} v_{zz} \right) + A_{22} v_{zz} \right) = -\frac{1}{e^p} v_z \]
We write this as
\[ \varepsilon^{1-2p} u(x) v_{zz} = -\varepsilon^{-p} v_z, \quad 0 < z < \infty. \]

where
\[ u(x) = A_{11} (y_T')^2 - 2 A_{12} y_T + A_{22}. \]

How to balance the power of \( \varepsilon \) we take \( 1-2p=-p \) or \( p=1 \).

We conclude that we have an ODE boundary layer and the leading order boundary layer profile \( v_0(x,z) \) satisfies
\[
\begin{align*}
\varepsilon^{-1} u(x) v_{0zz} &= -v_{0z} \quad \text{on} \quad 0 < z < \infty \\
v_0 &= u_T(x) \quad \text{on} \quad z = 0 \quad \text{as} \quad v_0 \to u_B(x) \quad \text{as} \quad z \to \infty \quad \text{to match to} \quad \text{outer limit}
\end{align*}
\]  

Now we claim that \( u(x) > 0 \) on upper boundary. At point \( C \) as shown we have \( u > 0 \) since \( A_{33} > 0 \). However, since ellipticity holds we have \( A_{11} > 0 \) and \( A_{12}^2 < A_{11} A_{22} \).

Thus \( u \) has one sign and \( \to -u \) on upper boundary.

Thus the solution to (2) is
\[ v_0 = u_B(x) + \left[ u_T(x) - u_B(x) \right] e^{-Z(u(x))} \]
where \( Z = [y_T(x) - y] e^{-1} \) and \( u(x) = A_{11} (y_T')^2 - 2 A_{12} y_T + A_{22} \).

Remark: The \( O(\varepsilon) \) boundary layer analysis breaks down near endpoints A and B since \( |y_T'| \to \infty \) at those points. We would need a separate analysis in the tiny corner region near A and B to describe solution there. These corner regions are where \( \bar{u} \) is born.

Since \( \bar{u}_B(x) \) has assumed continuity, the outer limit has no discontinuity. A thin layer inside the domain would be needed near any point \( x_0 \) on discontinuity of \( \bar{u}_B(x) \).

The subcharacteristics are defined as the characteristic curve of the first order PDE for reduced limit \( \varepsilon \to 0 \). So \( y_0 = \bar{x} \) constant is a subcharacteristic. If the boundary layer is not parallel to a subcharacteristic,
We consider the case where a boundary segment is parallel to a subcharacteristic. In this case, the BL equation is a parabolic PDE.

For a domain as shown:

\[ A_{11} V_{xx} + 2 A_{12} V_{xy} + A_{22} V_{yy} = V_y \text{ in } \Omega \]

For a domain as shown:

We assume ellipticity so that

\[ A_{11} > 0, \quad A_{22} > 0, \quad A_{11}^2 - 2 A_{11} A_{22} < 0 \]

Now as shown before, the outer limit is \( U_y = 0 \) so subcharacteristics are line \( x = \) constant as shown by dotted line.

- An outer boundary layer near upper boundary to allow the BC there to be satisfied. This is same as done earlier.
- Now since \( U_B(x, y) = U_5(y) \) in general, there will be an additional boundary layer localized near \( x = x_5 \). Since this boundary segment is \( \parallel \) to subcharacteristic, the BL equation will be a PDE.

Near \( x = x_5 \), we let \( s = \frac{x - x_5}{\varepsilon} \) and

\[ V[s, y] = U[x_5 - \varepsilon s, y] \]

We label the point \( C \) by \( x = x_5, \quad y = y_5 \). Now we substitute to obtain

\[ \frac{\varepsilon A_{11} V_{ss} + 2 \varepsilon A_{12} V_{sy} + \varepsilon A_{22} V_{yy}}{\varepsilon^2} = V_y \]

The boundary layer PDE balances underlined term so that \( \varepsilon = \frac{1}{2} \).

Hence to leading order

\[ A_{11} V_{ss} = V_y, \quad 0 < s < \infty \]

This is a parabolic PDE with time-like variable \( y \). It is well-posed if we integrate for increasing \( y \). Our problem is to solve

\[ \begin{cases} A_{11} V_{yy} = V_y, & 0 < s < \infty, \quad y > y_5 \quad \text{with } A_{11} > 0 \end{cases} \]

BC: \( V(0, y) = U_5(y) \), \( V(s, y_5) = U_5(x_5) \)

The solution is

\[ V_0 = U_B(x_5) + \frac{\varepsilon}{2 \sqrt{A_{11}}} \int_0^x \left[ \frac{U_5(\tilde{y}) + y_5 - U_5(x_5)}{\sqrt{y - \tilde{y} - y_5}} \right]^{3/2} \exp \left( \frac{-\varepsilon^2}{4A_{11}(y - y_5)} \right) d\tilde{y} \]
The BL equation is a heat equation with $y$ the time variable.

We must take the initial data $V_0(x, y_0)$, $U_0(y_0)$ corresponding to point C and then integrate forward in $y$ up to point D where $y = y_0(x_3)$. We cannot take the data at point D and integrate downward for smaller values of $y$, doing so would yield a backward heat equation.

There would be a very intricate analysis needed near point D, where the parabolic BL and ODE boundary layer merge.

Example: Find the boundary layer structure for the following problem:

$$\epsilon (u_{xx} + u_{yy}) = -u_x \text{ in } \Omega$$

Solution: This is steady-state for $u_x = u_y = \epsilon (u_{xx} + u_{yy})$. If $\epsilon = 0$, $u = f(x + t)$ so wind is blowing to the left.

This gives the BL structure as shown:

1. The outer limit are either $u = 0$ or $u = 3$ as shown.
2. $f$ is an ODE BL to allow $u = 2$ to be satisfied.
3. On top, bottom is a PDE BL where $u_{yy} = -u_x$ with $\hat{y} = y/\epsilon y$, or $\hat{y} = 1 - \frac{y}{\epsilon y}$.
4. A shear layer in middle of domain to smooth out continuity between $u = 0$ and $u = 3$ outer limits. This is a PDE $u_{yy} = -u_x$.
5. A BL ODE near the right end on the semi-circle.
Example Consider the convection-diffusion problem

\[ \varepsilon (u_{xx} + u_{yy}) + u_x + bu_y = 0 \quad \text{in} \quad 0 < x < \varepsilon, \quad 0 < y < 1 \]

with the boundary data as shown. Here \( b \) is a constant.

