We first recall a few basic results from potential theory. Suppose that \( \Delta u = \delta(x-x_0) \) in \( \mathbb{R}^n \) with \( n = 2 \), or \( n = 3 \).

Then
\[
\Delta u = \frac{1}{4\pi |x-x_0|^3} \text{ in } 3\text{-dim.}
\]
\[
\Delta u = \frac{1}{2\pi} \log |x-x_0|, \quad \text{in } 2\text{-dim.}
\]

We recall the derivation of this in 3-dim. We take a small sphere of radius \( \epsilon \) about \( x_0 \) so that \( \Omega_\epsilon = \{ |x-x_0| < \epsilon \} \). Then in a neighborhood of \( x_0 \) we let \( \Gamma = |x-x_0| \) and then
\[
\Delta u = \nabla u \cdot \nabla \Gamma = 0 \quad \text{for } \Gamma > 0 \quad \text{so that } \quad u = B/\Gamma.
\]

We then use the divergence theorem
\[
\int_{\Omega_\epsilon} \Delta u \, d\mathbf{x} = \int_{\partial \Omega_\epsilon} \nabla u \cdot \mathbf{n} \, d\mathbf{s} = 4\pi |\nabla u| = 1.
\]

Then let \( \epsilon \to 0 \)
\[
\int_{\partial \Omega_\epsilon} \nabla u \cdot \mathbf{n} \, d\mathbf{s} = \int_{\partial \Omega_\epsilon} \left( \frac{\partial u}{\partial \Gamma} \right) \, d\mathbf{s} = 4\pi \left( \frac{\partial u}{\partial \Gamma} \right)_{\Gamma = \epsilon} = 1.
\]

We calculate \( \frac{\partial u}{\partial \Gamma} = -B/\Gamma^2 \) so that
\[
4\pi \left( -\epsilon^2 \frac{B}{\epsilon^3} \right) = 1
\]
which yields \( B = -\frac{1}{4\pi} \), and so
\[
u \sim \frac{1}{4\pi |x-x_0|} \quad \text{as} \quad x \to x_0.
\]

In contrast in two-dimensions we let \( \Omega_\epsilon = \{ |x-x_0| < \epsilon \} \) and then for \( \Gamma = |x-x_0| > 0 \), \( \Delta u = \nabla u \cdot \nabla \Gamma = 0 \) so \( u = B \log \Gamma \) is the singular solution. Then using the divergence theorem
\[
\lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \Delta u \, d\mathbf{x} = \lim_{\epsilon \to 0} \int_0^{2\pi} \frac{\partial u}{\partial \Gamma} \epsilon \, d\theta = \lim_{\epsilon \to 0} \int_0^{2\pi} \frac{\partial (\frac{B}{\epsilon}) \theta}{\partial \Gamma} \epsilon \, d\theta = \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \left( \frac{\partial u}{\partial \Gamma} \right) \, d\mathbf{x} = 2\pi B = 1,
\]

This yields that \( 2\pi B = 1 \) and so
\[
u \sim \frac{1}{2\pi} \log |x-x_0|.
Remark

(i) If we want to solve in 3-D,

\[ \Delta u = 0 \text{ in } \Omega \setminus \{X_0\} \]
\[ u = 0 \text{ on } \partial \Omega \]

and \( u \sim \frac{A}{1|X - X_0|} \)

Then we recall \( -\frac{1}{4\pi} \frac{1}{|X - X_0|} \rightarrow \delta(X - X_0) \), and so

by scaling \( \frac{A}{|X - X_0|} \rightarrow -4\pi A \delta(X - X_0) \).

This yields that

\[ \begin{cases} \Delta u = -4\pi A \delta(X - X_0) \text{ in } \Omega \\ u = 0 \text{ on } \partial \Omega \end{cases} \]

(ii) Similarly, \( \Delta u = 0 \text{ in } \Omega \setminus \{X_0\} \), \( \Omega \) is two-dimensional

\[ u = 0 \text{ on } \partial \Omega \]
\[ u \sim A \log|X - X_0| \]

is equivalent to \( \Delta u = 2\pi A \delta(X - X_0) \) in \( \Omega \) with \( u = 0 \) on \( \partial \Omega \).

Now suppose that we have lower-order terms of the form

\[ \Delta u - p^2 u = \delta(X - X_0) \text{ in either 2-D or 3-D.} \]

Then we calculate

\[ u = -e^{-\frac{r}{2}} \text{ in } \mathbb{R}^3; \quad u = -\frac{1}{2\pi} \log|X_0| \text{ in } \mathbb{R}^2. \]

But critically, the form of the singularity to leading-order near the singular point is independent of the lower-order term \( p u \). In fact,

\[ u \sim -\frac{1}{4\pi|X - X_0|} \]
\[ u \sim -\frac{1}{2\pi} \log|X - X_0|. \]

This follows from the observation that

\[ \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} u \, d\varepsilon = 0. \]
EIGENVALUE ASYMPTOTICS IN 3-D

Let $\Omega$ be a 3-D bounded domain with a hole of "radius" $0(\varepsilon)$ removed from $\Omega$.

\[ \Delta u + \lambda u = 0 \text{ in } \Omega \setminus \Omega_\varepsilon \]
\[ u = 0 \text{ on } \partial \Omega \]
\[ u = 0 \text{ on } \partial \Omega_\varepsilon \]
\[ \int_{\Omega \setminus \Omega_\varepsilon} u^2 \, dx = 1 \]

We assume that $\Omega_\varepsilon$ shrinks to a point $x_0$ as $\varepsilon \to 0$.

For instance, $\Omega_\varepsilon$ could be the sphere $|x - x_0| < \varepsilon$.

The unperturbed problem is

\[ \Delta \phi + \mu \phi = 0 \text{ in } \Omega \]
\[ \phi = 0 \text{ on } \partial \Omega \]
\[ \int_{\Omega} \phi^2 \, dx = 1 \]

This problem has eigenpairs as $\phi_j(x)$, $\mu_j$ $j = 0, 1, 2, \ldots$

with $\int_{\Omega} \phi_j \phi_k \, dx = 0$ $j \neq k$ and $\phi_0(x) > 0$ for $x$ inside $\Omega$.

