STRONG LOCALIZED PERTURBATION THEORY

WE FIRST RECALL A FEW BASIC RESULTS FROM POTENTIAL THEORY. SUPPOSE THAT $\Delta u = \delta(x - x_0)$ IN $\mathbb{R}^N$ WITH $N=2$, OR $N=3$.

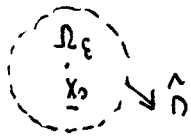
THEN $$u \sim -\frac{1}{4\pi|\underline{x}-\underline{x}_0|} \quad \text{AS } \underline{x} \to \underline{x}_0 \text{ IN 3-DIM.}$$

$$u \sim \frac{1}{2\pi}\log|\underline{x}-\underline{x}_0| \quad \text{AS } \underline{x} \to \underline{x}_0 \text{ IN 2-DIM.}$$

WE RECALL THE DERIVATION OF THIS IN 3-DIM. WE TAKE A SMALL SPHERE OF RADIUS ε ABOUT $\underline{x}_0$ SO THAT $\Omega_\varepsilon : \{|\underline{x}-\underline{x}_0| \le \varepsilon\}$. THEN IN A NEIGHBORHOOD OF $\underline{x}_0$ WE LET $r = |\underline{x}-\underline{x}_0|$ AND THEN

$$\Delta u = u_{rr} + \frac{2}{r}u_r = 0 \quad \text{FOR } r>0 \text{ SO THAT } u = B/r.$$

WE THEN USE THE DIVERGENCE THEOREM



$$\text{SO} \quad \int_{\Omega_\varepsilon} \Delta u \, d\underline{x} = \int_{\Omega_\varepsilon} \delta(\underline{x}-\underline{x}_0)\, d\underline{x} = 1$$

$$\text{THEN LET } \varepsilon \to 0 \quad \int_{\partial\Omega_\varepsilon} \nabla u \cdot \hat{n} \, dS = 4\pi\left[r^2 \frac{\partial u}{\partial r}\right]_{r=\varepsilon} = 1.$$

WE CALCULATE $\partial u/\partial r = -B/r^2$ SO THAT $4\pi\left(-\varepsilon^2 B/\varepsilon^2\right) = 1$

WHICH YIELDS $B = -1/4\pi$, AND SO $u \sim -\dfrac{1}{4\pi|\underline{x}-\underline{x}_0|}$ AS $\underline{x} \to \underline{x}_0$.

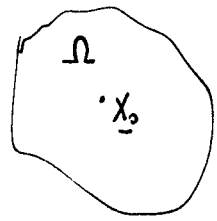
IN CONTRAST IN TWO-DIMENSIONS WE LET $\Omega_\varepsilon : |\underline{x}-\underline{x}_0| \le \varepsilon$ AND THEN FOR $r = |\underline{x}-\underline{x}_0| > 0$, $\Delta u = u_{rr} + \frac{1}{r}u_r = 0$ SO $u = B\log r$ IS THE SINGULAR SOLUTION. THEN USING THE DIVERGENCE THEOREM

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \Delta u \, d\underline{x} = \lim_{\varepsilon\to 0}\int_0^{2\pi} \left.\frac{\partial u}{\partial r}\right|_{r=\varepsilon} \varepsilon \, d\varphi = \lim_{\varepsilon\to 0}\int_0^{2\pi}\left(\frac{B}{\varepsilon}\right)\varepsilon \, d\varphi = \int_{\Omega_\varepsilon}\delta(\underline{x}-\underline{x}_0)\, d\underline{x} =$$

THIS YIELDS THAT $2\pi B = 1$ AND SO $u \sim \dfrac{1}{2\pi}\log|\underline{x}-\underline{x}_0|$

REMARK

(i) IF WE WANT TO SOLVE IN 3-D.



$$\Delta u = 0 \quad \text{IN } \Omega \setminus \{X_0\}$$
$$u = 0 \quad \text{ON } \partial\Omega$$

AND $u \sim \dfrac{A}{|\underline{X} - \underline{X}_0|}$ AS $\underline{X} \to \underline{X}_0$

THEN WE RECALL $-\dfrac{1}{4\pi}\dfrac{1}{|\underline{X} - \underline{X}_0|} \longrightarrow \delta(\underline{X} - \underline{X}_0)$, AND SO

BY SCALING $\dfrac{A}{|\underline{X} - \underline{X}_0|} \longrightarrow -4\pi A\,\delta(\underline{X} - \underline{X}_0)$.

THIS YIELDS THAT $\begin{cases} \Delta u = -4\pi A\,\delta(\underline{X} - \underline{X}_0) & \text{IN } \Omega \\ u = 0 & \text{ON } \partial\Omega \end{cases}$

(ii) SIMILARLY $\Delta u = 0$ IN $\Omega \setminus \{X_0\}$, Ω IS TWO-DIMENSIONAL

$$u = 0 \quad \text{ON } \partial\Omega$$
$$u \sim A \log|\underline{X} - \underline{X}_0|$$

IS EQUIVALENT TO $\Delta u = 2\pi A\,\delta(\underline{X} - \underline{X}_0)$ IN Ω WITH $u = 0$ ON $\partial\Omega$.

NOW SUPPOSE THAT WE HAVE LOWER-ORDER TERMS OF THE FORM

$$\Delta u - p^2 u = \delta(\underline{X} - \underline{X}_0) \quad \text{IN EITHER 2-D OR 3-D.}$$

THEN WE CALCULATE $u = -\dfrac{e^{-pr}}{4\pi r}$ IN $\mathbb{R}^3$; $u = \dfrac{-1}{2\pi} K_0(pr)$ IN $\mathbb{R}^2$.

BUT CRITICALLY THE FORM OF THE SINGULARITY TO LEADING-ORDER NEAR THE SINGULAR POINT IS INDEPENDENT OF THE LOWER-ORDER TERM pu. IN FACT,

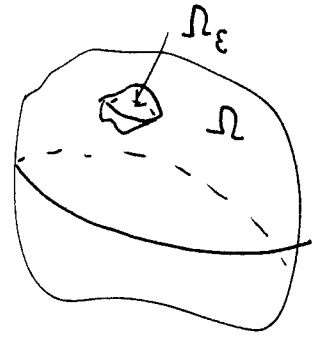
$$u \sim -\frac{1}{4\pi|\underline{X} - \underline{X}_0|} \quad \text{AS } \underline{X} \longrightarrow \underline{X}_0 \text{ IN 3-D}$$

$$u \sim \frac{1}{2\pi}\log|\underline{X} - \underline{X}_0|.$$

THIS FOLLOWS FROM THE OBSERVATION THAT $\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} u \, d\underline{X} = 0$.

EIGENVALUE ASYMPTOTICS IN 3-D

LET Ω BE A 3-D BOUNDED DOMAIN WITH A HOLE OF "RADIUS" $O(\varepsilon)$ IS REMOVED FROM Ω.



$$(*)\quad \begin{cases} \Delta u + \lambda u = 0 & \text{IN } \Omega \setminus \Omega_\varepsilon \\ u = 0 & \text{ON } \partial\Omega \\ u = 0 & \text{ON } \partial\Omega_\varepsilon \\ \displaystyle\int_{\Omega\setminus\Omega_\varepsilon} u^2 \, d\underline{x} = 1 \end{cases}$$

WE ASSUME THAT Ω_ε SHRINKS TO A POINT $\underline{x}_0$ AS $\varepsilon \to 0$.

FOR INSTANCE, Ω_ε COULD BE THE SPHERE $|\underline{x} - \underline{x}_0| \leq \varepsilon$.

THE UNPERTURBED PROBLEM IS

$$\begin{aligned} \Delta\phi + \mu\phi &= 0 \quad \text{IN } \Omega \\ \phi &= 0 \quad \text{ON } \partial\Omega \\ \int_\Omega \phi^2 \, d\underline{x} &= 1 \end{aligned}$$

THIS PROBLEM HAS EIGENPAIRS AS

$$\phi_j(x), \ \mu_j \quad j = 0, 1, 2, \ldots$$

WITH $\int_\Omega \phi_j \phi_k \, d\underline{x} = 0 \quad j \neq k$ AND $\phi_0(\underline{x}) > 0$ FOR x INSIDE Ω.

WE NOW LOOK FOR AN EIGENPAIR OF $(*)$ NEAR THE PRINCIPAL EIGENPAIR $\phi_0(\underline{x}), \mu_0$.

WE PROCEED BY THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS.

WE FIRST EXPAND THE EIGENVALUE FOR $(*)$ AS

$$\lambda \sim \mu_0 + \nu(\varepsilon)\lambda_1 + \cdots \qquad \nu(\varepsilon) \to 0 \text{ AS } \varepsilon \to 0.$$

IN THE OUTER REGION AWAY FROM THE HOLE, WE EXPAND

$$u = \phi_0(\underline{x}) + \nu(\varepsilon)\, u_1 + \cdots$$

NOW SINCE $\Omega_\varepsilon \to \{x_0\}$ AS $\varepsilon \to 0$, WE OBTAIN THAT

$$(1)\quad \begin{cases} \Delta u_1 + \mu_0 u_1 = -\lambda_1 \phi_0 & \text{IN } \Omega \setminus \{x_0\} \\ u_1 = 0 & \text{ON } \partial\Omega \\ \int_\Omega 2 u_1 \phi_0 \, d\underline{x} = 0 \end{cases}$$

NOW WE CONSTRUCT AN INNER EXPANSION NEAR THE HOLE.

WE LET $\underline{y} = \varepsilon^{-1}(\underline{x} - \underline{x}_0)$ AND $V(\underline{y};\varepsilon) = u(x_0 + \varepsilon y)$ SATISFIES

$$\Delta_y V + \lambda \varepsilon^2 V = 0 \quad \text{OUTSIDE } \Omega_0 \qquad \Omega_0 = \Omega_\varepsilon/\varepsilon$$

$$V = 0 \quad \text{ON } \partial\Omega_0$$

HOLE IS "MAGNIFIED"

THEN WE EXPAND $V = V_0 + \nu(\varepsilon) V_1 + \cdots$.