Consider the 3 cases: \( b > 0, \quad b < 0 \) and \( b = 0 \). Assume \( \varepsilon(x) \to 0 \) and \( f(x) \to 0 \) as \( x \to +\infty \).

\[ \begin{array}{c|c}
\varepsilon & \varepsilon(x) \\
\hline
0 & 0 \\
\end{array} \]

Solution

The outer limit \( \varepsilon = 0 \) satisfies \( u_{ox} + bu_{oy} = 0 \), to leading order.

The solution \( u = \phi_0(y - bx) \) for an arbitrary function \( \phi_0(z) \).

We can obtain higher order terms by expanding \( u = u_0 + \varepsilon u_1 + \ldots \).

We would then obtain that

\[ u_{xxxx} + bu_{xxyy} = -(u_{oxxx} + u_{oyyy}) = -(b^2 + 1) \phi_0''(z). \]

Thus \( y - bx = \text{constant} \),

\[ \frac{du}{dy} = -\frac{(b^2 + 1)}{b} \phi_0''(y - bx). \]

Hence

\[ u = -\frac{(1 + b^2)}{b} \phi_0''(y - bx) + \phi_0(y - bx). \]

Now we find the boundary layer for different ranges of \( b \).

Case I (\( b > 0 \)) Then the outer limit gives \( u = \text{constant} \) on line \( y = bx + \text{constant} \).

\[ \begin{array}{c|c|c}
\varepsilon & \varepsilon(x) & \phi_0(z) \\
\hline
0 & 0 & 0 \\
\end{array} \]

Since we need decay out of a BL we cannot put a BL along \( y = L \). Otherwise if \( z = (L - y) \to u_{zz} + bu_{z} = 0 \) \( u = c e^{-bz} \) which blow up as \( z \to +\infty \).

Also we must take outer limit to satisfy the BC on \( y = L \).

Hence \( u \sim u_0 \sim \phi_0 \left( x - \frac{(y - L)}{b} \right) \) is outer solution.

We then need BL near both \( y = 0 \) and \( x = 0 \) to satisfy the BC there. These will be ODE boundary layers.
Near \( y = 0 \): Let \( \hat{y} = y/\varepsilon \) and \( \bar{V}(x, \hat{y}) = u(x, \varepsilon \hat{y}) \).

Expand \( v = v_0 + \varepsilon v_1 + \cdots \); we obtain

\[
\begin{align*}
V_0 \hat{y} + b V_0 \hat{y} &= 0 \quad \text{in} \quad 0 < \hat{y} < \infty \\
V_0 &= f(x) \text{ at } \hat{y} = 0 \\
V_0 \to h \left( x + \frac{L}{b} \right) \text{ as } \hat{y} \to \infty \quad \text{matching condition} \quad \lim_{y \to 0} U_0 = \lim_{y \to 0} h \left( \frac{x - (y - L)}{b} \right) = h \left( \frac{x - L}{b} \right)
\end{align*}
\]

This is an ODE BL problem with solution

\[
V_0 = h \left( x + \frac{L}{b} \right) + \int f \left( \frac{x - (y - L)}{b} \right) e^{-b\hat{y}} \, \hat{y}.
\]

Decay exponentially as \( \hat{y} \to 0 \) since \( b > 0 \).

Near \( x = 0 \): Let \( \hat{x} = x/\varepsilon \) and \( \bar{W}(\hat{x}, y) = U(y, \varepsilon \hat{x}) = w_0 + \varepsilon w_1 + \cdots \).

To leading order we obtain

\[
\begin{align*}
w_0 \hat{x}^2 + w_0 \hat{x} &= 0, \quad 0 < \hat{x} < \infty \\
w_0 &= g(y) \text{ at } \hat{x} = 0; \quad w_0 \to \lim_{x \to 0} h \left( \frac{x - (y - L)}{b} \right) = h \left( \frac{L - y}{b} \right) \text{ at } \hat{x} \to \infty.
\end{align*}
\]

The solution is

\[
w_0 = h \left( \frac{L - y}{b} \right) + \left[ g(y) - h \left( \frac{L - y}{b} \right) \right] e^{-\hat{x}}
\]

which decays exponentially as \( \hat{x} \to \infty \).

Case 1: \( b > 0 \): In this case the subcharacteristics point in a different direction and we cannot put in a BL near \( y = 0 \).

The picture is, with subcharacteristics \( y = bx + \text{constant} \):

\[
\begin{align*}
\text{The outer limit } U_0 &= f(y, bx) \\
\text{must satisfy the BC on lower wall.}
\end{align*}
\]

Hence \( U \sim U_0 \equiv f \left( x - \frac{y}{b} \right) \) except in thin \( O(\varepsilon) \) BL near \( x = 0 \) and \( y = L \).

Near \( y = L \): Put \( \hat{y} = (L - y)/\varepsilon \) and \( \bar{V}(x, \hat{y}) = u(x, \varepsilon \hat{y}) = v_0 + \varepsilon v_1 + \cdots \).

The leading order ODE BL problem is

\[
\begin{align*}
V_0 \hat{y} - b V_0 \hat{y} &= 0 \quad \text{in} \quad 0 < \hat{y} < \infty \\
V_0 &= h(x) \text{ at } \hat{y} = 0; \quad V_0 \to f \left( x - \frac{L}{b} \right) \text{ as } \hat{y} \to \infty
\end{align*}
\]

The solution is

\[
v_0 = f \left( x - \frac{L}{b} \right) + \int h(x) e^{-b\hat{y}} \, \hat{y}, \quad \text{which decays as } \hat{y} \to \infty
\]
BL NEAR $x = 0$ put $\hat{x} = x/\epsilon$, $W(\hat{x}, y) = U[\epsilon \hat{x}, y] = W_0 + \epsilon W_1 + \ldots$ we obtain

$$W_0 \hat{x} + W_0 \hat{x} = 0, \ 0 < \hat{x} < \infty$$

$$W_0 = g(y) \ \text{on} \ \hat{x} = 0; \ \ W_0 \to \lim_{\hat{x} \to 0} U_0 = \frac{F(-y/b)}{x} \ \text{as} \ \hat{x} \to 0$$

The solution is

$$W_0 = \frac{F(-y/b)}{x} + \left[ g(y) - \frac{F(-y/b)}{x} \right] e^{-\hat{x}}, \ \text{which decays as} \ \hat{x} \to \infty$$

**Case 3 (b = 0).** $\epsilon (U_{xx} + U_{yy}) = -U_x$.

In this case the flow $u$ from right to left and so we take $U_0 x = 0$.

An outer solution and observe that we can put a BL near $x = 0$. The picture is

$$U = \text{constant on lines} \ y = \text{constant}$$

Since $b, f \to 0$ as $x \to +\infty$, the outer limit is $W_0 = 0$.