We now look for an eigenpair of $(\mathcal{e})$ near the principal eigenpair $\phi_0(x)$, $\mu_0$.

We proceed by the method of matched asymptotic expansions.

We first expand the eigenvalue for $(\mathcal{e})$ as

$\lambda \approx \mu_0 + \nu(\varepsilon) \lambda_1 + \ldots$

$\nu(\varepsilon) \to 0$ as $\varepsilon \to 0$.

In the outer region away from the hole, we expand
\[ U = \phi_0 (x) + \sqrt{\varepsilon} \, U_1 + \ldots \]

Now, since \( \Omega_\varepsilon \to \Omega \setminus \{X_0\} \) as \( \varepsilon \to 0 \), we obtain that

\[
\begin{align*}
\Delta U_1 + \mu_0 U_1 &= -A_1 \phi_0 \quad \text{in} \quad \Omega \setminus \{X_0\} \\
U_1 &= 0 \quad \text{on} \quad \partial \Omega \\
\int_{\Omega} 2U_1 \phi_0 \, dx &= 0
\end{align*}
\]

(1)

Now we construct an inner expansion near the hole. We let \( y = \varepsilon^{-1} (x - X_0) \) and \( v(y; \varepsilon) = u(x_0 + \varepsilon y) \) satisfies

\[
\Delta v + \frac{1}{\varepsilon^2} v = 0 \quad \text{outside} \quad \Omega_0, \quad \Omega_0 = \Omega_\varepsilon / \varepsilon
\]

\[
v = 0 \quad \text{on} \quad \partial \Omega_0.
\]

Then we expand \( v = v_0 + \sqrt{\varepsilon} \, v_1 + \ldots \).

We substitute to obtain

\[
\begin{align*}
\Delta v_0 &= 0 \quad \text{outside} \quad \Omega_0 \\
v_0 &= 0 \quad \text{on} \quad \partial \Omega_0.
\end{align*}
\]

(2)

\[
v_0 \to \phi_0 (x_0) \quad \text{as} \quad |y| \to \infty.
\]

The matching condition is that

\[
\begin{align*}
\phi_0 (x) + \sqrt{\varepsilon} \, U_1 + \ldots &\sim v_0 + \sqrt{\varepsilon} \, v_1 + \ldots \\
 x &\to X_0 \quad \text{as} \quad |y| \to \infty
\end{align*}
\]

Therefore,

\[
v_0 \to \phi_0 (x_0) \quad \text{as} \quad |y| \to \infty.
\]

Now we write the solution to (2) as

\[
v_0 = \phi_0 (x_0) \left( 1 - v_c (y) \right)
\]

Then we obtain that \( v_c (y) \) satisfies
\[ \Delta y \nu C = 0, \text{ y outside } \Omega_0 \]
\[ \nu C : 1 \quad \text{ y on } \partial \Omega_0 \]
\[ \nu C \to 0 \quad \text{ as } |y| \to \infty. \]

The solution has the well-known asymptotic behavior
\[ \nu C \sim \frac{C}{|y|} + \ldots, \quad \text{ as } |y| \to \infty, \]
where \( C > 0 \) is called the electrostatic capacitance of \( \Omega_0 \).

Remark (i) If \( \Omega_0 : |x-x_0| \leq \varepsilon \) then \( \Omega_0 : |y| \leq 1 \).

We let \( r = |y| \) so that in 3-D, \( \nu C = \nu C(r) \)
\[ \nu C'' + \frac{2}{r} \nu C' = 0, \quad r > 1 \]
\[ \nu C = 1 \quad \text{ on } \gamma = 1 \]
then \( \nu C = \frac{1}{r} \) for \( r \geq 1 \) and so \( C = 1 \).

(ii) \( C \) is known analytically for a wide variety of shapes
including ellipsoids etc.; otherwise we calculate it numerically.

(iii) A famous isoperimetric inequality of Szegő states that
of all bodies \( \Omega_0 \) of the same volume, the sphere has
the smallest capacitance.

Now we return to \( \nu_0 \) and note that
\[ \nu_0 \sim \phi(x_0) \left( \frac{1}{|y|} \right), \quad |y| \to \infty \]

Let \( y = e^{i}(x-x_0) \) and recall the matching condition (3)
\[ \phi_0(x_0) + \nu(e) u_1 \sim \phi_0(x_0) - \frac{\phi_0(x_0) e C}{|x-x_0|} + \ldots. \]

This yields that \( \nu(e) = e \), and \( u_1 \to -\frac{\phi_0(x_0) C}{|x-x_0|} \) as \( x \to x_0 \).
THEN WE RETURN TO (1) AND WRITE

\[ \Delta u_1 + \mu_0 u_1 = -\lambda_1 \phi_0 \quad \text{in} \quad \Omega \setminus \{ x_0 \} \]

\[ u_1 = 0 \quad \text{on} \quad \partial \Omega \]

\[ u_1 \rightarrow -\phi_0(x_0) \frac{c}{|x-x_0|} \quad \text{as} \quad x \rightarrow x_0. \]

\[ \int_{\Omega} u_1 \phi_0 \, d x = 0. \]

SINCE \(-\frac{1}{4\pi |x-x_0|} \rightarrow \delta(x-x_0)\), THEN WE OBTAIN

\[ \lambda_1 \phi_0 = \Delta u_1 + \mu_0 u_1 = -\lambda_1 \phi_0 + 4\pi c \phi_0(x_0) \delta(x-x_0) \quad \text{in} \quad \Omega \]

\[ u_1 = 0 \quad \text{on} \quad \partial \Omega \]