WE SUBSTITUTE TO OBTAIN

$$(2)\quad \begin{cases} \Delta_y V_0 = 0 & \underline{y} \text{ OUTSIDE } \Omega_0 \\ V_0 = 0 & \text{ON } \partial\Omega_0. \\ V_0 \to \phi(\underline{x}_0) & \text{AS } |\underline{y}| \to \infty. \end{cases}$$

THE MATCHING CONDITION IS THAT

$$(3)\quad \underset{x \to x_0}{\phi_0(\underline{x}) + \nu(\varepsilon) u_1 + \cdots} \sim \underset{|y| \to \infty}{V_0 + \nu(\varepsilon) V_1 + \cdots}$$

THEREFORE, $V_0 \to \phi_0(\underline{x}_0)$ AS $|y| \to \infty$.

NOW WE WRITE THE SOLUTION TO (2) AS

$$V_0 = \phi(\underline{x}_0)\,(1 - V_c(\underline{y}))$$

THEN WE OBTAIN THAT $V_c(\underline{y})$ SATISFIES

$$\Delta_y V_c = 0, \quad y \text{ OUTSIDE } \Omega_0$$

$$V_c = 1 \quad \text{ON } y \text{ ON } \partial\Omega_0$$

$$V_c \longrightarrow 0 \quad \text{AS } |y| \to \infty.$$

THE SOLUTION HAS THE WELL-KNOWN ASYMPTOTIC BEHAVIOR

$$V_c \sim \frac{C}{|\underline{y}|} + \cdots \quad \text{AS } |y| \to \infty,$$

WHERE $C > 0$ IS CALLED THE ELECTROSTATIC CAPACITANCE OF Ω_0.

<u>REMARK</u> (i) IF $\Omega_\varepsilon : |\underline{x} - \underline{x}_0| \leq \varepsilon$ THEN $\Omega_0 : |y| \leq 1$.

WE LET $r = |\underline{y}|$ SO THAT IN 3-D, $V_c = V_c(r)$

$$V_c'' + \frac{2}{r} V_c' = 0, \quad r \geq 1$$

$$V_c = 1 \quad \text{ON } r = 1$$

THEN $V_c = 1/r$ FOR $r \geq 1$ AND SO $C = 1$.

(ii) C IS KNOWN ANALYTICALLY FOR A WIDE VARIETY OF SHAPES INCLUDING ELLIPSOIDS ETC.. OTHERWISE WE CALCULATE IT NUMERICALLY.

(iii) A FAMOUS ISOPERIMETRIC INEQUALITY OF SZEGÖ STATES THAT OF ALL BODIES Ω_0 OF THE SAME VOLUME, THE SPHERE HAS THE SMALLEST CAPACITANCE.

NOW WE RETURN TO V_0 AND NOTE THAT

$$V_0 \sim \phi(\underline{x}_0)\left(1 - \frac{C}{|\underline{y}|} + \cdots\right), \quad |y| \longrightarrow \infty$$

LET $\underline{y} = \varepsilon^{-1}(\underline{x} - \underline{x}_0)$ AND RECALL THE MATCHING CONDITION (3)

$$\phi_0(\underline{x}_0) + \nu(\varepsilon) U_1 \sim \phi_0(\underline{x}_0) - \phi_0(\underline{x}_0) \frac{\varepsilon C}{|\underline{x} - \underline{x}_0|} + \cdots$$

AS $x \to x_0$

THIS YIELDS THAT $\nu(\varepsilon) = \varepsilon$, AND $U_1 \longrightarrow -\phi_0(\underline{x}_0) \dfrac{C}{|\underline{x} - \underline{x}_0|}$, $x \to x_0$

THEN WE RETURN TO (1) AND WRITE

$$\Delta u_1 + \mu_0 u_1 = -\lambda_1 \phi_0 \quad \text{IN } \Omega \setminus \{x_0\}$$

$$u_1 = 0 \quad \text{ON } \partial\Omega$$

$$u_1 \to -\phi_0(\underline{x}_0)\frac{C}{|\underline{x}-\underline{x}_0|} \quad \text{AS } \underline{x} \to \underline{x}_0 .$$

$$\int_\Omega u_1 \phi_0 \, d\underline{x} = 0 .$$

SINCE $-1/4\pi|\underline{x}-\underline{x}_0| \longrightarrow \delta(\underline{x}-\underline{x}_0)$, THEN WE OBTAIN

$$\mathcal{L}u_1 \equiv \Delta u_1 + \mu_0 u_1 = -\lambda_1 \phi_0 + 4\pi C \phi_0(\underline{x}_0)\,\delta(\underline{x}-\underline{x}_0) \quad \text{IN } \Omega$$

$$u_1 = 0 \quad \text{ON } \partial\Omega$$

NOW WE USE GREEN'S SECOND IDENTITY WITH $\mathcal{L}\phi_0 = 0$ TO OBTAIN

$$\int_\Omega (\phi_0 \mathcal{L}u_1 - u_1 \mathcal{L}\phi_0)\, d\underline{x} = \int_{\partial\Omega} (\phi_0 \partial_n u_1 - u_1 \partial_n \phi_0)\, dS$$

SINCE $\phi_0 = u_1 = 0$ ON $\partial\Omega$ AND $\mathcal{L}\phi_0 = 0$ WE OBTAIN

$$0 = \int_\Omega \phi_0 \mathcal{L}u_1 \, d\underline{x} = \int_\Omega \phi_0 \left(-\lambda_1 \phi_0 + 4\pi C \phi_0(\underline{x}_0)\,\delta(\underline{x}-\underline{x}_0)\right) d\underline{x}$$

THIS YIELDS THAT
$$\lambda_1 = \frac{4\pi C\,[\phi_0(\underline{x}_0)]^2}{\int_\Omega \phi_0^2 \, d\underline{x}}$$

IN SUMMARY WITH $\nu(\varepsilon) = \varepsilon$ WE OBTAIN THE TWO-TERM EXPANSION

$$(3) \quad \left\{ \lambda \sim \mu_0 + \varepsilon \lambda_1 + \dots \qquad \lambda_1 = \frac{4\pi C\,[\phi_0(\underline{x}_0)]^2}{\int_\Omega \phi_0^2 \, d\underline{x}} \right.$$

<u>REMARK</u> IF THERE ARE N SMALL HOLES THEN WE OBTAIN

$$\lambda_1 = \frac{4\pi}{\int_\Omega \phi_0^2 \, d\underline{x}} \sum_{j=1}^{N} C_j \phi_0^2(\underline{x}_j),$$ WHERE C_j = capacitance of the jth hole.

REMARK

(i) LET US ASSUME THAT $u = 0$ ON $\partial\Omega$ IS REPLACED BY NO-FLUX CONDITION $\partial_n u = 0$ ON $\partial\Omega$.

THEN
$$\Delta\phi + \mu\phi = 0 \quad \text{IN } \Omega$$
$$\partial_n \phi = 0 \quad \text{ON } \partial\Omega$$
$$\int_\Omega \phi^2 \, d\underline{x} = 1$$

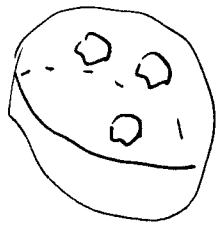
HAS THE PRINCIPAL EIGENVALUE $\mu_0 = 0$ AND $\phi_0 = \dfrac{1}{|\Omega|^{1/2}}$

WHERE $|\Omega|$ IS THE VOLUME OF Ω.

IN THIS CASE $\lambda \sim \dfrac{4\pi\varepsilon C}{|\Omega|}$ (AS SEEN FROM (3)).

NOTICE THAT THIS LEADING-ORDER TERM IS INDEPENDENT OF THE HOLE LOCATION.

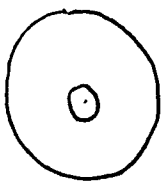
(ii) FOR MULTIPLE HOLES



Ω_{ε_j}, $j = 1, \ldots, N$

THEN
$$\lambda \sim \mu_0 + 4\pi\varepsilon \sum_{j=1}^{N} c_j \frac{[\phi_0(\underline{x}_j)]^2}{\int_\Omega \phi_0^2 \, d\underline{x}} + \ldots$$

CONSIDER THE SPECIAL CASE OF TWO CONCENTRIC SPHERES AS SHOWN



$$u_{rr} + \frac{2}{r} u_r + \lambda u = 0, \quad \varepsilon < r < 1$$
$$u(1) = 0, \quad u(\varepsilon) = 0.$$

THE EIGENFUNCTION IS $u = \dfrac{\sin\left(\sqrt{\lambda}\,(r - \varepsilon)\right)}{r}$

THEN WE OBTAIN $u(1) = 0$ SO THAT $\sqrt{\lambda}\,(1 - \varepsilon) = \pi$.

THIS YIELDS THAT

$$\lambda = \frac{\pi^2}{(1-\varepsilon)^2} \sim \pi^2 (1 + 2\varepsilon + \dots)$$

HENCE, $\lambda \sim \pi^2 + 2\varepsilon\pi^2 + \dots$

NOW USE THE ASYMPTOTIC FORMULA (3). IN (3) WE SET

$$\mu_0 = \pi^2, \quad \phi_0 = \frac{\sin(\pi r)}{r}$$

THEN $\phi_0(0) = \lim_{r \to 0} \frac{\sin(\pi r)}{r} = \pi.$

IN ADDITION,

$$\int_\Omega \phi_0^2 \, d\underline{x} = 4\pi \int_0^1 \frac{\sin^2(\sqrt{\pi}\, r)}{r^2} \, dr = 2\pi.$$

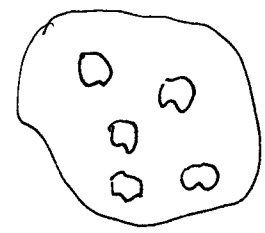
THEN (3) YIELDS $\lambda \sim \pi^2 + \frac{4\pi\varepsilon C \pi^2}{2\pi} \sim \pi^2 + 2\varepsilon C \pi^2 + \dots$

FINALLY $C = 1$ BY REMARK (i) ON PAGE (3). SO $\lambda \sim \pi^2 + 2\varepsilon\pi^2 + \dots$

AS WE WOULD HAVE EXPECTED.

<u>OPEN PROBLEM</u> CONSIDER N HOLES OF "RADIUS" ε FOR

$$\Delta u + \lambda u = 0 \text{ IN } \Omega \setminus \bigcup_{j=1}^{N} \Omega_{\varepsilon_j}$$
$$\partial_n u = 0 \text{ ON } \partial\Omega$$
$$u = 0 \text{ ON } \partial\Omega_{\varepsilon_j}$$



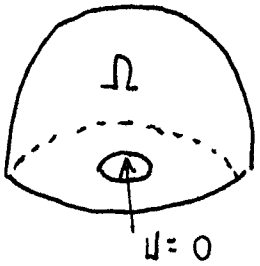
THEN SINCE $\mu_0 = 0$ AND $\phi_0 = 1/|\Omega|^{1/2}$ WE GET $\lambda \sim \frac{4\pi\varepsilon}{|\Omega|} \sum_{j=1}^{N} C_j$.