We will get an ODE boundary layer near $x = 0$ and PDE boundary layer near $y = 0$ and near $y = L$.

**BL near $x = 0$** let $\hat{x} = x/\epsilon$ with $W(\hat{x}, y) = U[\epsilon \hat{x}, y] = W_0 + \epsilon W_1 + \ldots$ we obtain to leading order that $W_0 \hat{x} + W_0 \hat{x} = 0, \ 0 < \hat{x} < \infty$.

Thus $W_0 = g(y) e^{-\hat{x}}$.

**BL near $y = 0$** put $\hat{y} = y/\epsilon^p$ with $p > 0$ and $V(x, \hat{y}) = U[x, \epsilon^p \hat{y}]$.

Then $\epsilon^{1-2p} V_{\hat{y}} + \epsilon V_{xx} + V_x = 0$. Choose $p = 1/2$ to balance term.

To leading order we get PDE problem

$$V_0 \hat{y} = -V_0 x \ \text{on} \ \hat{y} = 0, \ \hat{y} < \infty, \ x > 0$$

$$V_0 = \frac{f(x)}{x} \ \text{on} \ \hat{y} = 0, \ V_0 \to 0 \ \text{as} \ x \to \infty \ \text{for} \ \hat{y} \ \text{fixed}.$$

This is well posed so we are integrating backward in $x$, being the time variable. The solution is $V_0(x, \hat{y}) = \int_0^x \exp \left( -\frac{\hat{y}^2}{4(\sigma - x)} \right) \frac{f(\sigma)}{(\sigma - x)^{3/2}} d\sigma$.

A similar parabolic PDE BL near upper wall $y = L$. 
EXAMPLE (BOUNDARY LAYER TRANSITION)

For $E \to 0^+$, find the BL structure for solution to

$EAu + \Gamma \frac{\partial u}{\partial \Gamma} = 0$ in unit disk

with $u = f(q)$ on $\Gamma = 1$ with $f(q) = f(q + 2\pi)$ and $f$ continuous.

Solution: This is the steady-state for $\Gamma + x$ $\to$ $E u$ $\to$ for $E > 0$, $u = u(x) = 0$

so that $u = f(x + \gamma) \to$ wind blows from right to left.

We expect a BL near $\Gamma = 1$ on $\pi/2 < \gamma < 3\pi/2$.

Now subcharacteristic are lines $y =$ constant. Outer limit $u_x \to 0$ implies $u_y \to$ constant on $y =$ constant.

Now fix $\gamma$ in $-\pi/2 < \gamma < \pi/2$. The outer solution is with $y = \sin \gamma$, $q = \sin^{-1} y$

that to leading order $u \approx u_0 = f[sin^{-1}(y)] = f(q) - \pi/2 < \gamma < \pi/2$.

Now we construct BL near $\Gamma = 1$ for $\pi/2 < \gamma < 3\pi/2$. First convert to polar coordinates:

$u_x = u_r \Gamma + u_q q_x$ where $\Gamma = x \sin y$, $q = \tan^{-1}(y/x)$.

We have $\Gamma_x = x/\Gamma = \cos \gamma$, $q_x = -y/\Gamma = -\sin \gamma$

Thus $u_x = u_r \cos \gamma - u_q \sin \gamma/\Gamma$.

We obtain in polar coordinates

$E \left( u_{rr} + \frac{1}{\Gamma} u_r + \frac{1}{\Gamma^2} u_q q \right) + u_r \cos \gamma - u_q \sin \gamma = 0 \text{ in } 0 < \Gamma < 1$, $0 < \gamma < 2\pi$.

Now near $\Gamma = 1$ we put $\Gamma = (1 - \Gamma)/\epsilon$ and fix $\gamma$ in $\pi/2 < \gamma < 3\pi/2$. Let $\epsilon \to 0$ to find

Then

$E^{-1} \frac{1}{\epsilon^2} \frac{\partial^2 u}{\partial \Gamma^2} + E^{-1} \frac{\epsilon}{\Gamma - \epsilon \Gamma^2} \frac{\partial u}{\partial \Gamma} + E^{-1} \frac{\epsilon}{\Gamma - \epsilon \Gamma^2} v \frac{\partial u}{\partial \Gamma} - E^{-1} \frac{\epsilon}{\Gamma - \epsilon \Gamma^2} \frac{\partial u}{\partial \Gamma} = 0$

Balancing the underlined terms we get $\epsilon = 1$ and to leading order

$V_{\Gamma \Gamma} = \cos \gamma V_\Gamma$ in $0 < \Gamma < 1$ for fixed $\gamma$ in $\pi/2 < \gamma < 3\pi/2$.

Notice that since $\pi/2 < \gamma < 3\pi/2$, $\cos \gamma < 0$ and we have decay as $\Gamma \to 0$

out of the BL as we must have. Moreover, but writing the outer limit as $f(\pi - \gamma)$ for $\pi/2 < \gamma < 3\pi/2$ we obtain the value of $f$ on the

limit as $f(\pi - q)$ for $\pi/2 < q < 3\pi/2$ we obtain the value of $f$ on the

outer limit.
The solution to (v) is simply
\[ V_0 = f(\bar{\pi} - \phi) + \left[ f(\phi) - f(\bar{\pi} - \phi) \right] e^{\rho \cos \phi}, \quad \bar{\pi}/2 < \phi < 3\pi/2. \]

In terms of \( \xi \) we have
\[ V_0 = f(\bar{\pi} - \phi) + \left[ f(\phi) - f(\bar{\pi} - \phi) \right] e^{(1-\xi) \cos \phi}, \quad \text{for} \quad \xi = 1 + o(1) \]
and \( \pi/2 < \phi < 3\pi/2 \).

Remark (i) Our assumption that \( f \) is continuous and \( 2\pi \) periodic means that there is no interior discontinuity to be smoothed out by a shear layer.

(ii) The BL theory breaks down near \( \phi = \pi/2 \) and \( \phi = 3\pi/2 \) (point A, B) where a more specialized analysis is needed. In this region near point A, \( \rho = \frac{1-\xi}{\xi} \), \( \phi = \pi/2 \) is needed. This is rather intricate but not so important.
EXAMPLE (QUASI 1-D HEAT CONDUCTION)

We will consider the following 1-D heat conduction problem in a cylinder of variable cross-sectional area. Find \( U(x, r) \) that satisfies

\[
\begin{align*}
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + \frac{\partial^2 U}{\partial x^2} &= 0 \quad \text{in} \quad 0 < x < L, \quad 0 < r < R_o \frac{f(x)}{L} \\
U = g_o \left( \frac{R}{R_o} \right) & \quad \text{on} \quad x = 0 \\
U = g \left( \frac{R}{R_o} \right) & \quad \text{on} \quad x = L \\
\nabla U \cdot \hat{n} &= 0 \quad \text{on} \quad r = R_o \frac{f(x)}{L}
\end{align*}
\]

Thus \( f(.) \) describes the shape of the tube. Find an approximation for the solution when \( R_o/L \ll 1 \).