NOW WE USE GREEN’S SECOND IDENTITY WITH \( \lambda \phi_0 = 0 \) TO OBTAIN

\[ \int_{\Omega} (\phi_0 \Delta u_1 - \nabla \phi_0 \cdot \nabla \phi_0) \, d x = \int_{\partial \Omega} (\phi_0 \frac{\partial u_1}{\partial n} - u_1 \frac{\partial \phi_0}{\partial n}) \, d s \]

SINCE \( \phi_0 = u_1 = 0 \) ON \( \partial \Omega \) AND \( \lambda \phi_0 = 0 \) WE OBTAIN

\[ 0 = \int_{\Omega} \phi_0 \Delta u_1 \, d x = \int_{\Omega} \phi_0 \left( -\lambda_1 \phi_0 + 4\pi c \phi_0(x_0) \delta(x-x_0) \right) \, d x \]

THIS YIELDS THAT

\[ \lambda_1 = \frac{4\pi c \left[ \phi_0(x_0) \right]^2}{\int_{\Omega} \phi_0^2 \, d x} \]

IN SUMMARY WITH \( \psi(\varepsilon) = \varepsilon \) WE OBTAIN THE TWO-TERM EXPANSION

(3) \[ \lambda \sim \mu_0 + \varepsilon \lambda_1 + \ldots \]

\[ \lambda_1 = \frac{4\pi c \left[ \phi_0(x_0) \right]^2}{\int_{\Omega} \phi_0^2 \, d x} \]

REMARK IF THERE ARE N SMALL HOLES THEN WE OBTAIN

\[ \lambda_j = \frac{4\pi}{\int_{\Omega} \phi_0^2 \, d x} \sum_{j=1}^{N} C_j \phi_0^2(x_j), \quad \text{where} \quad C_j = \text{capacitance of the} \ j^{\text{th}} \text{ hole.} \]
REMARK

(i) Let \( \mathcal{U} \) assume that \( \mathcal{U} = 0 \) on \( \partial \Omega \) is replaced by no-flux condition \( \partial_\nu \mathcal{U} = 0 \) on \( \partial \Omega \).

Then \[ \Delta \phi + \mathcal{U} \phi = 0 \quad \text{in} \quad \Omega, \]
\[ \partial_\nu \phi = 0 \quad \text{on} \quad \partial \Omega, \]
\[ \int_{\Omega} \phi^2 \, dx = 1 \]
has the principal eigenvalue \( \mathcal{U}_0 \) and \( \phi_0 = \frac{1}{\left( \int_{\Omega} \right)^{1/2}} \)
where \( |\Omega| \) is the volume of \( \Omega \).

In this case \[ \lambda \sim \frac{4\pi \mathcal{E}}{|\Omega|} \quad (\text{as seen from (3)}). \]

Notice that this leading-order term is independent of the hole location.

(ii) For multiple holes \( \Omega_{s,j}, \quad j=1, \ldots, N \)

then \[ \lambda \sim \mathcal{U}_0 + 4\pi \varepsilon \sum_{j=1}^{N} c_j \left[ \phi_0(x_j) \right]^2 + \cdots \]
\[ \int_{\Omega} \phi_0^2 \, dx \]

Consider the special case of two concentric spheres.

As shown

\[ \mathcal{U}_{rg} + \frac{2}{r} \mathcal{U}_r + \mathcal{U}_g = 0, \quad 0 < r < 1 \]
\[ \mathcal{U}(1) = 0, \quad \mathcal{U}(\varepsilon) = 0. \]

The eigenfunction is \( \mathcal{U} = \frac{\sin \left( \sqrt{\varepsilon} \left( r - \varepsilon \right) \right)}{r} \)

Then we obtain \( \mathcal{U}(1) = 0 \) so that \( \sqrt{\varepsilon} \left( 1 - \varepsilon \right) = \pi \).
This yields that

$$\Lambda = \frac{\pi^2}{1 - \varepsilon} \simeq \pi^2 \left(1 + 2\varepsilon + \ldots\right)$$

Hence, \(\Lambda \sim \pi^2 + 2\varepsilon \pi^2 + \ldots\).

Now use the asymptotic formula (3). In (3) we set

\(\mu_0 = \pi^2\), \(\phi_0 = \sin (\pi \Gamma) \frac{1}{\Gamma}\)

Then \(\phi_0(0) = \lim_{\Gamma \to 0} \frac{\sin (\pi \Gamma)}{\Gamma} = \pi\).

In addition,

$$\int_0^1 \phi_0^2 \, d\lambda = 4\pi \int_0^1 \frac{\sin^2 (\sqrt{\pi} \Gamma)}{\Gamma} \, d\Gamma = 2\pi.$$  

Then (3) yields \(\Lambda \sim \pi^2 + 4\pi \varepsilon \pi^2 \text{ Re} \pi^2 + 2\varepsilon \pi^2 + \ldots\).

Finally \(C = 1\) by Remark (i) on page (3), so \(\Lambda \sim \pi^2 + 2\varepsilon \pi^2 + \ldots\).

As we would have expected.

**Open Problem:** Consider \(N\) holes of "radius" \(r\) for

$$\Delta u + \lambda u = 0 \text{ in } \Omega \setminus \bigcup_{j=1}^N \Omega_j$$

\(\partial u = 0\) on \(\partial \Omega_j\)

\(u = 0\) on \(\partial \Omega \setminus \bigcup_{j=1}^N \Omega_j\)

Then since \(\mu_0 = 0\) and \(\phi_0 = \frac{1}{\Omega} \int_\Omega \phi_0\), we get

$$\Lambda \sim \frac{4\pi \varepsilon}{\Omega} \sum_{j=1}^N c_j.$$

If we want to maximize \(\Lambda\) (in other words, get fastest decay to zero for the heat equation) we need the next term in the expansion, i.e., perhaps

$$\Lambda \sim \frac{4\pi \varepsilon}{\Omega} \sum_{j=1}^N c_j + \frac{4\pi \varepsilon^2}{\Omega^2} \sum_{i=1}^N \sum_{j=1}^N c_i c_j \rho(x_i, \ldots, x_N)$$

and so we need to find maximum of \(\rho(x_1, \ldots, x_N)\).
Suppose that a small patch of boundary allows for the leakage of heat. The model is

\[
\begin{align*}
\Delta u + \lambda u &= 0 \quad \text{in } \Omega \\
\partial_{n} u &= 0 \quad \text{on } \partial \Omega \setminus \partial \Omega_{\varepsilon} \\
u &= 0 \quad \text{on } \partial \Omega_{\varepsilon}
\end{align*}
\]

We assume that \( \partial \Omega_{\varepsilon} \to \frac{1}{\varepsilon} \mathcal{X}_{0} \) as \( \varepsilon \to 0 \).