IF WE WANT TO MAXIMIZE λ (IN OTHER WORDS GET FASTEST DECAY TO ZERO FOR HEAT EQUATION WE NEED THE NEXT TERM IN THE EXPANSION. I.e. PERHAPS $\lambda \sim \frac{4\pi\varepsilon}{|\Omega|} \sum_{j=1}^{N} + \frac{4\pi\varepsilon^2}{|\Omega|^2} \sum_{i=1}^{N}\sum_{j=1}^{N} C_i C_j \, p(\underline{X}_1, \dots, X_N)$

AND SO WE NEED TO FIND MAXIMUM OF $p(X_1, \dots, X_N)$.

SMALL HOLES ON BOUNDARY

SUPPOSE THAT A SMALL PATCH OF BOUNDARY ALLOWS FOR THE LEAKAGE OF HEAT. THE MODEL IS



$$\Delta u + \lambda u = 0 \quad \text{IN } \Omega \qquad \int_\Omega u^2\, dx = 1$$

$$\partial_n u = 0 \quad \text{ON } \partial\Omega \setminus \partial\Omega_\varepsilon$$

$$u = 0 \quad \text{ON } \partial\Omega_\varepsilon$$

WE ASSUME THAT $\partial\Omega_\varepsilon \to \{\underline{x}_0\}$ AS $\varepsilon \to 0$.

WE CAN REPEAT THE ENTIRE INNER/OUTER ANALYSIS TO GET THAT THE INNER PROBLEM SATISFIES WITH $\underline{y} = \varepsilon^{-1}(\underline{x} - \underline{x}_0)$

$$\Delta_y V_0 = 0 \quad \text{IN } y_3 \geq 0$$

$$\partial_n V_0 = 0 \qquad (y_1, y_2) \notin \partial D_0 \qquad \partial D_0 = \partial\Omega_\varepsilon/\varepsilon .$$

$$V_0 = 0 \qquad (y_1, y_2) \in \partial D_0$$

$$V_0 \to \frac{1}{|\Omega|^{1/2}} \quad \text{AS } |y| \to \infty .$$

NOTICE THAT $\mu_0 = 0$ AND $\phi_0 = 1/|\Omega|^{1/2}$.

NOW $V_0 = \frac{1}{|\Omega|^{1/2}}(1 - V_c)$ WHERE $V_c(\underline{y})$ SATISFIES

$V_c(y) \sim C/|\underline{y}|$ AS $|y| \to \infty$.

THEN WE END UP WITH

$$\Delta u_1 + \mu_0 u_1 = 0 \quad \text{IN } \Omega$$

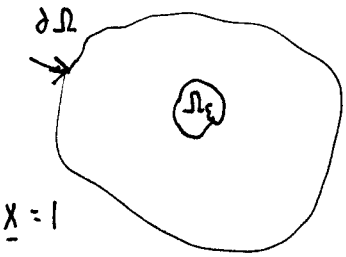
$$\partial_n u_1 = 0 \quad \text{ON } \partial\Omega \setminus \{x_0\}$$

$$u \sim -\frac{1}{|\Omega_0|^{1/2}} \frac{C}{|\underline{x} - \underline{x}_0|} \quad \text{AS } x \to x_0$$

SINCE A DELTA FUNCTION ON $\partial\Omega$ CONTRIBUTES ONLY $1/2$ OF AN INTERIOR δ-FUNCTION, WE OBTAIN

$$\lambda \sim 2\pi\varepsilon C/|\Omega| + \cdots$$

TWO-DIMENSIONS



WE CONSIDER

$$(*)\begin{cases} \Delta u + \lambda u = 0 & \text{IN } \Omega\setminus\Omega_\varepsilon \\ u = 0 & \text{ON } \partial\Omega \\ u = 0 & \text{ON } \partial\Omega_\varepsilon\,; \quad \int_{\Omega\setminus\Omega_\varepsilon} u^2\,d\underline{x} = 1 \end{cases}$$

WITH Ω_ε A SMALL HOLE OF "RADIUS" ε FOR WHICH $\Omega_\varepsilon \to \{\underline{x}_0\}$ AS $\varepsilon \to 0$.

LET μ_0, ϕ_0 BE THE PRINCIPAL (OR FIRST) EIGENPAIR OF THE UNPERTURBED PROBLEM SATISFYING

$$(1)\begin{cases} \Delta\phi_0 + \mu_0\phi_0 = 0 & \text{IN } \Omega \\ \phi_0 = 0 & \text{ON } \partial\Omega\,; \quad \int_\Omega \phi_0^2\,d\underline{x} = 1. \end{cases}$$

NOW WE WILL EXPAND THE EIGENVALUE OF $(*)$ THAT IS CLOSE TO μ_0 AS

$$\lambda \sim \mu_0 + \nu(\varepsilon)\lambda_1 + \cdots \qquad \nu(\varepsilon) \to 0 \text{ AS } \varepsilon \to 0.$$

IN THE OUTER REGION AWAY FROM THE HOLE WE EXPAND

$$u = \phi_0 + \nu u_1 + \cdots$$

UPON SUBSTITUTING INTO $(*)$ WE OBTAIN

$$(2)\begin{cases} \Delta u_1 + \mu_0 u_1 = -\lambda_1\phi_0 & \text{IN } \Omega\setminus\{x_0\} \\ u_1 = 0 & \text{ON } \partial\Omega\,; \quad \int_\Omega u_1\phi_0\,d\underline{x} = 0. \end{cases}$$

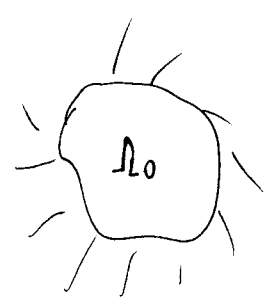
WITH SOME SINGULARITY CONDITION AS $\underline{x} \to \underline{x}_0$ TO BE FOUND.

NOW IN THE INNER REGION NEAR THE HOLE WE LET $\underline{y} = \varepsilon^{-1}(\underline{x} - \underline{x}_0)$ AND WE EXPAND

$$u = \nu(\varepsilon)\, V_0(y) + \cdots$$

WHERE $\Delta_y V_0 = 0$. WE WANT $V_0(y) \sim A_0 \log|y|$ AS $|y| \to \infty$ AND SO WE WRITE

$$V_0(y) = A_0 V_c(y)$$



$\Omega_0 = \Omega_\varepsilon/\varepsilon$

WHERE $V_c(y)$ SATISFIES

$$\Delta_y V_c = 0, \quad \underline{y} \text{ outside } \Omega_0$$
$$V_c = 0, \quad y \text{ ON } \partial\Omega_0$$
$$V_c \sim \log|y| \text{ AS } |y| \to \infty.$$

THIS PROBLEM FOR $V_c(\underline{y})$ HAS A UNIQUE SOLUTION, AND THE BEHAVIOR AT ∞ IS

$$V_c(y) \sim \log|y| - \log d + O(1/|y|) \quad \text{AS } |y| \to \infty$$

WHERE d IS CALLED THE "LOGARITHMIC CAPACITANCE" OF Ω_0. IT DEPENDS ON THE SHAPE OF Ω_0 AND NOT ITS ORIENTATION WITHIN

THERE ARE SOME KEY EXAMPLES

Ω_0	d
circle, radius a	a $\longrightarrow V_c = \log(\lvert y\rvert/a)$
ellipse, semi-axes a, b	$(a+b)/2$
equilateral triangle side h	$\sqrt{3}\,[\Gamma(1/3)]^3 h/(8\pi^2) \approx 0.422\,h$
square side h	$[\Gamma(1/4)]^2 h/(4\pi^{3/2}) \approx .5902\,h$

THESE VALUES OF d ARE FOUND BY CONFORMAL MAPPING, T. RANSFORD (CAMBRIDGE U. PRESS 1995).

NOW WE WRITE INNER EXPANSION IN OUTER VARIABLES

$$U \sim \nu(\varepsilon) A_0 [\log|y| - \log d] \sim \nu(\varepsilon) A_0 [-\log(\varepsilon d) + \log|X - X_0|].$$

THUS THE MATCHING CONDITION BECOMES:

$$\phi_0(\underline{X}_0) + \cdots + \nu(\varepsilon)\, U_1(X \to X_0) \sim (-\log(\varepsilon d))\, A_0\, \nu(\varepsilon) + A_0\, \nu(\varepsilon) \log|\underline{X} - \underline{X}_0| + \cdots$$

THEREFORE, WE TAKE

$$\nu(\varepsilon) = -1/\log(\varepsilon d), \quad A_0 = \phi_0(\underline{X}_0)$$

AND WE LET $U_1(X) \longrightarrow A_0 \log|\underline{X} - \underline{X}_0| = \phi_0(\underline{X}_0) \log|\underline{X} - \underline{X}_0|$ AS $\underline{X} \to \underline{X}_0$

THEREFORE (2) BECOMES

$$\Delta U_1 + \mu_0 U_1 = -\lambda_1 \phi_0 \quad \text{IN } \Omega \setminus \{X_0\}$$

$$U_1 = 0 \text{ ON } \partial\Omega$$

$$U_1 \sim \phi_0(\underline{X}_0) \log|\underline{X} - \underline{X}_0| \quad \text{AS } \underline{X} \to \underline{X}_0 \; ; \quad \int_\Omega U_1 \phi_0 \, d\underline{X} = 0 .$$

THIS CAN BE WRITTEN AS

$$\mathcal{L} u_1 \equiv \Delta u_1 + \mu_0 u_1 = -\lambda_1 \phi_0 + 2\pi \phi_0(\underline{x}_0)\,\delta(\underline{x}-\underline{x}_0) \quad \text{IN } \Omega$$

$$u_1 = 0 \quad \text{ON } \partial\Omega$$

$$\int_\Omega u_1 \phi_0 \, d\underline{x} = 0$$

NOW USE GREEN'S SECOND IDENTITY

$$\int_{\partial\Omega} (\phi_0 \partial_n u_1 - u_1 \partial_n \phi_0)\, dS = \int_\Omega (\phi_0 \mathcal{L} u_1 - u_1 \mathcal{L}\phi_0)\, d\underline{x}.$$

BUT $\phi_0 = u_1 = 0$ ON $\partial\Omega$ AND $\mathcal{L}\phi_0 = 0$ WE OBTAIN $\int_\Omega \phi_0 \mathcal{L} u_1 \, d\underline{x} = 0$.