Solution \( \nabla U = (\frac{U_r}{r}, \frac{U_x}{L}) \) \( \hat{n} = \left( 1, -\frac{R_o}{L} \frac{f'(x/L)}{\sqrt{1 + \left( \frac{R_o}{L} \frac{f'(x/L)}{L} \right)^2}} \right) \).

Let \( \Gamma = \frac{R}{R_o}, \quad x = \frac{x}{L} \) so that \( U_{rr} = \frac{U_{rr}}{R_o^2} \).

We obtain that \( \nabla U \cdot \hat{n} = \left( \frac{1}{R_o}, \frac{R_o}{L} \frac{U_x}{L} \right) \cdot \left( 1, -\frac{R_o}{L} \frac{f'(x)}{\sqrt{k+\ldots}} \right) = 0 \).

Hence \( \Gamma \frac{R_o}{L^2} \frac{U_x}{L} f'(x) \) on \( \Gamma = f(x) \).

The problem that we must solve is

\[
\frac{1}{\Gamma} \frac{\partial U}{\partial \Gamma} + \frac{\partial U}{\partial x} + \varepsilon^2 U_{xx} = 0 \quad \text{in} \quad 0 < x < L, \quad 0 < \Gamma < f(x) \\
U = \varepsilon^2 f'(x) U_x & \quad \text{on} \quad \Gamma = f(x) \quad \text{U bounded as} \quad \Gamma \to 0.
\]

\( U(0, \Gamma) = g_o(\Gamma), \quad U(1, \Gamma) = g(\Gamma) \)

with \( f(x) > 0 \) given, \( g_o, g, \) specified and \( \varepsilon = \frac{R_o}{L} \).

In outer region away from \( x = 0 \):

\( U = \phi + \varepsilon^2 \zeta \ldots \)

so that

\[
\begin{align*}
\phi_{rr} + \frac{1}{\Gamma} \phi_{r} &= 0 \quad \text{in} \quad 0 < \Gamma < f(x) \\
\phi_r &= 0 \quad \text{on} \quad \Gamma = 1, \quad \phi \text{ bounded as} \quad \Gamma \to 0.
\end{align*}
\]

\( \zeta \)
\[
O(\varepsilon^3) \left\{ \begin{aligned}
\varepsilon_{11} r + \frac{1}{r} \varepsilon_{1r} &= -\varepsilon_{0xx} \\
\varepsilon_{1r} &= F'(x) \varepsilon_{0x} \quad \text{on} \quad r = 1 \\
\varepsilon_{1} &= \text{bounded at} \ r = 0.
\end{aligned} \right.
\]

We conclude \( \varepsilon_0 : \text{span} \{ 1, \log r \} \rightarrow \varepsilon_0 : \varepsilon_0(x) \) constant independent of \( r \) with \( \varepsilon_0 \) to be found. Then

\[
\left\{ \begin{aligned}
\varepsilon_{11} &= (r \varepsilon_{1r})_r = -r \varepsilon_{0xx} \\
\varepsilon_{1r} &= F'(x) \varepsilon_{0x} \quad \text{on} \quad r = 1; \quad \varepsilon_{1} \text{ bounded at} \ r = 0.
\end{aligned} \right.
\]

Since homogeneous problem \( L \phi = 0 \)

\( \phi_{1r} = 0 \quad \text{on} \quad r = 1; \quad \phi, \text{ bounded at} \ r = 0 \)

has a nontrivial solution \( \phi_1 = 1 \) then \( \exists \) a solvability condition for \((\Psi)\).

We use IBP

\[
(1, L \varepsilon_1) = \int_0^1 \left( r \varepsilon_{1r} / r \right) dr = \int_0^1 \frac{F'(x) \varepsilon_{0x}}{F(x)}
\]

Thus

\[
\int_0^1 \left( -r \varepsilon_{0xx} \right) dx = F\left( \frac{F'(x)}{F(x)} \right) \varepsilon_{0x}.
\]

\[
\rightarrow \quad \frac{\left[ F(x) \right]^2}{2} \varepsilon_{0xx} = F(x) F'(x) \varepsilon_{0x}.
\]

Thus we deduce that \( \varepsilon_0(x) \) satisfies

\[
[ F^2(x) \varepsilon_{0x} ]_x \quad \text{in} \quad 0 < x < 1.
\]

Recognizing that \( A(x) = \pi \left[ F(x) \right]^2 \) is the cross-sectional area

we conclude that

\[
[ A(x) \varepsilon_{0x} ]_x = 0 \quad \text{in} \quad 0 < x < 1.
\]

In region where \( A \) is large (wide part of channel or tube) then the effective diffusivity \( \text{Deff} \sim A(x) \) is large.

Now to find the boundary condition for \( \varepsilon_0 \) at \( x \to 0^+ \) and \( A \to \infty \) we put in BL near endpoint.
**BL NEAR X:** \( \hat{X} = X/E \) \( \text{and} \) \( V(\hat{X}, \Gamma) = \sum_{\epsilon \in \hat{X}, \Gamma}. \)

Now \( F(X) \equiv F(1) \) \text{and we obtain to leading order that}

\[ V_{\Gamma r} + \Gamma_{\Gamma r} + V_{\Gamma r} = 0 \quad \text{in} \quad 0 < \hat{X} < \infty, \quad 0 < \Gamma < F(0) \]

\[ V_{\Gamma r} = 0 \quad \text{on} \quad \Gamma = F(0) \]

"local picture". \( \Gamma \rightarrow \hat{X} \)

Then, in terms of this solution we have by matching that

\[ \lim_{\hat{X} \rightarrow \infty} V_{0}(\hat{X}, \Gamma) = V_{0, \infty} \equiv U_{0}(1) \]

We now identify \( V_{0}(\hat{X}, \Gamma) \) and \( V_{0, \infty} \).