We can repeat the entire inner/outer analysis to get that the inner problem satisfies with \( y = \varepsilon^{2} (x - x_{0}) \)

\[
\begin{align*}
\Delta y v_{0} &= 0 \quad \text{in } y_{3} > 0 \\
\partial_{n} y v_{0} &= 0 \quad (y_{1}, y_{2}) \notin \partial \Omega_{0} \\
v_{c} &= 0 \quad (y_{1}, y_{2}) \in \partial \Omega_{0} \\
v_{0} &\to \frac{1}{|y|^{1/2}} \quad \text{as } |y| \to \infty.
\end{align*}
\]

Notice that \( \mu_{0} = 0 \) and \( \phi_{0} = \frac{1}{|y|^{1/2}} \).

Now \( v_{0} = \frac{1}{|y|^{1/2}} (1 - v_{c}) \) where \( v_{c} (y) \) satisfies

\[ v_{c} (y) \sim c / |y| \quad \text{as } |y| \to 0. \]

Then we end up with

\[
\begin{align*}
\Delta u_{1} + \lambda u_{1} &= 0 \quad \text{in } \Omega \\
\partial_{n} u_{1} &= 0 \quad \text{on } \partial \Omega \setminus \frac{1}{\varepsilon} \mathcal{X}_{0} \\
u &= -\frac{1}{|\Omega_{0}|^{1/2}} \frac{c}{|x - x_{0}|} \quad \text{as } \gamma \to x_{0}
\end{align*}
\]

Since a delta function on \( \partial \Omega \) contributes only \( 1/2 \) of an interior \( \delta \)-function, we obtain

\[ a \sim 2 \pi \varepsilon c / |\Omega| + \ldots \]
WE CONSIDER

\begin{align*}
\Delta u + \lambda u &= 0 \quad \text{in } \Omega \setminus \Omega_\varepsilon \\
\lambda &\to \lambda_0 \quad \text{as } \varepsilon \to 0 \\
\lambda &= \lambda_0 \quad \text{on } \partial \Omega \\
\lambda &= \lambda_0 \quad \text{on } \partial \Omega_\varepsilon \\
\int_{\Omega_\varepsilon} \lambda^3 \, d\varepsilon &= 1
\end{align*}

WITH \( \Omega_\varepsilon \) A SMALL HOLE OF "RADIUS" \( \varepsilon \) FOR WHICH \( \Omega_\varepsilon \to \{ x_0 \} \) AS \( \varepsilon \to 0 \).

LET \( \lambda_0, \phi_0 \) BE THE PRINCIPAL (OR FIRST) EIGENPAIR OF THE UNPERTURBED PROBLEM SATISFYING

\begin{align*}
\Delta \phi_0 + \lambda_0 \phi_0 &= 0 \quad \text{in } \Omega \\
\phi_0 &= 0 \quad \text{on } \partial \Omega \\
\int_{\Omega} \phi_0^2 \, d\Omega &= 1
\end{align*}

NOW WE WILL EXPAND THE EIGENVALUE OF (1) THAT IS CLOSE TO \( \lambda_0 \)

\[ \lambda \sim \lambda_0 + \sqrt{\varepsilon} \lambda_1 + \cdots \quad \sqrt{\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0. \]

IN THE OUTER REGION AWAY FROM THE HOLE WE EXPAND

\[ u = \phi_0 + \sqrt{\varepsilon} u_1 + \cdots \]

UPON SUBSTITUTING INTO (1) WE OBTAIN

\begin{align*}
\Delta u_1 + \lambda_0 u_1 &= -\lambda_1 \phi_0 \quad \text{in } \Omega \setminus \{ x_0 \} \\
\Delta u_1 &= 0 \quad \text{in } \Omega \\
\int_{\Omega} u_1 \phi_0 \, d\Omega &= 0
\end{align*}

WITH SOME SINGULARITY CONDITION AS \( x \to x_0 \) TO BE FOUND.

NOW IN THE INNER REGION NEAR THE HOLE WE LET \( y = \varepsilon^{-1} (x - x_0) \)

AND WE EXPAND

\[ u = \sqrt{\varepsilon} \right) v_0(y) + \cdots \]

WHERE \( \Delta y v_0 = 0 \). WE WANT \( v_0 y) \sim A_0 \log y \) AS \( y \to 0 \) AND SO WE WRITE

\[ v_0(y) = A_0 v_c(y) \]

WHERE \( v_c(y) \) SATISFIES

\begin{align*}
\Delta y v_c &= 0, \quad y \text{ outside } \Omega_0 \\
v_c &= 0, \quad y \text{ on } \partial \Omega_0 \\
v_c &\sim \log y \quad \text{as } |y| \to 0.
\end{align*}
This problem for $V_c(y)$ has a unique solution, and the behavior at $\infty$ is

$$V_c(y) \sim \log |y| - \log d + O\left(\frac{1}{|y|}\right) \quad \text{as} \quad |y| \to \infty$$

where $d$ is called the "logarithmic capacitance" of $\Omega_0$. It depends on the shape of $\Omega_0$ and not its orientation within $\Omega$.