THUS $$\int_\Omega \phi_0 (-\lambda_1 \phi_0 + 2\pi \phi_0(\underline{x}_0)\,\delta(\underline{x}-\underline{x}_0))\, d\underline{x} = 0.$$

THIS YIELDS THAT $$\lambda_1 = \frac{2\pi [\phi_0(\underline{x}_0)]^2}{\int_\Omega \phi_0^2 \, d\underline{x}}.$$

THEREFORE, WE OBTAIN A TWO-TERM EXPANSION

$$\lambda \sim \mu_0 + \frac{2\pi \nu [\phi_0(\underline{x}_0)]^2}{\int_\Omega \phi_0^2 \, d\underline{x}} + \cdots \qquad \nu = -1/\log[\varepsilon d].$$

d: logarithmic capacitance

<u>REMARK</u> (i) FURTHER TERMS IN THE EXPANSION HAVE THE FORM

$$\lambda \sim \mu_0 + A_1 \nu + A_2 \nu^2 + A_3 \nu^3 + \cdots$$

WHICH IS AN INFINITE-LOGARITHMIC EXPANSION IN POWERS OF ν. SINCE $(\log(\varepsilon d))^{-1}$ DECREASES VERY SLOWLY AS ε DECREASES, WE'D LIKE TO "SUM" THE SERIES.

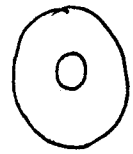
(ii) IF $u = 0$ ON $\partial\Omega$ IS REPLACED BY $\partial_n u = 0$ ON $\partial\Omega$

THEN $\mu_0 = 0$ AND $\phi_0 = \dfrac{1}{|\Omega_0|^{1/2}}$ SO THAT $\int_\Omega \phi_0^2 \, d\underline{x} = 1$.

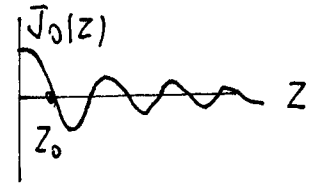
THIS YIELDS $$\lambda \sim \frac{2\pi\nu}{|\Omega_0|} \quad \text{AS } \varepsilon \to 0.$$

<u>EXAMPLE</u> CONSIDER A CIRCULAR DOMAIN WITH A CONCENTRIC HOLE OF RADIUS ε AS

$$\Delta u + \lambda u = 0 \quad \text{IN} \quad \varepsilon < r < 1$$
$$u = 0 \quad \text{ON} \quad r = 1$$
$$u = 0 \quad \text{ON} \quad r = \varepsilon$$



THE UNPERTURBED SOLUTION IS $\phi_0 = J_0(\sqrt{\mu_0}\, r)$ WHERE $J_0(\sqrt{\mu_0}) = 0$ AND $\sqrt{\mu_0} = z_0$, WITH z_0 THE FIRST ZERO OF $J_0(z)$



USING THE PERTURBATION FORMULA WE HAVE $V_c(y) = \log|y|$ SINCE $\Delta_y V_c = 0$, $V_c = 0$ ON $|\underline{y}| = 1$, SO THAT $d = 1$. THEN $\underline{X}_0 = 0$ AND

$$\phi_0(\underline{X}_0) = J_0(0) = 1.$$

THEREFORE,
$$\lambda \sim \mu_0 + \frac{2\pi\nu}{\int_\Omega \phi_0^2(x)\,dx} \sim \mu_0 + \frac{2\pi\nu}{2\pi \int_0^1 r J_0^2(\sqrt{\mu_0}\, r)\,dr}$$

BUT WE RECALL $\int_0^1 r J_0^2(\sqrt{\mu_0}\, r)\,dr = \frac{1}{2}\left(J_0'(\sqrt{\mu_0})\right)^2$ WHEN $J_0(\sqrt{\mu_0}) = 0$

THIS YIELDS THAT

$$(*) \qquad \lambda \sim \mu_0 + \left(-\frac{1}{\log\varepsilon}\right)\left(\frac{2}{[J_0'(\sqrt{\mu_0})]^2}\right) + \cdots$$

NOW WE COMPARE $(*)$ WITH THE EXACT SOLUTION. WE WRITE

$$u = J_0(\sqrt{\lambda}\, r) + c\, Y_0(\sqrt{\lambda}\, r)$$

NOW $u(1) = 0$ SO
$$u = J_0(\sqrt{\lambda}\, r) - \frac{J_0(\sqrt{\lambda})}{Y_0(\sqrt{\lambda})} Y_0(\sqrt{\lambda}\, r)$$

NEXT $u(\varepsilon) = 0$ GIVES THE EIGENVALUE RELATION

$$J_0(\sqrt{\lambda}\,\varepsilon) = \frac{J_0(\sqrt{\lambda})}{Y_0(\sqrt{\lambda})} Y_0(\sqrt{\lambda}\,\varepsilon).$$

WE WRITE THIS AS
$$J_0(\sqrt{\lambda}) = Y_0(\sqrt{\lambda})\,\frac{J_0(\sqrt{\lambda}\,\varepsilon)}{Y_0(\sqrt{\lambda}\,\varepsilon)}$$

NOW WE RECALL THAT

$$J_0(z) \sim 1 + O(z^2) \quad \text{AS } z \to 0$$

$$Y_0(z) \sim \frac{2}{\pi}\left[\log z - \log 2 + \gamma\right] + \cdots \quad z \to 0 \qquad \gamma = \text{euler's constant.}$$

THEREFORE WITH $z = \sqrt{\lambda}$ WE OBTAIN,

$$J_0(z) \sim Y_0(z)\,\frac{\pi}{2}\left[\log(\varepsilon z) - \log 2 + \gamma\right]^{-1} \quad \text{FOR } \varepsilon \ll 1.$$

TO FIND THE ROOT WE LET

$$z = z_0 - \frac{1}{\log \varepsilon} z_1 + \cdots \qquad \text{WITH } J_0(z_0) = 0 \text{ AND } z_0 = \sqrt{\lambda_0}$$

WHERE z_0 IS THE FIRST ROOT OF $J_0(z_0) = 0$. SO $z_0 = \sqrt{\mu_0}$

THEN WE USE TAYLOR SERIES TO OBTAIN

$$J_0(z_0) - \frac{1}{\log \varepsilon} J_0'(z_0) z_1 + \cdots \sim \frac{\pi\, Y_0(z_0)}{2 \log \varepsilon} + \cdots$$

THIS YIELDS THAT $$z_1 = -\frac{\pi}{2}\,\frac{Y_0(z_0)}{J_0'(z_0)}.$$

NOW WE WRITE $$\sqrt{\lambda} = z = z_0 + \left(-\frac{1}{\log \varepsilon}\right) z_1 + \cdots$$

HENCE $$\lambda \sim z_0^2 + \left(-\frac{1}{\log \varepsilon}\right) 2 z_0 z_1 + \cdots$$

THIS YIELDS $$\lambda \sim \mu_0 + \left(-\frac{1}{\log \varepsilon}\right) 2\sqrt{\mu_0}\, z_1$$

THEREFORE $$\lambda \sim \mu_0 + \left(\frac{-1}{\log \varepsilon}\right)\lambda_1 + \cdots$$

WITH $$\lambda_1 = 2\sqrt{\mu_0}\, z_1 = 2\sqrt{\mu_0}\left(-\frac{\pi}{2}\,\frac{Y_0(\sqrt{\mu_0})}{J_0'(\sqrt{\mu_0})}\right) = -\pi\sqrt{\mu_0}\,\frac{Y_0(\sqrt{\mu_0})}{J_0'(\sqrt{\mu_0})}$$

TO WRITE THIS IN A FORM TO COMPARE WITH THE GENERAL THEORY RESULT WE NEED AN IDENTITY.

ASIDE CONSIDER THE SECOND ORDER EQUATION

$$Ly = y'' + p(x)y' + q(x)y$$

THEN DEFINE y_1, y_2 AS TWO SOLUTIONS TO $Ly = 0$.

THEN LET $W = y_1' y_2 - y_1 y_2'$ = WRONSKIAN OF y_1, y_2.

WE DERIVE $W' + p(x) W = 0$

SO $W = c \exp\left(-\int^x p(s)\,ds\right)$.

FOR BESSEL EQUATION $y'' + \frac{1}{r} y' + \lambda y = 0$

WE OBTAIN $p = 1/r$, $y_1 = J_0(\sqrt{\lambda} r)$, $y_2 = Y_0(\sqrt{\lambda} r) \rightarrow W = C/r$.

THEN $\left(\frac{d}{dr} J_0(\sqrt{\lambda} r)\right) Y_0(\sqrt{\lambda} r) - \left(\frac{d}{dr} Y_0(\sqrt{\lambda} r)\right) J_0(\sqrt{\lambda} r) = C/r$.

NOW AS $z \to 0$, $Y_0'(z) \sim 2/\pi z$ AND $J_0(z) = 1$.

THIS GIVES $C = -2/\pi$.

NOW EVALUATE AT $r = 1$ AND SET $\lambda = \mu_0$ WHERE $J_0(\sqrt{\mu_0}) = 0$,

WE GET $\sqrt{\mu_0}\, J_0'(\sqrt{\mu_0})\, Y_0(\sqrt{\mu_0}) = -\frac{2}{\pi}$.