By separating variables

\[ V_{0}(\hat{X}, \Gamma) = b_{0} + \sum_{n \geq 0} b_{n} e^{-\lambda_{n} \hat{X}} J_{0}(\lambda_{n} \Gamma), \quad \text{so} \quad V_{0, \infty} = b_{0} \]

Where \( J_{0}'(\lambda_{n} F(1)) = 0 \) for \( n = 1, 2, \ldots \) since by orthogonality we have

\[ \int_{0}^{F(1)} 1 \cdot J_{0} / J_{0} \Gamma \Gamma d\Gamma = 0 \]

We set \( \hat{X} = 0 \) and integrate:

\[ b_{0} \int_{0}^{F(1)} \Gamma d\Gamma = \int_{0}^{F(1)} \Gamma g_{0}(\Gamma) d\Gamma \rightarrow b_{0} = \frac{2}{[F(1)]^{2}} \int_{0}^{F(1)} \Gamma g_{0}(\Gamma) d\Gamma. \]

We conclude that

\[ U_{0}(1) = \frac{2}{[F(1)]^{2}} \int_{0}^{F(1)} \Gamma g_{0}(\Gamma) d\Gamma \]

And likewise

\[ U_{0}(1) = \frac{2}{[F(1)]^{2}} \int_{0}^{F(1)} \Gamma g_{1}(\Gamma) d\Gamma. \]

We then have

\[ \int (X(\hat{X}))^{2} U_{0, X} \hat{X} = 0 \]

Thus

\[ U_{0} = C \int_{0}^{X} \frac{1}{(F(\xi))^{2}} d\xi + U_{0}(1) \]

Where

\[ C = \frac{U_{0}(1) - U_{0}(1)}{1 - \frac{1}{d\xi}} \]
Remark: If we had a time-dependent PDE of form \( u_{t}(r, x, t) \)

satisfying

\[
D \left( \frac{u_{rr}}{r} + \frac{1}{r} u_{r} + \frac{1}{\lambda} u_{xx} \right) = u_{t}
\]

with same BC, then we can derive a limiting PDE on an appropriate time scale. Scaling as before yields

\[
\frac{1}{R_{0}^{\frac{1}{2}}} \left( \frac{u_{rr}}{r} + \frac{1}{r} u_{r} \right) + \frac{1}{L^{2}} u_{xx} = \frac{1}{D} u_{t}.
\]

so

\[
\frac{u_{rr}}{r} + \frac{1}{r} u_{r} + \frac{R_{0}^{\frac{1}{2}}}{L^{2}} u_{xx} = \frac{R_{0}^{\frac{1}{2}}}{D} u_{t}.
\]

Now let \( T = L^{2}/D \), diffusion time for axial flow. Then \( \frac{u}{T} = \frac{u_{t}}{D/L^{2}} \).

In this way we obtain

\[
\begin{cases}
    u_{rr} + \frac{1}{r} u_{r} = -\epsilon^{2} u_{xx} + \epsilon^{3} u_{t}, & 0 < x < 1, \ 0 < r < F(x), \\
    u_{r} = \epsilon^{2} F'(x) u_{x}, & \text{on} \ r = F(x) \\
    u = g_{0}(r) \text{ at } x = 0, \ u = g_{1}(r) \text{ at } x = 1.
\end{cases}
\]

Repeating the same calculation as before we obtain that the outer solution satisfies the PDE:

\[
\frac{F^{2}}{2} u_{ot} - \frac{F^{2}}{2} u_{oxx} = F F' u_{ox}.
\]

This becomes

\[
\frac{F^{2}}{2} u_{ot} = \left( \frac{F^{2}}{2} u_{ox} \right)_{X}
\]

since \( A = \frac{F^{2}}{2} \) cross-sectional area we conclude that \( u_{0} \) satisfies

\[
u_{0t} = \frac{1}{A(x)} \left[ A(x) u_{0x} \right]_{X}
\]

Diffusion equation hold on time-scale of \( L^{2}/D \).
A conceptual equivalent in acoustics to the quasi 1-d heat conduction problem considered on p17-20 is Webster's horn.

Consider the wave equation for \( u(R, Z, t) \) satisfying

\[
\dot{u} + \gamma u = \frac{c^2}{\gamma} \Delta u \quad \text{in} \quad Z > 0, \quad \gamma R < R_0 S(Z/L)
\]

\[
\ddot{u} = e^{-i\omega t} u_0 \left( \frac{R}{R_0} \right) \quad \text{on} \quad Z = 0
\]

with \( \nabla u \cdot \hat{n} = 0 \) on \( R = R_0 S(Z/L) \).

Here \( \Delta u = \frac{\partial^2 u}{\partial R^2} + \frac{1}{R} \frac{\partial u}{\partial R} + \frac{\partial^2 u}{\partial Z^2} \), the signal generated on \( Z = 0 \) is hopefully propagated along the length \( 0 < Z < L \) of the horn.

Now put \( \tilde{u} = e^{-i\omega t} \phi (R, Z) \) so that

\[
\begin{cases}
\dot{\phi} + \gamma \frac{\partial \phi}{\partial R} = 0 & \text{in} \quad 0 < R < R_0 S(Z/L) \\
\nabla \phi \cdot \hat{n} = 0 & \text{on} \quad R = R_0 S(Z/L) \\
\phi = \phi_0 \frac{R}{R_0} & \text{on} \quad Z = 0.
\end{cases}
\]

Here \( \eta = \omega/c \) is wavenumber, which sets the spatial scale of the wave.

Now non-dimensionalize \( \gamma = R/R_0, \quad Z = Z/L, \quad \phi = \phi R/R_0 \) with \( \phi = \phi(R, Z) \).

We obtain that

\[
\begin{align*}
\Phi_{RR} + \frac{1}{\gamma} \Phi_R + E^2 \Phi_{ZZ} + \eta^2 R_0^2 \Phi & = 0 & \text{in} \quad 0 < R < S(Z) \\
\Phi & = \Phi_0(R) & \text{on} \quad Z = 0, \quad \Phi & = E^2 B'(Z) \Phi_Z & \text{on} \quad R = S(Z).
\end{align*}
\]

We will consider the low-frequency asymptotic limit where \( \eta^2 R_0^2 = O(E^2) \).

We define \( \Phi_0 \) by \( \eta^2 R_0^2 = E^2 \Phi_0 \). Then

\[
\begin{align*}
\Phi_{RR} + \frac{1}{\gamma} \Phi_R & = -E^2 \Phi_{ZZ} - \Phi_0 E^2 \Phi & \text{in} \quad 0 < R < S(Z) \\
\Phi & = \Phi_0(R) & \text{on} \quad Z = 0, \quad \Phi & = E^2 \Phi'(Z) \Phi_Z & \text{on} \quad R = S(Z).
\end{align*}
\]

In S. Riemer's "Webber's Horn Revisited", SIAM J. APPL. MATH., 65 (6), (2005), p. 1981-2004, the limit \( E \to 0 \) was analyzed. For \( Z \gg O(E) \), one can readily derive that \( \Phi \to \Phi_0(Z) \) where

\[
\begin{pmatrix}
\frac{1}{A[Z]} [ A[Z] \Phi_0 ]' + \Phi_0 \Phi_0' = 0 & \text{in} \quad O(E) < Z < Z, \\
\Phi_0 & = \frac{1}{A[Z]} \Phi_0 & \text{on} \quad Z = 0, \quad \Phi_0 & = \frac{1}{A[Z]} \Phi_0 & \text{on} \quad R = S(Z).
\end{pmatrix}
\]

This is the Webster Horn equation.