There are some key examples:

| $\Omega_0$                     | $d$                                | $V_c = \log \left(\frac{|y|}{a}\right)$ |
|--------------------------------|------------------------------------|----------------------------------------|
| circle, radius $a$             | $a$                                | $\frac{\pi}{4} \cdot a^2$            |
| ellipse, semi-axes $a, b$      | $(a+b)/2$                          | $\frac{\pi}{4} \cdot \frac{a^2 + b^2}{2}$ |
| equilateral triangle side $h$ | $\sqrt{3} \cdot \frac{h}{\sqrt{3}} $ | $0.422 \cdot h$                        |
| square side $h$                | $\frac{\pi}{4} \cdot h$           | $0.5902 \cdot h$                      |

These values of $d$ are found by conformal mapping, T. RANJFORD (Cambridge U. Press 1995).

Now we write inner expansion in outer variables:

$$U \sim V(\epsilon) A_0 \left[ \log |y| - \log d \right] \sim V(\epsilon) A_0 \left[ - \log (\epsilon d) + \log |x-x_0| \right].$$

The matching condition becomes:

$$\phi_0 (x_0) \ldots + V(\epsilon) U_1 (x \to x_0) \sim (- \log (\epsilon d)) A_0 V(\epsilon) + A_0 V(\epsilon) \log |x-x_0| + \ldots$$

Therefore, we take

$$V(\epsilon) = -\frac{1}{\log (\epsilon d)} \quad \text{and} \quad A_0 = \phi_0 (x_0)$$

and we let $U_1 (x) \to A_0 \log |x-x_0| = \phi_0 (x_0) \log |x-x_2|$. This becomes

$$\Delta U_1 + \mu_0 U_1 = - \lambda_1 \phi_0 \quad \text{in} \quad \Omega \setminus \{x_0\}$$

$U_1 = 0$ on $\partial \Omega$

$$U_1 \sim \phi_0 (x_0) \log |x-x_2| \quad \text{as} \quad x \to x_2; \quad \int_{\Omega} U_1 \phi_0 \, dx = 0.$$
This can be written as

\[ \nabla u_i = \Lambda u_i + \mu_0 u_i = \Lambda, \quad \phi_0 + 2\pi \phi_0(x_0) \delta(x - x_0) \quad \text{in } \Omega \]

\[ u_i = 0 \quad \text{on } \partial \Omega \]

\[ \int_{\Omega} u_i \phi_0 \, dx = 0 \]

Now we Green's second identity

\[ \int_{\partial \Omega} (\phi_0 \partial_n u_i - u_i \partial_n \phi_0) \, ds = \int_{\Omega} (\phi_0 \partial_n u_i - u_i \partial_n \phi_0) \, dx \]

But \( \phi_0 = u_i = 0 \) on \( \partial \Omega \) and \( \partial \phi_0 = 0 \) we obtain \( \int_{\Omega} \phi_0 \partial_n u_i \, dx = 0 \).

Thus

\[ \int_{\Omega} \phi_0 (\Lambda, \phi_0 + 2\pi \phi_0(x_0) \delta(x - x_0)) \, dx = 0. \]

This yields that

\[ \Lambda = \frac{2\pi [\phi_0(x_0)]^2}{\int_{\Omega} \phi_0^2 \, dx} \]

Therefore, we obtain a two-term expansion

\[ \Lambda \sim \mu_0 + \frac{2\pi \sqrt{\int_{\Omega} \phi_0^2 \, dx}}{\mu_0 \sqrt{\int_{\Omega} \phi_0^2 \, dx}} \quad \text{or} \quad \nu = -\frac{1}{\mu_0 \sqrt{\int_{\Omega} \phi_0^2 \, dx}} \quad d \text{ logarithmic capacitance} \]

Remark (i) Further terms in the expansion have the form

\[ \Lambda \sim \mu_0 + A_1 \nu + A_2 \nu^2 + A_3 \nu^3 + \cdots \]

which is an infinite-logarithmic expansion in powers of \( \nu \).

Since \( \log(e^d) \) decreases very slowly as \( e \) decreases, we'd like to sum

(ii) If \( u_i = 0 \) on \( \partial \Omega \) is replaced by \( \phi_0(x_0) = 0 \) on \( \partial \Omega \), then \( \mu_0 = 0 \) and \( \phi_0 = \frac{1}{1/\mu_0} \)? so that \( \int_{\Omega} \phi_0^2 \, dx = 1 \).

Thus

\[ \Lambda \sim \frac{2\pi \nu}{1/\mu_0} \quad A_j \to 0. \]
EXAMPLE

Consider a circular domain with a concentric hole of radius $\epsilon$ and a hole of radius $\epsilon/2$ in $0 < r < 1$.

$u = 0$ on $\Gamma = 1$

$u = 0$ on $\Gamma = \epsilon$

The unperturbed solution is $\phi_0 = J_0(\sqrt{\mu_0} r)$ where $J_0(\sqrt{\mu_0}) = 0$ and $\sqrt{\mu_0} = z_0$, with $z_0$ the first zero of $J_0(z)$.

Using the perturbation formula we have $v_c(y) = \log |y|$

Since $\Delta v_c = 0$, $v_c = 0$ on $|y| = 1$, so that $d = 1$. Then $x_0 = 0$ and $\phi_0(x_0) = J_0(0) = 1$.

Therefore,

$$\lambda \sim \mu_0 + \frac{2\pi}{\int_0^1 \phi_0^2(x) dx} \mu_0 + \frac{2\pi}{\int_0^1 \int_0^1 \frac{J_0'}{J_0(\sqrt{\mu_0} r)} dr dt}$$

But we recall $\int_0^1 \frac{J_0'}{J_0(\sqrt{\mu_0} r)} dr = \frac{1}{2}(J_0'(\sqrt{\mu_0}))^2$ when $J_0'(\sqrt{\mu_0}) = 0$.