SO $Y_0(\sqrt{\mu_0}) = \dfrac{-2}{\pi \sqrt{\mu_0}\, J_0'(\sqrt{\mu_0})}$

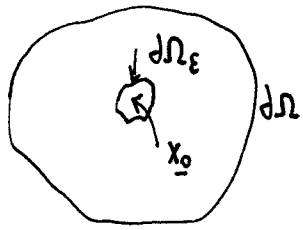
SUBSTITUTING THIS INTO THE RESULT ON THE PREVIOUS PAGE WE OBTAIN

$$\lambda_1 = -\pi \sqrt{\mu_0}\, \frac{Y_0(\sqrt{\mu_0})}{J_0'(\sqrt{\mu_0})} = \frac{2}{(J_0'(\sqrt{\mu_0}))^2}$$

THIS YIELDS $\lambda \sim \mu_0 + \left(\frac{-1}{\log \varepsilon}\right) \frac{2}{(J_0'(\sqrt{\mu_0}))^2} + \cdots$

SUMMING ALL LOGARITHMIC TERMS

WE CONSIDER POISSON'S EQUATION IN A DOMAIN WITH ONE SMALL HOLE GIVEN BY



$$\Delta W = -B \quad \text{IN } \Omega \setminus \Omega_\varepsilon$$
$$W = 0 \quad \text{ON } \partial\Omega$$
$$W = 0 \quad \text{ON } \partial\Omega_\varepsilon$$

WE THEN EXPAND IN THE OUTER REGION

$$W(\underline{x};\varepsilon) = W_0(\underline{x};\nu) + \sigma(\varepsilon)\, W_1(\underline{x};\nu) + \dots \quad \text{WITH } \nu = -1/\log(\varepsilon d)$$

AND WITH $\sigma \ll \nu^K$ FOR ANY $K > 0$.

WE OBTAIN THAT

$$\Delta W_0 = -B \quad \text{IN } X \in \Omega \setminus \{X_0\}$$
$$W_0 = 0 \quad \text{ON } \partial\Omega$$
$$W_0 \text{ IS SINGULAR AS } \underline{X} \to \underline{X}_0 .$$

IN THE INNER REGION WE WRITE

$$\underline{y} = \varepsilon^{-1}(\underline{x} - \underline{x}_0), \quad V(\underline{y};\varepsilon) = W[\underline{x}_0 + \varepsilon \underline{y};\varepsilon]$$

WE THEN EXPAND

$$V(\underline{y};\varepsilon) = V_0(\underline{y};\nu) + \mu_0(\varepsilon)\, V_1(\underline{y};\nu) + \dots$$

WITH $\mu_0 \ll \nu^K$ FOR ANY $K > 0$.

NOW WE GET THAT V_0 SATISFIES

$\Omega_0 = \Omega_\varepsilon / \varepsilon$

$$\Delta_y V_0 = 0 \quad \text{OUTSIDE } \Omega_0$$
$$V_0 = 0 \quad \text{ON } \partial\Omega_0$$

THE MATCHING CONDITION TO LEADING ORDER WILL BE THAT

$$W_0(X;\nu) \underset{X \to X_0}{\sim} W_0(X_0) \underset{|y| \to \infty}{\sim} V_0(y;\nu)$$

THEREFORE, WE TAKE IN TERMS OF AN UNKNOWN FUNCTION $\gamma = \gamma(\nu)$

WITH $\gamma(0) = O(1)$, $\quad V_0(\underline{y};\nu) = \nu\gamma V_C(\underline{y})$

WE THEN OBTAIN THAT $V_c(\underline{y})$ SATISFIES

$$\begin{cases} \Delta_y V_c = 0 & \text{OUTSIDE } \Omega_0 \\ V_c = 0 & \text{ON } \partial\Omega \\ V_c \sim \log|\underline{y}| & \text{AS } |y| \to \infty \end{cases}$$

THERE IS A UNIQUE SOLUTION TO THIS PROBLEM, AND $V_c(\underline{y})$ HAS THE ASYMPTOTIC BEHAVIOR

$$V_c(\underline{y}) \sim \log|y| - \log d + O(1/|\underline{y}|) \quad \text{FOR } |\underline{y}| >> 1.$$

HERE d IS THE LOGARITHMIC CAPACITANCE.

NOW WE WRITE THE FAR-FIELD BEHAVIOR OF V_0 AS $|\underline{y}| \to \infty$ IN THE FORM,

$$V_0(\underline{y};\nu) \sim \nu\gamma\left[\log\left(\frac{|X-X_0|}{\varepsilon}\right) - \log d\right] \sim \nu\gamma\left[-\log(\varepsilon d) + \log|\underline{X}-\underline{X}_0|\right].$$

NOW WITH $\nu = -1/\log[\varepsilon d]$ WE OBTAIN,

$$V_0(\underline{y};\nu) \sim \gamma + \nu\gamma\log|\underline{X}-\underline{X}_0|$$

THE MATCHING CONDITION GIVES $W_0 \sim \nu\gamma\log|\underline{X}-\underline{X}_0| + \gamma$ AS $\underline{X} \to \underline{X}_0$.

THEN WE WRITE

$$\begin{aligned} \Delta W_0 &= -B && \text{IN } \Omega\setminus\{X_0\} \\ W_0 &= 0 && \text{ON } \partial\Omega \\ W_0 &\sim \gamma + \gamma\nu\log|\underline{X}-\underline{X}_0| && \text{AS } \underline{X} \to \underline{X}_0. \end{aligned}$$

NOW WE INTRODUCE $W_{0H}(X)$ AND $G(X;X_0)$, WHICH SATISFY

$$\begin{cases} \Delta W_{0H} = -B & \text{IN } \Omega \\ W_{0H} = 0 & \text{ON } \partial\Omega, \end{cases}$$

AND THE GREEN'S FUNCTION,

$$\begin{aligned} \Delta G &= \delta(\underline{X}-\underline{X}_0) && \text{IN } \Omega \\ G &= 0 && \text{ON } \partial\Omega \end{aligned}$$

IN TERMS OF THIS SOLUTION, $G(\underline{X};X_0) = \frac{1}{2\pi}\log|\underline{X}-\underline{X}_0| + R_0(\underline{X}_0) + o(1)$ AS $\underline{X} \to \underline{X}_0$.

<u>REMARK</u> (i) IF WE WRITE

$$G(X;X_0) = \frac{1}{2\pi}\left[\log|\underline{X}-\underline{X}_0| + R(\underline{X};\underline{X}_0)\right]$$

WHERE $R(\underline{X};\underline{X}_0)$ IS THE "REGULAR PART" OF THE GREEN'S FUNCTION. AS $\underline{X}\to\underline{X}_0$ THEN $R_0(\underline{X}_0) = R(\underline{X}_0;\underline{X}_0)$.

THE FUNCTION $R(\underline{X};X_0)$ DEPENDS ON THE SHAPE OF THE DOMAIN.

THEN WE CAN SOLVE FOR W_0 TO OBTAIN

$$W_0(\underline{X};\nu) = W_{0H}(\underline{X}) + 2\pi\gamma\nu G(\underline{X};\underline{X}_0)$$

NOW AS $\underline{X}\to\underline{X}_0$ WE OBTAIN

$$W_{0H}(\underline{X}_0) + 2\pi\gamma\nu\left[\frac{1}{2\pi}\log|\underline{X}-\underline{X}_0| + R_0(\underline{X}_0)\right] = \gamma + \gamma\nu\log|\underline{X}-\underline{X}_0|.$$

THEREFORE, WE OBTAIN THAT γ SATISFIES

$$\gamma\left[1 - 2\pi\nu R_0(\underline{X}_0)\right] = W_{0H}(\underline{X}_0).$$

THIS YIELDS THAT

$$\gamma = \frac{W_{0H}(\underline{X}_0)}{1-2\pi\nu R_0(\underline{X}_0)}$$

THEREFORE, THE <u>OUTER EXPANSION</u> SATISFIES FOR $|\underline{X}-\underline{X}_0| = O(1)$

$$W \sim W_0(\underline{X};\nu) = W_{0H}(\underline{X}) + \frac{2\pi\nu W_{0H}(\underline{X}_0)}{1-2\pi\nu R_0(\underline{X}_0)} G(\underline{X};\underline{X}_0)$$

IN CONTRAST IN THE <u>INNER REGION</u> FOR $|X-X_0| = O(\varepsilon)$ THAT

$$W \sim V_0(\underline{y};\nu) \sim \frac{\nu W_{0H}(\underline{X}_0)}{1-2\pi\nu R_0(\underline{X}_0)} V_c(y)$$

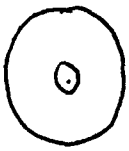
THIS RESULT IS DERIVED FOR $\nu \ll 1$ AND REQUIRES THAT $2\pi\nu R_0(X_0) < 1$ FOR THIS RESULT TO PROVIDE AN APPROXIMATION.

<u>EXAMPLE</u> CONSIDER A PIPE OF RADIUS r_0 CONTAINING A CORE CENTERED AT THE ORIGIN. ASSUME THAT THE CORE IS A CIRCLE OF RADIUS ε.

FOR THIS PROBLEM

$$W_{OH}(r) = \frac{B}{4}(r_0^2 - r^2), \quad G(r;0) = \frac{1}{2\pi}\log r - \frac{1}{2\pi}\log r_0$$

WHICH GIVES $R_0(0) = -\frac{1}{2\pi}\log r_0$, $W_{OH}(0) = B r_0^2/4$, $d = 1$.



THEN $$\gamma = \frac{B r_0^2/4}{1 - 2\pi\nu R_0(0)} = \frac{B r_0^2}{4}\,\frac{1}{1 + \nu\log r_0} \qquad \nu = -1/\log\varepsilon$$

THIS YIELDS THAT $$\gamma = \frac{B r_0^2}{4}\,\frac{\log\varepsilon}{\log(\varepsilon/r_0)}.$$

NOW IN THE OUTER SOLUTION WE OBTAIN

$$W \sim W_0(\underline{X};\nu) = W_{OH} + 2\pi\gamma\nu\, G(\underline{X};0) = \frac{B}{4}(r_0^2 - r^2) + \gamma\nu\log(r/r_0).$$

THEN $$W \sim W_0(\underline{X};\nu) = \frac{B}{4}(r_0^2 - r^2) - \frac{B r_0^2}{4}\,\frac{\log(r_0/r)}{\log(r_0/\varepsilon)}.$$

NOW THE EXACT SOLUTION TO

$$W_{rr} + \frac{1}{r}W_r = -B \qquad \varepsilon < r < r_0$$

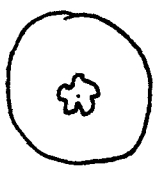
$$W(r_0) = 0, \quad W(\varepsilon) = 0$$

IS GIVEN BY $$W = \frac{B}{4}(r_0^2 - r^2) - \frac{B}{4}(r_0^2 - \varepsilon^2)\,\frac{\log(r_0/r)}{\log(r_0/\varepsilon)},$$

WHICH, BESIDES THE $O(\varepsilon^2)$ TERM, IS THE SAME AS GIVEN BY $W_0(\underline{X};\nu)$.