To analyze (xy) we eliminate first derivative term. We write

\[
\Phi_0'' + \frac{A'}{A} \Phi_0' + \Phi_0 = 0.
\]

We put \( \Phi_0 = p \psi \) to get

\[
p \psi'' + \psi [ 2 p' + \frac{A'}{A} p ] + \psi [ p'' + \frac{A'}{A} p' + \frac{A^2}{A} p ] = 0.
\]
DIVIDING BY $P$:

$$
\psi'' + \psi \left[ \frac{2p'}{p} + \frac{A'}{A} \right] = \psi \left[ \frac{P''}{P} + \frac{A'}{A} \frac{p'}{p} + \frac{\eta_0^2}{2} \right] = 0.
$$

WE TAKE $p = A^{1/2}$ SO THAT $\frac{2p'}{p} = \frac{A'}{A} \Rightarrow \frac{p'}{p} = -\frac{1}{2} \frac{A'}{A}.$

THEN WE CALCULATE:

$$
\frac{p''}{p} + \frac{A'}{A} \frac{p'}{p} = \left[ -\frac{1}{2} \frac{A''}{A} \right] + \frac{A'}{A} \left( \frac{1}{2} \frac{A'}{A} \right) = \frac{1}{2} \frac{A''}{A} + \frac{A'}{A} \frac{A'}{A} = \frac{1}{2} \frac{A''}{A} - \frac{1}{2} \frac{A'}{A} \frac{A'}{A} = \frac{1}{2} \frac{A''}{A} - \frac{1}{2} \frac{A'}{A} \frac{A'}{A}.
$$

WE GET

$$
\frac{p''}{p} + \frac{A'}{A} \frac{p'}{p} = -\frac{1}{2} \frac{A''}{A} + \frac{3}{4} \frac{A'}{A}^2 - \frac{1}{2} \frac{A'}{A}^2 = \left( \frac{1}{2} \frac{A''}{A} - \frac{1}{2} \frac{A'}{A} \frac{A'}{A} \right) \Rightarrow \frac{p''}{p} + \frac{A'}{A} \frac{p'}{p} = -\frac{1}{2} \frac{A''}{A} + \frac{1}{4} \frac{A'}{A}^2.
$$

WE THEN HAVE THE ODE

$$
\psi'' + \psi \left[ \eta_0^2 + \frac{1}{4} \left( \frac{A'}{A} \right)^2 - \frac{1}{2} \frac{A''}{A} \right] = 0.
$$

NOW TO FURTHER SIMPLIFY, WE WRITE $A(z) = \pi \bar{A} S(z)^2$. THUS, $A' = \frac{2SS'}{A} = \frac{2SS'}{S^2}. \Rightarrow \frac{A'}{A} = \frac{SS'}{S^2}.$

MOREOVER, $A'' = \frac{2S'S'' + 2SS''}{S^2} = \frac{2S'S'' + 2SS''}{S^2}$. WE GET

$$
\frac{1}{4} \left( \frac{A'}{A} \right)^2 - \frac{1}{2} \frac{A''}{A} = \left( \frac{S'}{S} \right)^2 \left( \frac{S'}{S} \right)^2 - \frac{S''}{S}.
$$

IN TERMS OF THE NON-DIMENSIONAL RADIO OF THE TUBE WE OBTAIN SCHRODINGER'S EQUATION:

$$
(1) \quad \psi'' + \left[ \eta_0^2 - \bar{v} \pi S(z)^2 \right] \psi = 0 \quad \text{IN} \quad 0(=) < Z < \text{WHRE} \quad \bar{v} = \frac{S''}{S(z)}. \quad \text{TO RECOVER THE POTENTIAL,} \quad \psi = e^{i\omega t} \phi = e^{i\omega t} A^{-1/2} \psi \quad \text{WITH} \quad A = \pi S^2.
$$

IN SUMMARY, WE GET $\psi \sim \frac{\phi}{\sqrt{\pi} S(z)} \quad \text{AT} \quad z \geq 0 \quad \text{AND} \quad z \rightarrow \infty$.

GIVEN SOME VALUE $\psi(0)$ WE WANT TO FIND A SOLUTION TO (1) THAT IS AN OUTGOING WAVE AS $Z \rightarrow \infty$, i.e. $\psi \sim e^{i\omega t} \phi(z)$ WITH $\phi > 0$ AT $Z \rightarrow \infty$. HERE WE TAKE THE HORN TO HAVE INFINITY EXTENT.

* VALUES OF $\eta_0$ FOR WHICH $\psi \rightarrow 0$ AS $Z \rightarrow \infty$ CORRESPOND TO TRAPPED OR EVANESCENT MODES, AS THE ENERGY DOES NOT RADIATE TO $Z \rightarrow \infty$, i.e. $\int_0^\infty \psi^2 dz < \infty \Rightarrow$ TRAPPED MODE.

* FOR WHAT $\eta_0$, GIVEN A "SHAPE" $S(z)$ WILL YIELD TRAPPED MODES? THIS IS AN EIGENVALUE PROBLEM. TECHNICALLY, WE ARE LOOKING FOR DISCRETE EIGENVALUES FOR $\eta_0$.

EMBEDDED IN THE CONTINUOUS SPECTRUM

AN EXPONENTIAL HORN HAS $S = e^{mZ}$ TAND YIELDS $\psi'' + \left[ \eta_0^2 - m^2 \right] \psi = 0$. WE GET

PROPAGATION ONLY IF $\eta_0 > m \Rightarrow \eta_0^2 R_0^2 > e^{2mZ} \Rightarrow k > e^m R_0$. THEN $\frac{\omega}{c} > \frac{e^m R_0}{c}$ IS NEEDED FOR PROPAGATION. FOR $S = e^{mZ}$, $\psi = c e^{i\sqrt{k^2 - m^2} z} e^{i\eta_0^2 - m^2} z$ WAVE PROPAGATING TO THE RIGHT.
**HYPERBOLIC EQUATIONS**

We now consider singular perturbation of hyperbolic PDE: Let $u(x,t)$ satisfy

$$
\epsilon (u_{xx} - u_{tt}) = au_x + bu_t, \quad a, b \text{ constants with } b > 0
$$

in $-\infty < x < \infty$, $t > 0$ with $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

We take $b > 0$ so that we have stability in time.