This yields that

$$(x) \quad \lambda \sim \mu_0 + \left( -\frac{1}{\log \epsilon} \right) \left( \frac{2}{[J_0'(\sqrt{\mu_0})]^2} \right) + \ldots$$

Now we compare (x) with the exact solution. We write

$u = J_0(\sqrt{\lambda} r) + c Y_0(\sqrt{\lambda} r)$

Now $u(1) = 0$ so

$u = J_0(\sqrt{\lambda} r) - \frac{J_0'(\sqrt{\lambda})}{Y_0(\sqrt{\lambda})} Y_0(\sqrt{\lambda} r)$

Next $u(\epsilon) = 0$ gives the eigenvalue relation

$J_0(\sqrt{\lambda} \epsilon) = \frac{J_0(\sqrt{\lambda})}{Y_0(\sqrt{\lambda})} Y_0(\sqrt{\lambda} \epsilon)$

We write this as

$J_0(\sqrt{\lambda}) = \frac{J_0(\sqrt{\lambda} \epsilon)}{Y_0(\sqrt{\lambda} \epsilon)}$
\[ J_0(z) \sim 1 + O(z^2) \text{ as } z \to 0 \]
\[ Y_0(z) \sim \frac{2}{\pi} \left[ \log z - \log 2 + \gamma \right]^{1} \text{ as } z \to 0 \quad \gamma = \text{Euler's constant}. \]

Therefore with \( z = \sqrt{\lambda} \) we obtain,
\[ J_0(z) \sim Y_0(z) \frac{\pi}{2} \left[ \log (e z) - \log 2 + \gamma \right]^{1} \quad \text{for } \varepsilon \ll 1. \]

To find the root we let
\[ z = z_0 - \frac{1}{\log e} z_1 + \ldots \quad \text{with } J_0(z_0) = 0 \quad \text{and} \quad z_0 = \sqrt{\lambda_0}. \]

Where \( z_0 \) is the first root of \( J_0(z_0) = 0 \), so \( z_0 = \sqrt{\mu_0}. \)

Then we use Taylor series to obtain
\[ J_0(z_0) - \frac{1}{\log e} J_0'(z_0) z_1 + \ldots \sim \frac{\pi}{2} \frac{Y_0(z_0)}{\log e}. \]

This yields that
\[ z_1 = -\frac{\pi}{2} \frac{Y_0(z_0)}{J_0'(z_0)}. \]

Now we write \( \sqrt{\lambda} = z = z_0 + \left( -\frac{1}{\log e} \right) z_1 + \ldots. \)

Hence
\[ \lambda \sim z_0^2 + \left( -\frac{1}{\log e} \right) 2 z_0 z_1 + \ldots. \]

This yields
\[ \lambda \sim \mu_0 + \left( -\frac{1}{\log e} \right) 2 \sqrt{\mu_0} z_1. \]

Therefore
\[ \lambda \sim \mu_0 + \left( -\frac{1}{\log e} \right) \lambda_1 + \ldots, \]

with \( \lambda_1 = 2 \sqrt{\mu_0} z_1 = 2 \sqrt{\mu_0} \left( -\frac{\pi}{2} \frac{Y_0(\sqrt{\mu_0})}{J_0'(\sqrt{\mu_0})} \right). \)

To write this in a form to compare with the general theory result we need an identity.
ASIDE CONSIDER THE SECOND ORDER EQUATION
\[ Ly = y'' + p(x)y' + q(x)y \]

THEN DEFINE \( y_1, y_2 \) AS TWO SOLUTIONS TO \( Ly = 0 \).

THEN LET \( W = y_1'y_2 - y_1y_2' = \text{WROKNIAN OF } y_1, y_2 \).

WE DERIVE \( W' + p(x)W = 0 \)

So \( W = C \exp \left(-\int p(x) \, ds\right) \).

FOR BESSEL EQUATION \( y'' + \frac{1}{\gamma} y' + \lambda y = 0 \)

WE OBTAIN \( p = \frac{1}{\gamma}, \quad y_1 = J_0(\sqrt{\lambda} \gamma), \quad y_2 = Y_0(\sqrt{\lambda} \gamma) \) \( \rightarrow W = C/\gamma \).

Then \( \left(\frac{d}{d \gamma} J_0(\sqrt{\lambda} \gamma)\right) Y_0(\sqrt{\lambda} \gamma) - \left(\frac{d}{d \gamma} Y_0(\sqrt{\lambda} \gamma)\right) J_0(\sqrt{\lambda} \gamma) = C/\gamma \).

NOW AT \( \gamma \to 0 \), \( Y_0'(\gamma) \sim 2/\pi \gamma \) AND \( J_0(\gamma) = 1 \).

THIS GIVES \( C = -2/\pi \).

NOW EVALUATE AT \( \gamma = 1 \) AND SET \( \lambda = \mu_0 \) WHERE \( J_0(\sqrt{\mu_0}) = 0 \)

WE GET \( \sqrt{\mu_0} J_0'(\sqrt{\mu_0}) Y_0(\sqrt{\mu_0}) = -\frac{2}{\pi} \).

\( J_0 Y_0(\sqrt{\mu_0}) = -\frac{2}{\pi \sqrt{\mu_0} J_0'(\sqrt{\mu_0})} \)

SUBSTITUTING THIS INTO THE RESULT ON THE PREVIOUS PAGE WE OBTAIN
\[ \lambda_1 = -\pi \sqrt{\mu_0} \frac{Y_0(\sqrt{\mu_0})}{J_0'(\sqrt{\mu_0})} = \frac{2}{(J_0'(\sqrt{\mu_0}))^2} \]

This yields \( \lambda = \mu_0 + \left(\frac{-1}{\log e}\right) \frac{2}{(J_0'(\sqrt{\mu_0}))^2} + \ldots \)
WE CONSIDER POISSON'S EQUATION IN A DOMAIN WITH ONE SMALL HOLE GIVEN BY

\[ \Delta W = -B \text{ in } \Omega \setminus \Omega_\varepsilon \]
\[ W = 0 \text{ on } \partial \Omega \]
\[ W = 0 \text{ on } \partial \Omega_\varepsilon \]

WE THEN EXPAND IN THE OUTER REGION

\[ W(x; \varepsilon) = W_0(x; \varepsilon) + \sigma(\varepsilon) W_1(x; \varepsilon) + \cdots \text{ with } \sigma = -1/\log(\varepsilon) \]

AND WITH \( \sigma \ll \varepsilon^k \) FOR ANY \( K > 0 \).