<u>REMARK</u>

(i) IMAGINE THAT THE INTERIOR CIRCLE $r = \varepsilon$ IS REPLACED BY A HOLE OF ARBITRARY SHAPE CENTERED AT THE ORIGIN.



THEN ALL WE NEED TO DO IS SIMPLY REPLACE ε BY εd WHERE d IS THE LOGARITHMIC CAPACITANCE OF THE BODY.

RECALL THAT d WAS DEFINED BY

$$\Delta_y V_c = 0 \quad \text{OUTSIDE } \Omega_0$$

$$V_c = 0 \quad \text{ON } \partial\Omega_0$$

$$V_c \sim \log|y| - \log d \quad \text{AS } |y| \to \infty$$

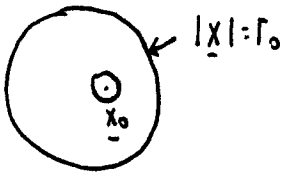
$$\Omega_0 = \Omega_\varepsilon/\varepsilon$$

FOR INSTANCE IF Ω_ε: $x_1^2 + x_2^2/5 = \varepsilon^2 \to \Omega_0$: $y_1^2 + y_2^2/5 = 1$.

THUS THE OUTER SOLUTION IS

$$W \sim W_0 \simeq \frac{B}{4}(r_0^2 - r^2) - \frac{B}{4} r_0^2 \frac{\log(r_0/r)}{\log(r_0/\varepsilon d)}.$$

(ii) NOW SUPPOSE THAT THE HOLE IS OFF-CENTER AT SOME POINT $\underline{x}_0$ IN Ω. ASSUME Ω IS A CIRCLE OF RADIUS r_0.



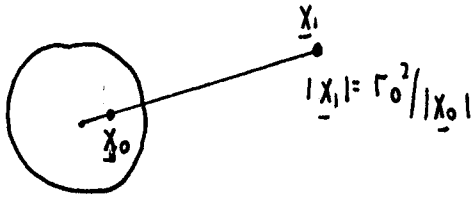
THEN WE MUST SOLVE

$$\Delta G = \delta(\underline{x} - \underline{x}_0) \text{ IN } \Omega; \quad G = 0 \text{ ON } \partial\Omega.$$

$$G(x; x_0) = \frac{1}{2\pi}\log|x - x_0| + R(\underline{x}; \underline{x}_0)$$

WE NEED TO CALCULATE $R_0(\underline{x}_0) = R(\underline{x}_0; \underline{x}_0)$.

TO SOLVE FOR $G(x; x_0)$ WE USE THE <u>METHOD OF IMAGES</u>:



THEN $|\underline{x}_0||\underline{x}_1| = r^2$

THE SOLUTION IS

$$G(x; x_0) = \frac{1}{2\pi}\log|\underline{x} - \underline{x}_0| - \frac{1}{2\pi}\log\left|\,|x - x_1|\,|x_0|/r_0\right|$$

THIS YIELDS THAT $\quad R(\underline{x}; \underline{x}_0) = R_0(\underline{x}_0) = \dfrac{1}{2\pi}\log\left(|\underline{x}_0 - \underline{x}_1|\,\dfrac{|x_0|}{r_0}\right).$

THEN THE OUTER SOLUTION IS SIMPLY

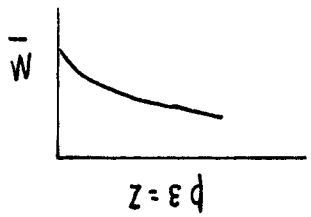
$$W \sim W_0 = \frac{B}{4}(r_0^2 - r^2) + \frac{2\pi\nu\,(B r_0^2/4)}{1 - 2\pi\nu R_0(\underline{x}_0)}\, G(\underline{x}; \underline{x}_0) \quad \text{FOR } |\underline{x} - \underline{x}_0| = O(1)$$

WITH $\quad \nu = -1/\log(\varepsilon d)$.

NOW IF WE DEFINE

$$\bar{W} = \frac{1}{|\Omega_0|} \int_{\Omega_0} W_0 \, d\underline{x}$$

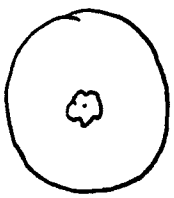
WE COULD PLOT $\bar{W}$ VERSUS εd TO OBTAIN



KAPLUN'S EQUIVALENCE PRINCIPLE:

CORRESPONDING TO EVERY NON-CIRCULAR DOMAIN Ω_0 THERE EXISTS A CORRESPONDING CIRCULAR DOMAIN Ω_{0c} WITH THE SAME $\bar{W}$.

(iii) CONSIDER THE FOLLOWING EIGENVALUE PROBLEM IN A CIRCULAR DOMAIN Ω_0: $r_0 = 1$ CONTAINING A HOLE OF ARBITRARY CROSS-SECTION CENTERED AT THE ORIGIN.

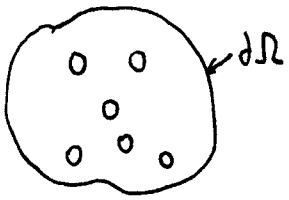


THE SUM OF ALL OF THE LOGARITHMIC TERMS FOR THE PRINCIPAL EIGENVALUE SATISFIES

$$J_0(\sqrt{\lambda^*}) = \frac{\pi}{2} Y_0(\sqrt{\lambda^*}) \left[\log(z\sqrt{\lambda^*}) - \log 2 + \gamma \right]^{-1}$$

WHERE $z = \varepsilon d$ AND γ IS EULER'S CONSTANT. ALSO $\lambda^* \to \mu_0^2$ WHERE $J_0(\mu_0) = 0$ IS THE FIRST ROOT OF $J_0(x) = 0$.

<u>EXAMPLE</u> FIND AN ASYMPTOTIC SOLUTION IN 3-D TO



$$\Delta u = M(\underline{x}) \quad \text{IN} \quad \Omega \setminus \bigcup_{j=1}^{N} \Omega_{\varepsilon_j} \qquad \Omega \text{ IN } \mathbb{R}^3.$$

$$u = \alpha_j \quad \text{ON} \quad \partial\Omega_{\varepsilon_j}, \quad j = 1, \dots, N$$

$$u = 0 \quad \text{ON} \quad \partial\Omega$$

EACH Ω_{ε_j} IS A HOLE OF "RADIUS" ε WITH $\Omega_{\varepsilon_j} \to \{\underline{x}_j\}$ AS $\varepsilon \to 0$. FOR INSTANCE IF $\Omega_{\varepsilon_j}: |\underline{x} - \underline{x}_j| \le \varepsilon$ THEN WE HAVE A SPHERE OF RADIUS ε CENTERED AT $\underline{x} = \underline{x}_j$.

IN THE <u>OUTER REGION</u> WE EXPAND

$$u = u_0 + \varepsilon u_1 + \dots$$

WE HAVE

$$\left.\begin{aligned} \Delta u_0 &= M(\underline{x}) \quad \text{IN } \Omega \\ u_0 &= 0 \quad \text{ON } \partial\Omega \end{aligned}\right\} \text{ UNPERTURBED PROBLEM}$$

AND THAT u_1 SATISFIES

$$\Delta u_1 = 0 \quad \text{IN } \Omega \setminus \{\underline{x}_1, \dots, \underline{x}_N\}$$

$$u_1 = 0 \quad \text{ON } \partial\Omega$$

$$u_1 \text{ SINGULAR AS } \underline{x} \to \underline{x}_j, \quad j = 1, \dots, N.$$

NOW IN THE INNER REGION NEAR $\underline{x} = \underline{x}_j$ WE WRITE $\underline{y} = \varepsilon^{-1}(\underline{x} - \underline{x}_j)$ AND $V(y;\varepsilon) = u(x_j + \varepsilon y, \varepsilon) = V_0(y) + \dots$. THE MATCHING CONDITION YIELDS THAT $V_0 \to u_0(\underline{x}_j)$ AS $|y| \to \infty$ SO THAT

$$\Delta_y V_0 = 0 \quad \underline{y} \text{ OUTSIDE } \Omega_j = \Omega_{\varepsilon_j}/\varepsilon$$

$$V_0 = \alpha_j \quad \text{ON } \partial\Omega_j$$

$$V_0 \to u_0(\underline{x}_j) \quad \text{AS } |y| \to \infty$$

THE SOLUTION IS SIMPLY

$$V_0 = u_0(\underline{x}_j) + (\alpha_j - u_0(\underline{x}_j))\, V_c(\underline{y})$$

WHERE $V_c(\underline{y})$ SATISFIES

$$\Delta_y V_c = 0, \quad \underline{y} \text{ OUTSIDE } \Omega_j$$

$$V_c = 1, \quad \underline{y} \text{ ON } \Omega_j$$

$$V_c \sim C_j / |\underline{y}| \quad \text{AS } |y| \to \infty.$$

C_j CAPACITANCE OF THE jth hole.

THIS YIELDS THAT

$$V_0 \sim u_0(\underline{x}_j) + (\alpha_j - u_0(\underline{x}_j))\, C_j / |\underline{y}| \quad \text{AS } |\underline{y}| \to \infty.$$

NOW THE MATCHING CONDITION IS SIMPLY

$$u_0(\underline{x}_j) + \varepsilon u_1 \underset{x \to x_j}{\sim} u_0(\underline{x}_j) + (\alpha_j - u_0(\underline{x}_j)) \frac{\varepsilon C_j}{|\underline{x} - \underline{x}_j|} + \cdots$$

THIS YIELDS THAT $\quad u_1 \sim (\alpha_j - u_0(\underline{x}_j))\, C_j / |\underline{x} - \underline{x}_j| \quad$ AS $\underline{x} \to \underline{x}_j$.

