MOTION BY CURVATURE (INTERFACE)

\[ u^*_t = \varepsilon^2 \Delta u + Q(u) \quad \text{in} \quad \Omega \]
\[ \partial_\nu u = 0 \quad \text{on} \quad \partial \Omega. \]

ASSUME \( Q(u^*_t) = 0 \), \( Q'(u^*_t) < 0 \), \( \int_{\Omega} Q(u) \, du = 0 \).

ON AN \( O(1) \) TIME SCALE \( u \to u^*_+ \) OR \( u \to u^- \) SEPARATED BY AN INTERFACE \( \Gamma \).

\[ \Gamma \] IS CENTERLINE OF INTERFACE OF WIDTH \( O(\varepsilon) \), AND IT LEVEL CURVE
\[ u|_{\Gamma} = (u^*_+ + u^-)/2. \]

WHAT IS EVOLUTION OF \( \Gamma \)?

IN OUTER REGION \( u \sim u^- \) OR \( u \sim u^*_+ \).

NEAR \( \Gamma \) WE INTRODUCE \((\eta, s)\) COORDINATES

\[ u^*_t = \varepsilon^2 \left( u_{\eta\eta} - \eta \varepsilon \frac{\eta}{1-\eta^2} u_{\eta} + \frac{1}{1-\eta^2} \frac{d}{ds} \left( \frac{1}{1-\eta^2} \phi_s u \right) \right) + Q(u) \]

NOW \( \eta < 0 \)

\[ \Gamma \] WITH \( \eta \) CURVATURE OF \( \Gamma \), \( \eta : \xi(s) \)

\( s \) IS ARC LENGTH ON \( \Gamma \)
\( \eta \) IS DISTANCE FROM \( \Gamma \).

NOW WE LET \( \hat{\eta} = \eta / \varepsilon \) THEN \( u \to \hat{u} \) WITH \( \eta = \eta(\tau) \), \( \tau = \varepsilon \rho^t \)

\[ \varepsilon^{p-1} \hat{u}^\hat{\eta} + \varepsilon^p \hat{u}^s \hat{\phi} = \hat{u}^\hat{\eta} - \kappa \varepsilon \hat{u}^\hat{\eta} + \varepsilon^2 \hat{u}^{ss} + Q(\hat{u}) \]

WHERE \( \hat{\eta} = d\eta/d\tau \)
We choose $p = 2$ and expand

\[ \bar{U} = U_0 + \varepsilon U_1 + \cdots \]

\[ \begin{cases} 
\bar{U}_0^{\Lambda} + Q(\bar{U}_0) = 0, & -\infty < \Lambda < \infty \\
\bar{U}_0 \to U_- \quad \Lambda \to +\infty \\
\bar{U}_0 \to U_+ \quad \Lambda \to -\infty 
\end{cases} \]

A solution to (1), and it is unique by specifying $\bar{U}_0(0) = \frac{U_+ + U_-}{2}$.

Now at next order,

\[ \begin{cases} 
\bar{U}_1^{\Lambda} + Q'(\bar{U}_0) \bar{U}_1 = \bar{U}_0^{\Lambda} \quad \Lambda \to +\infty \\
\bar{U}_1 \to 0 \quad \Lambda \to \pm \infty 
\end{cases} \]

Notice $\Lambda = \bar{V}$ normal velocity to $F$. Notice $\bar{V} > 0$ is expanding.

Now define $\bar{U}_1 = U_1^{\Lambda} + Q'(\bar{U}_0) \bar{U}_1$.

Since $\Lambda \bar{U}_0^{\Lambda} = (U_0^{\Lambda})^{\Lambda} + Q'(\bar{U}_0) \bar{U}_0^{\Lambda} = 0$

we must have for solvability of (2) that

\[ \Lambda \left[ \int_{-\infty}^{0} (U_0^{\Lambda})^2 \, d\Lambda = -\int_{0}^{\infty} \bar{U}_0^{\Lambda} \left( \bar{U}_0^{\Lambda} \right)^2 \, d\Lambda \right. \]

thus

\[ \bar{V} = -\frac{d\Lambda}{d\bar{V}}. \]

In terms of original time-scale $\tau = \varepsilon^2 \bar{t}$, then

\[ \bar{V} = -\varepsilon^2 \bar{V}. \]
Thus if \( r \) is a sphere of radius \( \rho \), we have \( \nabla = 1/\rho \).

And
\[
\nabla = \frac{dp}{dt}.
\]

Thus
\[
\frac{dp}{d\rho} = -\frac{1}{\rho}
\]

If \( p(0) = \rho_0 \), then \( p \) vanishes in finite time,
\[
\rho^2 - \rho_0^2 = -t
\]

Or
\[
\rho = \sqrt{\rho_0^2 - t} \quad \text{for} \quad 0 < t < \rho_0^2.
\]