The problem for any $\epsilon > 0$ has real characteristics which are given by $y = t - x$ and $\lambda = t + x$ with $s, \lambda$ fixed. The full PDE allows for jumps in $u$ across either of the characteristics.

The subcharacteristics (corresponding to $\epsilon = 0$ problem) are $bx - \lambda t = \text{constant}$.

For full PDE for any $\epsilon > 0$ the domain of dependence at point $P$ is the initial data between points $A$ and $B$ as shown.

The dotted line in this figure is the subcharacteristic through point $P$, which lies inside the triangular domain only when $b/|a| > 1$.

Thus if $G = F = 0$ for $x < x_1$ and $x > x_2 \rightarrow u = 0$ outside triangular wedge for $\epsilon (u_{xx} - u_{tt}) = 0$.

When $b/|a| < 1$, the subcharacteristic lies outside the domain of dependence for hyperbolic PDE as shown.
We will now show that for a well-posed limit as $\varepsilon \to 0^+$, we must have that $b/|a| > 1$, i.e. that (1) holds.

Suppose to the contrary that $b/|a| < 1$ with $q > 0$ as shown in Fig. 2. Consider the evolution of a jump discontinuity in $U_2$ along the curve $\gamma: \gamma_0$.

Such a jump could originate from specifying $U = F(X)$, $U_x = 0$ at $t = 0$ with $F(x) = 0$ for $x < x_1$ and $F(x_1^+) = 0$, but $F(x_1^-) \neq 0$.

We now change to characteristic coordinates: let $\xi = \frac{t}{\varepsilon} - X$ and $\eta = \frac{t}{\varepsilon} + X$.

We calculate that $U_x = -U_3 + U_2$, $U_3 = U_5 + U_4$, $U_{xx} = U_5 - 2U_4 + U_2$, $U_{33} = U_5 + 2U_4 + U_2$. We substitute to obtain that (1) becomes

$$-4\varepsilon U_{33} = (b-a)U_3 + (b+a)U_2.$$  

Suppose that $U$, $U_2$ continue across $\gamma: \gamma_0$ but that $U_3\big|_{\gamma_0} \neq U_3\big|_{\gamma_0}$.

Define $\eta(\eta) = U_3(\gamma_0, \eta) - U_3(\gamma_0, \eta^{-})$. We obtain by evaluating (1) for $\eta = \eta^+$ and $\eta = \eta^{-}$ and subtracting that

$$-4\varepsilon \frac{d\eta}{d\eta} = (b-a)\eta.$$  Thus $\eta = \eta_0 \exp\left[\frac{(b-a)(\eta - \eta_0)}{\varepsilon}\right].$

We conclude that $\eta$ diverges to $0$ as $\varepsilon \to 0^+$ when $b < a$. Thus if $b > a$ this is ill-posed as a limit.

A similar discussion shows that for jumps in $U_2/d\eta$ across $\lambda = \lambda_0$ lead to blowup (unstable behavior) when $b < 1$ and $a < 0$.

Hence for a well-posed limit in problem as $\varepsilon \to 0$ we must have that $b/|a| > 1$, i.e. that sub-characteristics lie within the WFD (i.e., domain of dependence of initial data).
Example 1 (Signalling Problem)

Consider
\[ E(\mu_{xx} - \mu_{tt}) = a \mu_{x} + b \mu_{t} \quad \text{in} \quad 0 < x < \infty, \quad t > 0 \]

\[ \mu(0,t) = F(t), \quad \mu(x,0) = \mu_{t}(x,0) = 0 \]

Here \( b > 0 \) and \( b/|q| > 1 \). Find BL structure for \( q > 0 \) and \( q < 0 \). Assume \( F(0) \neq 0 \).

Solution

We have the following two pictures:

**Case I** \( q > 0 \)

**Case II** \( q < 0 \) (Incoming Characteristics)

Suppose that \( q < 0 \). The characteristics take data given at \( t = 0 \), i.e.

\[ \mu = \mu_{t} = 0 \]

and propagate up to the boundary near \( x = 0 \).

Outer solution is \( \mu_{0} = \frac{\Phi}{b} (x - at/b) \) and since \( \mu = \mu_{t} \) at \( x > 0 \) for \( t = 0 \),

we take \( \Phi = 0 \). This gives the BL structure

Now near \( x = 0 \) we put \( y = x/e \) and \( \nu(y,t) = \mu[e^{y},t] \). We substitute to obtain that

\[ E \left( \frac{1}{e^{2}} \nu_{yy} - \nu_{tt} \right) = \frac{a}{c} \nu_{y} + b \nu_{t} \]

Expanding \( \nu = \nu_{0} + e \nu_{1} + \ldots \) we obtain to leading order that

\[ \nu_{0yy} - a \nu_{0yy} = 0, \quad 0 < y < \infty \]

\[ \nu_{0}(F(t)) \text{ on } y = 0, \quad \nu_{0} \to 0 \text{ as } y \to \infty. \]

Since \( q < 0 \) this has exponential decay. The solution is

\[ \nu_{0}(F(t)e^{ay} = F(t)e^{ax/e} \quad \text{for } x = O(e). \]
CASE I (OUTGOING CHARACTERISTICS)

We let $a > 0$. The leading order outer solution is shown below is to let $a U_0 + b U_0^+$ $= 0$ satisfy $U_0 = F(t)$ at $x = 0$, with $U_0 = 0$ for $x > a t / b$, or $t < b x / a$. Since $U_0 = \gamma (x - a t / b)$ we have by

**Method of Characteristic** that

\[
U_0 = \begin{cases} 
0, & t < b x / a \\
F(t - b x / a), & t > b x / a
\end{cases}
\]

Now if $F(t) \neq 0$ or in $F(0) \neq 0$ but $F'(0) \neq 0$ then we have a discontinuity across $t = b x / a$. However for the full hyperbolic PDE discontinuities can only occur across the characteristic $x \equiv t = c$, we must smooth out the discontinuity along the sub-characteristic.