THEREFORE, THE PROBLEM FOR u_1 IS SIMPLY

$$\Delta u_1 = -4\pi \sum_{j=1}^{N} (\alpha_j - u_0(\underline{x}_j))\, C_j\, \delta(\underline{x} - \underline{x}_j) \quad \text{IN } \Omega.$$

$$u_1 = 0 \quad \text{ON } \partial\Omega.$$

THE SOLUTION IS SIMPLY

$$u_1 = -4\pi \sum_{j=1}^{N} (\alpha_j - u_0(\underline{x}_j))\, C_j\, G(x; x_j)$$

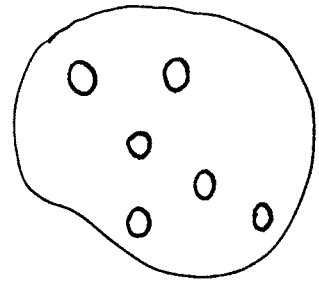
WHERE $\quad G(\underline{x}; \underline{x}_j)$ IS THE GREEN'S FUNCTION SATISFYING

$$\Delta G = \delta(x - x_j) \quad \text{IN } \Omega$$

$$G = 0 \quad \text{ON } \partial\Omega$$

<u>EXAMPLE</u> NOW CONSIDER THE CORRESPONDING NEUMANN PROBLEM

$$(*)\quad \begin{cases} \Delta u = M(\underline{x}) \text{ IN } \Omega \setminus \bigcup_{j=1}^{N} \Omega_{\varepsilon_j} \qquad \Omega \subset \mathbb{R}^3. \\ \partial_n u = 0 \text{ ON } \partial\Omega \\ u = \alpha_j \text{ ON } \partial\Omega_{\varepsilon_j}, \; j = 1, \dots, N. \end{cases}$$



WE ASSUME FOR SIMPLICITY THAT $\Omega_{\varepsilon_j} : |x - x_j| = \varepsilon r_j$

SO THAT WE HAVE N-SMALL SPHERES OF RADIUS εr_j.

<u>REMARK</u> i) WE CANNOT EXPAND $u = u_0 + \varepsilon u_1 + \cdots$

SINCE $\Delta u_0 = M(\underline{x})$ IN Ω

$\partial_n u_0 = 0$ ON $\partial\Omega$

HAS NO SOLUTION IN GENERAL UNLESS $\int_\Omega M(\underline{x})\,d\underline{x} = 0$.

(ii) ALSO RECALL THAT THE EIGENVALUE PROBLEM

$$\Delta\phi + \lambda\phi = 0 \text{ IN } \Omega \setminus \bigcup_{j=1}^{N} \Omega_{\varepsilon_j}$$

$$\partial_n \phi = 0 \text{ ON } \partial\Omega$$

$$\phi = 0 \text{ ON } \partial\Omega_{\varepsilon_j}, \; j = 1, \dots, N.$$

HAS THE PRINCIPAL EIGENVALUE (FROM PAGE 7) THAT

$$\lambda \sim \frac{4\pi\varepsilon}{|\Omega|} \sum_{j=1}^{N} C_j.$$

NOTICE THAT IN THE jth INNER REGION

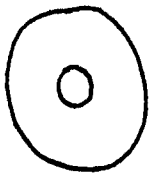
$$\Delta_y V_c = 0, \quad |\underline{y}| \geq r_j$$

$$V_c = 1, \quad |\underline{y}| = r_j, \quad V_c \sim C_j/|\underline{y}| \text{ AS } |\underline{y}| \to \infty$$

$\longrightarrow$ $V_c = r_j/|y|$. THIS IMPLIES THAT $C_j = r_j$

THIS EIGENVALUE WITH $\lambda = O(\varepsilon)$ SUGGESTS THAT THE EXPANSION FOR u SHOULD BE $u = u_0/\varepsilon + u_1 + \varepsilon u_2 + \cdots$

(iii) CONSIDER THE SPECIAL CASE



$$\Delta u = M \quad \text{IN} \quad \varepsilon < r < 1 \qquad \text{WITH } M = \text{CONSTANT}$$

$$u_r = 0 \quad \text{ON} \quad r = 1$$

$$u = 1 \quad \text{ON} \quad r = \varepsilon$$

THEN WE CALCULATE $u = \dfrac{Mr^2}{6} + \dfrac{A}{r} + B.$

NOW $u_r = 0$ ON $r = 1 \longrightarrow \dfrac{M}{3} - A = 0$ SO $A = M/3$

$u = 1$ ON $r = \varepsilon \rightarrow \dfrac{M\varepsilon^2}{6} + \dfrac{M}{3\varepsilon} + B = 1$

THIS YIELDS THAT $B = 1 - \dfrac{M}{3\varepsilon} - \dfrac{M\varepsilon^2}{6}.$

THEN $u = \dfrac{M}{6}(r^2 - \varepsilon^2) + \dfrac{M}{3}\left(\dfrac{1}{r} - \dfrac{1}{\varepsilon}\right) + 1.$

NOTICE THAT IN THE <u>OUTER REGION</u>

$$u = u_0/\varepsilon + u_1 + \cdots$$

AND IN THE <u>INNER REGION</u> $v = v_0/\varepsilon + v_1 + \cdots$

WE RETURN TO THE GENERAL PROBLEM (*)

IN THE OUTER REGION WE EXPAND

$$u = u_0/\varepsilon + u_1 + \varepsilon u_2 + \cdots$$

WE OBTAIN THAT $\Delta u_0 = 0$ AND SO $u_0 = \mu$ WHERE μ IS A CONSTANT.

THE PROBLEMS FOR u_1 AND u_2 ARE

$$\Delta u_1 = M(\underline{x}) \quad \text{IN} \quad \Omega \setminus \{\underline{x}_1, \ldots, \underline{x}_N\}$$

$$\partial_n u_1 = 0 \quad \text{ON} \quad \partial\Omega$$

$$u_1 \text{ SINGULAR AS } \underline{x} \rightarrow \underline{x}_j$$

THEN U_2 SATISFIES

$$\Delta U_2 = 0 \quad \text{IN} \quad \Omega \setminus \{ \underline{X}_1, \dots, \underline{X}_N \}$$

$$\partial_n U_2 = 0 \quad \text{ON} \quad \partial\Omega$$

U_2 SINGULAR AS $\underline{X} \to \underline{X}_j$.

NOW IN THE INNER REGION WE LET $\underline{y} = \varepsilon^{-1}(\underline{X} - \underline{X}_j)$ AND WE EXPAND

$$V = \frac{V_0}{\varepsilon} + V_1 + \varepsilon V_2 + \dots$$

WE THEN OBTAIN THAT, UPON USING MATCHING CONDITION $V_0 \to U_0$ AT ∞ THAT

$$\Delta_y V_0 = 0 \quad |\underline{y}| \geq r_j$$

$$V_0 = 0 \quad |\underline{y}| = r_j$$

$$V_0 \to \mu \quad \text{AS} \quad |\underline{y}| \to \infty$$

THE SOLUTION IS WRITTEN AS $V_0 = \mu(1 - V_C)$

WHERE $V_C(\underline{y})$ SATISFIES

$$\Delta_y V_C = 0, \quad |\underline{y}| \geq r_j$$

$$V_C = 1, \quad |\underline{y}| = r_j$$

$V_C \sim C_j / |\underline{y}|$ AS $|\underline{y}| \to \infty$. WE GET $V_C = \dfrac{r_j}{|y|}$ SO $C_j = r_j$.

THEREFORE, THE MATCHING CONDITION BECOMES NEAR $X = X_j$

$$\mu/\varepsilon + U_1 + \varepsilon U_2 + \dots \underset{\underline{X} \to \underline{X}_j}{\sim} \frac{V_0}{\varepsilon} + V_1 + \dots \underset{y \to \infty}{=} \frac{\mu}{\varepsilon}\left(1 - \frac{C_j}{|\underline{X} - X_j|}\varepsilon\right) + V_1$$

THEREFORE, WE OBTAIN

$$U_1 \longrightarrow -\frac{\mu C_j}{|\underline{X} - X_j|} \quad \text{AS} \quad \underline{X} \to \underline{X}_j$$

THE PROBLEM FOR U_1 IS SIMPLY:

$$\Delta U_1 = M(\underline{X}) \quad \text{IN } \Omega \setminus \{\underline{X}_1, \dots, \underline{X}_N\}$$

$$\partial_n U_1 = 0 \quad \text{ON } \partial\Omega$$

$$U_1 \sim \frac{-\mu C_j}{|\underline{X} - \underline{X}_j|} \quad \text{AS } X \to X_j, \; j = 1, \dots, N$$

THIS PROBLEM IS EQUIVALENT TO

$$\begin{cases} \Delta U_1 = M(\underline{X}) + 4\pi\mu \sum_{j=1}^{N} C_j \, \delta(\underline{X} - \underline{X}_j) & \text{IN } \Omega \\ \partial_n U_1 = 0 & \text{ON } \partial\Omega \end{cases}$$

THEN USING THE DIVERGENCE THEOREM

$$\int_\Omega M(\underline{X}) \, d\underline{X} + 4\pi\mu \sum_{j=1}^{N} C_j = 0$$

THIS YIELDS $\quad \mu = -\dfrac{1}{4\pi} \dfrac{\int_\Omega M(\underline{X}) \, d\underline{X}}{\sum_{j=1}^{N} C_j}$,

WHICH DETERMINES THE LEADING ORDER OUTER SOLUTION

$$U \sim \mu/\varepsilon$$

NOW WE PROCEED TO SECOND ORDER. WE NEXT SOLVE FOR U_1 EXPLICITLY. WE INTRODUCE THE NEUMANN GREEN'S FUNCTION $G(X; X_j)$ DEFINED BY THE SOLUTION TO

$$\begin{cases} \Delta G = \dfrac{1}{|\Omega|} - \delta(X - X_j), & \text{IN } \Omega \\ \partial_n G = 0 & \text{ON } \partial\Omega \\ \int_\Omega G \, d\underline{X} = 0 \end{cases}$$

NOTICE THAT $G(X; X_j)$ IS UNIQUE AND EXISTS SINCE $\int_\Omega \left(\frac{1}{|\Omega|} - \delta(X - X_j)\right) dX = 0$

THEN WE HAVE

$$G(\underline{x};\underline{x}_j) = \frac{1}{4\pi|\underline{x}-\underline{x}_j|} + R(\underline{x};\underline{x}_j) \qquad R = \text{Regular part of NEUMANN GREEN'S FUNCTION.}$$

THEREFORE AS $\underline{x} \to \underline{x}_j$ WE OBTAIN

$$G(x;x_j) \sim \frac{1}{4\pi|\underline{x}-\underline{x}_j|} + R_j, \qquad R_j \equiv R(\underline{x}_j;\underline{x}_j).$$