WE OBTAIN THAT

\[ \Delta W_0 = -B \text{ in } \Omega \setminus \{x_0\} \]
\[ W_0 = 0 \text{ on } \partial \Omega \]

\( W_0 \) IS SINGULAR AS \( x \to x_0 \).

IN THE INNER REGION WE WRITE

\[ y = \varepsilon^{-1}(x - x_0), \quad V(y; \varepsilon) = W(\varepsilon x_0 + \varepsilon y; \varepsilon) \]

WE THEN EXPAND

\[ V(y; \varepsilon) = V_0(y; \varepsilon) + \mu_0(\varepsilon) V_1(y; \varepsilon) + \cdots \]

WITH \( \mu_0 \ll \varepsilon^k \) FOR ANY \( K > 0 \).

NOW WE GET THAT \( V_0 \) SATISFIES

\[ \Delta y V_0 = 0 \text{ outside } \Omega_0 \]
\[ V_0 = 0 \text{ on } \partial \Omega_0 \]

THE MATCHING CONDITION TO LEADING ORDER WILL BE THAT

\[ W_0(x; \varepsilon) \sim W_0(x_0) \sim V_0(y; \varepsilon) \]

\( x \to x_0 \)
\( 1/|y| \to \infty \)

THEOREFORE, WE TAKE IN TERMS OF AN UNKNOWN FUNCTION \( \chi = \chi(\varepsilon) \)

WITH \( \chi(0) = 0(1) \),
\[ V_0(y; \varepsilon) = \nabla \chi V_0(y) \]
WE THEN OBTAIN THAT $V_c(y)$ SATISFIES

\[
\begin{cases}
\Delta y V_c = 0 & \text{outside } \Omega_0 \\
V_c = 0 & \text{on } \partial \Omega \\
V_c \sim \log |y| & \text{as } |y| \to \infty
\end{cases}
\]

THERE IS A UNIQUE SOLUTION TO THIS PROBLEM, AND $V_c(y)$ HAS THE
ASYMPTOTIC BEHAVIOR

\[V_c(y) \sim \log |y| - \log d + O\left(\frac{1}{|y|}\right) \quad \text{for } |y| \gg 1.\]

HERE $d$ IS THE LOGARITHMIC CAPACITANCE.

NOW WE WRITE THE FAR-FIELD BEHAVIOR OF $V_0$ AS $|y| \to \infty$
IN THE FORM,

\[V_0(y; y) \sim \frac{\log \left(\frac{|x - x_0|}{\varepsilon}\right)}{-\log d} \sim \frac{\log |x - x_0|}{-\log (\varepsilon d) + \log |x - x_0|}.
\]

NOW WITH $v = -\frac{1}{\log (\varepsilon d)}$ WE OBTAIN,

\[V_0(y; y) \sim \gamma + \gamma \gamma \log |x - x_0|.
\]

THE MATCHING CONDITION GIVE $W_0 \sim \gamma \gamma \log |x - x_0| + \gamma$ AS $X \to x_0$.

THEN WE WRITE

\[
\begin{align*}
\Delta W_0 &= -B & \text{in } \Omega \setminus \{ x_0 \} \\
W_0 &= 0 & \text{on } \partial \Omega \\
W_0 &\sim \gamma + \gamma \gamma \log |x - x_0| & \text{as } X \to x_0.
\end{align*}
\]

NOW WE INTRODUCE $W_{0H}(X)$ AND $G(X; x_0)$, WHICH SATISFY

\[
\begin{cases}
\Delta W_{0H} = -B & \text{in } \Omega \\
W_{0H} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

AND THE GREEN'S FUNCTION,

\[
\begin{align*}
\Delta G &= \delta(x - x_0) & \text{in } \Omega \\
G &= 0 & \text{on } \partial \Omega.
\end{align*}
\]

IN TERMS OF THIS SOLUTION, $G(X; x_0) = \frac{1}{2\pi} \log |x - x_0| + R(x_0) + c(1)$ AS $X \to x_0.$
Remark (i) If we write
\[ G(x; x_0) = \frac{1}{2\pi} \log |x - x_0| + R(x; x_0) \]
where \( R(x; x_0) \) is the "regular part" of the Green's function. As \( x \to x_0 \), then \( R_0(x_0) = R(x_0; x_0) \).

The function \( R(x; x_0) \) depends on the shape of the domain.

Then we can solve for \( \omega_0 \) to obtain
\[ \omega_0(x; \gamma) = \omega_{OH}(x) + 2\pi \gamma \log \frac{1}{2\pi} \log |x - x_0| + R_0(x_0) \]

Therefore, as \( x \to x_0 \), we obtain
\[ \omega_{OH}(x_0) + 2\pi \gamma \log \left[ \frac{1}{2\pi} \log |x - x_0| + R_0(x_0) \right] = \gamma + \gamma \log |x - x_0| \]

Therefore, we obtain that \( \gamma \) satisfies
\[ \gamma \left[ 1 - 2\pi \gamma R_0(x_0) \right] = \omega_{OH}(x_0) \]

This yields that
\[ \gamma = \frac{\omega_{OH}(x_0)}{1 - 2\pi \gamma R_0(x_0)} \]

Therefore, the outer expansion satisfies for \( |x - x_0| = o(1) \)
\[ W \sim W_0(x; \gamma) = \omega_{OH}(x) + 2\pi \gamma \frac{\omega_{OH}(x_0)}{1 - 2\pi \gamma R_0(x_0)} \]

In contrast in the inner region for \( |x - x_0| = o(\epsilon) \) that
\[ W \sim V_0(y; \gamma) \sim \gamma \frac{\omega_{OH}(x_0)}{1 - 2\pi \gamma R_0(x_0)} V_G(y) \]

This result is derived for \( \gamma << 1 \) and requires that \( 2\pi \gamma R_0(x_0) < 1 \) for this result to provide an approximation.
Consider a pipe of radius \( R_0 \) containing a core centered at the origin. Assume that the core is a circle of radius \( \varepsilon \).