Let $y = x - a t / b$ and $V(y, t) = U[qt / b + e^p y, t]$. We calculate

\[
\begin{align*}
U_x &= \frac{1}{\epsilon^p} V_y, \\
U_t &= V_t - \frac{a}{b} \frac{e^p}{\epsilon^p} V_y, \\
U_{xx} &= \frac{1}{\epsilon^p} V_{yy}, \\
U_{tt} &= \frac{a^2}{b^2} \frac{e^p}{\epsilon^p} V_{yy} - \frac{2a}{b} \frac{e^p}{\epsilon^p} V_{yt} + V_{tt}.
\end{align*}
\]

Substitute into the PDE: $e^p U_{xx} - U_{tt} = a U_{xx} + b U_t$

\[
\frac{e^p}{\epsilon^p} \left(1 - \frac{a^2}{b^2}\right) V_{yy} + \frac{2a}{b} \frac{e^p}{\epsilon^p} V_{yt} - e V_{tt} = \frac{a}{b} \frac{e^p}{\epsilon^p} \sqrt{V_{yy} + \frac{V_t - a}{b e^p} \frac{V_y}{\epsilon^p}.}
\]

Some terms cancel and we balance underlined term. This gives

\[
p = \frac{1}{2}
\]

And to leading order we get

\[
V_0 \approx \begin{cases} 
U_0 V_{0yy}, & y < 0, \ t > 0 \\
F(0^+), & y \rightarrow -\infty
\end{cases}
\]

\[
V_0 \rightarrow \begin{cases} 
U_0, & y \rightarrow +\infty \\
0, & y \rightarrow -\infty
\end{cases}
\]

We take the initial condition

\[
V_0 = \begin{cases} 
F(0^+), & \hat{y} < 0 \text{ at } \hat{t} = 0. \\
0, & \hat{y} > 0
\end{cases}
\]

Since $b / a > 1$ we have $\gamma > 0$; i.e., a well-posed heat equation. This again shows that $b / a > 1$ is essential. The solution is

\[
V_0(y, t) = F(0^+) \text{ erf} C \left( \frac{y}{2 \sqrt{\eta_1 t}} \right) \Rightarrow V_0 = F(0^+) \text{ erf} C \left( \frac{y - at / b}{2 \sqrt{\eta_1 t}} \right) \text{ erf} C(z) = 2 \int_0^z e^{-k^2} dk.
\]
**Example (Initial Value Problem)**

We consider \( \varepsilon (U_{xx} - U_{tt}) = aU_x + bU_t \) with \( b > 0, \; \varepsilon / |a| > 1 \)
on \(-\infty < x < \infty, \; t > 0 \) with \( U(x, 0) = F(x), \; U_t(x, 0) = G(x) \).

Find an asymptotic solution for \( \varepsilon \ll 1 \).

**Solution**

We will expand \( U = U_0 + \varepsilon U_1 + \ldots \).

We collect power of \( \varepsilon \) to obtain

\[
aU_{0x} + bU_{0t} = 0 \quad \Rightarrow \quad U_0 = \frac{\phi_0(x - at/b)}{b}
\]

\[
aU_{1x} + bU_{1t} - U_{0xx} - U_{0tt} \varepsilon = (1 - a^2/b^2) \phi_0''(z) \quad \text{with} \quad z = x - at/b.
\]

Thus on \( dx/dt = a/b \) we have \( du/dt = (1 - a^2/b^2) \phi_0''(z) \).

We obtain \( U_1 = (1 - a^2/b^2) \frac{\phi_0''(z)}{b} + \phi_1(z) \).

Thus a 2-term outer expansion is in terms of unknown function \( \phi_0, \phi_1 \):

\[
U = \phi_0(z) + \varepsilon \left[ (1 - a^2/b^2) \frac{\phi_0''(z)}{b} + \phi_1(z) \right] + \ldots \quad \text{with} \quad z = x - at/b.
\]

Now construct an initial layer near \( t = 0 \). Let \( \tau = t/\varepsilon \rho, \; W(x, \tau) = U[x, \varepsilon \rho \tau] \).

Then \( \varepsilon (W_{xx} - \frac{1}{\varepsilon \rho} W_{\tau\tau}) = aW_x + bW_\tau \)

with \( W = F(x), \; W_\tau = \varepsilon^p G(x) \) at \( \tau \rightarrow 0 \).

To balance underlined term take \( \rho = 1 \) so \( \tau = t/\varepsilon \). Expand \( W = W_0 + \varepsilon W_1 + \ldots \).

We obtain that

\[
W_{0\tau\tau} + bW_{0\tau} = 0 \quad \text{in} \quad \tau > 0; \quad W_0 = F(x), \; W_{0\tau} = 0 \text{ at } \tau = 0
\]

\[
W_{1\tau\tau} + bW_{1\tau} = -aW_0 \quad \text{in} \quad \tau > 0; \quad W_1 = 0, \; W_{1\tau} = G(x) \text{ at } \tau = 0.
\]

The solution is \( W_0 = F(x) \) and

\[
W_1 = \frac{1}{b} \int \left( G(x) + \frac{a}{b} F'(x) \right) \left( 1 - e^{-b\tau} \right) - \frac{a}{b} F'(x) \tau.
\]

Now let \( \tau \rightarrow 0 \) and write it in outer variables \( \frac{t}{\varepsilon} \).

We obtain the matching condition

\[
(x, \tau) \quad \int \quad W = F(x) + \varepsilon \left[ \frac{1}{b} G(x) + \frac{a}{b} F'(x) \right] - \frac{a}{b} F'(x) \tau.
\]

Now in \( (x) \) let \( \tau \rightarrow 0 \), we obtain that the outer limit becomes

\[
U \sim \phi_0(x) + \varepsilon \left[ \phi_1(x) \right] - \frac{a}{b} \phi_0'(x) \tau.
\]
COMPARING THIS WITH \((x\,x')\) GIVES

\[ g_0(x) = f(x) \]

\[ g_1(x) = \frac{1}{b} g(x) + \frac{a}{b^2} f'(x). \]

IN THIS WAY, FOR \( t >> O(\varepsilon) \), WE HAVE FROM \((x)\) THE OUTER SOLUTION

\[ u \sim f\left(x - \frac{at}{b}\right) + \varepsilon \left[ \left(1 - \frac{a^2}{b^2}\right) \frac{d}{dx} f\left(x - \frac{at}{b}\right) + \frac{1}{b} \left( g\left(x - \frac{at}{b}\right) + \frac{a}{b} f'(x - \frac{at}{b}) \right) \right] \]

FOR SHORT-TIME \( t = O(\varepsilon) \) WE HAVE THE INITIAL TRANSIENT

\[ u \sim f(x) + \varepsilon \left[ \frac{1}{b} \left( g(x) + \frac{a}{b} f'(x) \right) \left(1 - e^{-\frac{b^2}{4\varepsilon}}\right) - \frac{a}{b} f'(x) \frac{t}{\varepsilon} \right]. \]