NOW WE WRITE THE PROBLEM FOR u_1 AS

$$\Delta u_1 = \left(M(\underline{x}) - \frac{1}{|\Omega|}\int_\Omega M\,d\underline{x}\right) + \left(\frac{1}{|\Omega|}\int_\Omega M(\underline{x})\,d\underline{x} + 4\pi\mu\sum_{j=1}^N C_j\,\delta(\underline{x}-\underline{x}_j)\right) \text{ IN } \Omega$$

$$\partial_n u_1 = 0 \quad \text{ON} \quad \partial\Omega$$

NOW WE WRITE THE SOLUTION AS

$$(*) \quad u_1 = u_{1p} - 4\pi\mu\sum_{i=1}^N C_i\,G(x;x_i) + \bar{u}_1$$

HERE $\bar{u}_1$ IS A CONSTANT AND u_{1p} SATISFIES

$$\Delta u_{1p} = \left(M(\underline{x}) - \frac{1}{|\Omega|}\int M\,d\underline{x}\right) \text{ IN } \Omega$$

$$\partial_n u_{1p} = 0 \quad \text{ON} \quad \partial\Omega$$

$$\int_\Omega u_{1p}\,d\underline{x} = 0$$

NOTICE THAT

$$\Delta\left[-4\pi\mu\sum_{i=1}^N C_i G\right] = -4\pi\mu\sum_{i=1}^N C_i\left(\frac{1}{|\Omega|} - \delta\right) = 4\pi\mu\sum_{i=1}^N C_i\,\delta + \frac{1}{|\Omega|}\int_\Omega M\,d\underline{x}$$

NOTICE THAT u_{1p} IS UNIQUELY DETERMINED. NOW SINCE $\int_\Omega u_{1p}\,d\underline{x} = 0$ AND $\int_\Omega G(\underline{x};\underline{x}_j)\,d\underline{x} = 0$ THEN $\int_\Omega u_1\,d\underline{x} = \bar{u}_1|\Omega|$, AND SO $\bar{u}_1 = \frac{1}{|\Omega|}\int_\Omega u_1\,d\underline{x}$.

NOW WE EXPANSION $(*)$ AS $\underline{x} \to \underline{x}_j$ FOR EACH $j = 1, \ldots, N$ TO OBTAIN

$$u_1 \sim u_{1p}(\underline{x}_j) - 4\pi\mu\left(\sum_{i\neq j}^N C_i\,G(\underline{x}_j;\underline{x}_i) + C_j\left(\frac{1}{4\pi|\underline{x}-\underline{x}_j|} + R_j\right)\right) + \bar{u}_1.$$

WE WRITE $\quad u_1 \sim B_j + \bar{u}_1 - \dfrac{\mu C_j}{|\underline{x}-\underline{x}_j|} \quad$ AS $\underline{x} \to \underline{x}_j$

WHERE $$B_j = u_{1p}(\underline{x}_j) - 4\pi\mu\left(C_j R_j + \sum_{i\neq j}^{N} C_i\, G(\underline{x}_j ; x_i)\right).$$

THEN THE MATCHING CONDITION IS

$$\frac{\mu}{\varepsilon} + u_1 + \varepsilon u_2 + \cdots \sim \frac{V_0}{\varepsilon} + V_1 + \cdots$$

WRITING THIS OUT WE GET

$$\frac{\mu}{\varepsilon} + \bar{u}_1 + B_j - \frac{\mu C_j}{|x - x_j|} + \varepsilon u_2 \sim \frac{\mu}{\varepsilon}\left(1 - \frac{C_j \varepsilon}{|\underline{x} - \underline{x}_j|}\right) + V_1$$

THIS IMPLIES THAT V_1 SATISFIES, FOR EACH $j = 1, \ldots, N$,

$$\Delta_y V_1 = 0, \quad |\underline{y}| \geq r_j$$

$$V_1 = \alpha_j \quad |\underline{y}| = r_j$$

$$V_1 \sim \bar{u}_1 + B_j \quad \text{AS } |\underline{y}| \to \infty$$

THEN WE HAVE $$V_1 = (\bar{u}_1 + B_j) - [(\bar{u}_1 + B_j) - \alpha_j]\, V_c$$

WHERE $$\Delta_y V_c = 0, \quad |\underline{y}| \geq r_j$$

$$V_c = 1, \quad |y| = r_j$$

$$V_c \sim C_j / |\underline{y}| \quad \text{AS } |y| \to \infty.$$

THEREFORE, $$V_1 \sim (\bar{u}_1 + B_j) - [(\bar{u}_1 + B_j) - \alpha_j]\frac{C_j}{|\underline{y}|} \quad \text{AS } |\underline{y}| \to \infty$$

THIS IMPLIES THAT u_2 SATISFIES

$$\Delta u_2 = 0 \quad \text{IN } \Omega \setminus \{x_1, \ldots, x_N\}$$

$$\partial_n u_2 = 0 \quad \text{ON } \partial\Omega$$

$$u_2 \sim [\alpha_j - (\bar{u}_1 + B_j)]\frac{C_j}{|\underline{x} - \underline{x}_j|} \quad \text{AS } \underline{x} \to \underline{x}_j.$$

THEREFORE, $$\Delta u_2 = -4\pi \sum_{j=1}^{N} [\alpha_j - (\bar{u}_1 + B_j)]\, C_j\, \delta(\underline{x} - \underline{x}_j)$$

FINALLY, WE DETERMINE $\bar{u}_1$ BY DIVERGENCE THEOREM,

$$\sum_{j=1}^{N} \left[\alpha_j - (\bar{u}_1 + B_j) \right] C_j = 0$$

THIS YIELDS $\bar{u}_1 \sum_{j=1}^{N} C_j = \sum_{j=1}^{N} (\alpha_j - B_j) C_j$.

THIS YIELDS THAT
$$\bar{u}_1 = \frac{\sum_{j=1}^{N} (\alpha_j - B_j) C_j}{\sum_{j=1}^{N} C_j}$$

IN SUMMARY, A TWO-TERM OUTER EXPANSION IS GIVEN BY

$$u \sim \frac{\mu}{\varepsilon} + u_1 + \cdots$$

WITH
$$\mu = -\frac{1}{4\pi} \frac{\int_\Omega M(\underline{x})\, d\underline{x}}{\sum_{j=1}^{N} C_j}$$

WHILE
$$u_1 = u_{1p}(\underline{x}) + \bar{u}_1 - 4\pi\mu \sum_{i=1}^{N} C_i\, G(X; X_i)$$

WITH $\bar{u}_1$ IS GIVEN ABOVE

AND
$$B_j = u_{1p}(\underline{x}_j) - 4\pi\mu \left(C_j R_j + \sum_{i \neq j}^{N} C_i\, G(X_i; X_j) \right).$$

NOW CONSIDER THE SPECIAL CASE WITH M CONSTANT AND

$$\Delta u = M \quad \text{IN } \varepsilon < r < 1$$
$$u_r = 0 \quad \text{ON } r = 1$$
$$u = 1 \quad \text{ON } r = \varepsilon$$

EXACT SOLUTION
$$u = \frac{M}{6}(r^2 - \varepsilon^2) + \frac{M}{3}\left(\frac{1}{r} - \frac{1}{\varepsilon}\right) + 1$$

THEN $j = 1$, $\underline{X}_1 = \underline{0}$, $C_1 = 1$, $\alpha_1 = 1$ AND $u_{1p} \equiv 0$ SINCE $M =$ CONSTANT.

NOW
$$\Delta G = \frac{1}{|\Omega|} \quad \text{IN } \Omega \setminus \{0\}$$
$$G_r = 0 \quad \text{ON } r = 1$$
$$G \sim +\frac{1}{4\pi r} + R_1 \quad \text{AS } r \to 0$$
$$\int G\, d\underline{x} = 0$$

WE CALCULATE
$$G = D_0 + \frac{D_1}{r} + \frac{r^2}{|\Omega| 6}.$$

NOW $D_1 = \frac{1}{4\pi}$, $|\Omega| = 4\pi/3$

THEN $G = D_0 + \frac{1}{4\pi r} + \frac{r^2}{8\pi}$.

NOTICE THAT $G_r = 0$ ON $r = 1$. NOW $\int_0^1 r^2 G\, dr = 0$ SO THAT

$$\int_0^1 \left(D_0 r^2 + \frac{r}{4\pi} + \frac{r^4}{8\pi}\right) dr = 0 \rightarrow \frac{D_0}{3} + \frac{1}{8\pi} + \frac{1}{40\pi} = 0$$

THUS,
$$\frac{D_0}{3} + \frac{6}{40\pi} = 0 \qquad D_0 = -\frac{9}{20\pi}.$$

THEN
$$G = -\frac{9}{20\pi} + \frac{1}{4\pi r} + \frac{r^2}{8\pi}.$$

NOW AS $r \to 0$, $G \sim \frac{1}{4\pi r} + R_1$, WITH $R_1 = -9/20\pi$.

NOW WE CALCULATE $\mu = -\frac{1}{4\pi C_1}\int_\Omega M\, d\underline{x} = -\frac{M}{4\pi(1)}\left(\frac{4\pi}{3}\right) = -\frac{M}{3}$.

NOW WE CALCULATE THAT

$$B_1 = -4\pi\mu C_1 R_1 = -4\pi\left(-\frac{M}{3}\right)\left(-\frac{9}{20\pi}\right) = -\frac{3M}{5\pi}.$$

THEN $\bar{U}_1 = (\alpha_1 - B_1)\frac{C_1}{C_1} = \left(1 + \frac{3M}{5}\right)$.

THEN
$$U_1 = U_{1P} + \bar{U}_1 - 4\pi\mu C_1 G = 1 + \frac{3M}{5} - 4\pi\left(-\frac{M}{3}\right)\left[-\frac{9}{20\pi} + \frac{1}{4\pi r} + \frac{r^2}{8\pi}\right]$$

$$U_1 = 1 + \frac{4\pi}{3} M\left(\frac{1}{4\pi r} + \frac{r^2}{8\pi}\right) = 1 + \frac{M}{3r} + \frac{Mr^2}{6}.$$

THEREFORE TO TWO-TERMS $U \sim \frac{\mu}{\varepsilon} + U_1 + \ldots$

WITH
$$U \sim 1 + \frac{M}{3}\left(\frac{1}{r} - \frac{1}{\varepsilon}\right) + \frac{Mr^2}{6}$$

WHICH AGREES WITH THE EXACT SOLUTION UP TO TERMS OF $O(\varepsilon^2)$.