For this problem

\[
W_{OH}(\gamma) = \frac{B}{A} (\gamma_0^2 - \gamma^2), \quad G(\gamma; 0) = \frac{1}{2\pi} \log \gamma - \frac{1}{2\pi} \log \gamma_0
\]

which gives

\[
R_0(0) = -\frac{1}{2\pi} \log \gamma_0, \quad W_{OH}(0) = B \gamma_0^2/A, \quad d = 1.
\]

Then

\[
\chi = \frac{B \gamma_0^2}{A} = \frac{B \gamma_0^2}{1 - 2\pi \nu R_0(0)} \frac{1}{4} \frac{1}{1 + \nu \log \gamma_0} \quad \nu = -\frac{1}{2} \log \varepsilon
\]

This yields that

\[
\chi = \frac{B \gamma_0}{A} \frac{\log \varepsilon}{\log (\gamma_0/\varepsilon)}.
\]

Now in the outer solution we obtain

\[
W \sim W_0(x; \gamma) = W_{OH} + 2\pi \chi \nu G(x; 0) = \frac{B}{4} (\gamma_0^2 - \gamma^2) + \chi \nu \log \left( \frac{\gamma}{\gamma_0} \right).
\]

Then

\[
W \sim W_0(x; \gamma) = \frac{B}{4} (\gamma_0^2 - \gamma^2) - \frac{B \gamma_0^2}{4} \frac{1}{\log (\gamma_0/\varepsilon)} \frac{1}{\log (\gamma_0/\varepsilon)}
\]

Now the exact solution to

\[
\frac{\partial^2 W}{\partial \gamma^2} + \frac{1}{\gamma} \frac{\partial W}{\partial \gamma} = -B \quad \varepsilon < \gamma < \gamma_0
\]

\[
W(\gamma_0) = 0, \quad W(\varepsilon) = 0
\]

is given by

\[
W = \frac{B}{4} (\gamma_0^2 - \gamma^2) - \frac{B}{4} \left( \gamma_0^2 - \varepsilon^2 \right) \frac{1}{\log \left( \gamma_0/\gamma \right)}
\]

which, besides the \( O(\varepsilon^2) \) term, is the same as given by \( W_0(x; \gamma) \).

Remark

(i) Imagine that the interior circle \( \gamma = \varepsilon \) is replaced by a hole of arbitrary shape centered at the origin.

Then all we need to do is simply replace \( \varepsilon \) by \( \varepsilon \) where \( d \) is the logarithmic capacitance of the body.
RECALL THAT d WAS DEFINED BY
\[ \Delta y V_C = 0 \text{ outside } \Omega \]
\[ V_C = 0 \text{ on } \partial \Omega \]
\[ V_C \sim \log |y| - \log d \quad \text{as } |y| \to \infty \]

FOR INSTANCE IF \( \Omega = \{ x^2 + y^2 / 5 = \epsilon^2 \} \to \Omega_0 = \{ y^2 + y^2 / 5 = 1 \} \)

THUS THE OUTER SOLUTION IS
\[ W \sim W_0 = \frac{B}{4} \left( r^2 - r^2 \right) - \frac{B}{4} \log \left( \frac{r_0}{r} \right) \]
\[ + \frac{r^2}{\log \left( \frac{r_0}{\epsilon d} \right)} \]

(i) NOW SUPPOSE THAT THE HOLE IS OFF-CENTER AT SOME POINT \( x_0 \) IN \( \Omega \). ASSUME \( \Omega \) IS A CIRCLE OF RADIUS \( r_0 \).

\[ |x| = r_0 \]

THEN WE MUST SOLVE
\[ \Delta G = \delta(x - x_0) \text{ in } \Omega ; \quad G = 0 \text{ on } \partial \Omega \]
\[ G(x; x_0) = \frac{1}{2\pi} \log |x - x_0| + R(x; x_0) \]

WE NEED TO CALCULATE \( R_0(x_0) = R(x_0; x_0) \).

TO SOLVE FOR \( G(x; x_0) \) WE USE THE METHOD OF IMAGES:

\[ |x| = \frac{r_0^2}{|x_0|} \]

THEN \( |x_0| |x_1| = r^2 \)

THE SOLUTION IS
\[ G(x; x_0) = \frac{1}{2\pi} \log |x - x_0| - \frac{1}{2\pi} \log |x - x_1| |x_0| / r_0 \]

THIS YIELDS THAT
\[ R(x; x_0) = R_0(x_0) = \frac{1}{2\pi} \log \left( \frac{1}{|x_0 - x|} \frac{|x_0|}{r_0} \right) \]

THEN THE OUTER SOLUTION IS SIMPLY
\[ W \sim W_0 = \frac{B}{4} \left( r_0^2 - r^2 \right) + \frac{2\pi \sqrt{B r_0^2 / 4}}{1 - 2\pi \sqrt{R_0(x_0)}} \log \left( \frac{r_0}{\epsilon d} \right) \]

WITH \( \nu = -1 / \log (\epsilon d) \).
Now if we define

\[ \bar{w} = \frac{1}{\Omega_0^1} \int_{\Omega_0} w_0 \, dx \]

we could plot \( \bar{w} \) versus ed to obtain Kaplan's equivalence principle:

Corresponding to every non-circular domain \( \Omega_0 \) there exists a corresponding circular domain \( \Omega_0^c \) with the same \( \bar{w} \).

(iii) Consider the following eigenvalue problem in a circular domain \( \Omega_0 \): \( \Gamma_0 = 1 \) containing a hole of arbitrary cross-section centered at the origin.

The sum of all of the logarithmic terms for the principal eigenvalue satisfies

\[ J_0(\sqrt{\lambda^*}) = \frac{\pi}{2} Y_0(\sqrt{\lambda^*}) \left[ \log \left( \frac{z}{\sqrt{\lambda^*}} \right) - \log 2 + \gamma \right]^{-1} \]

where \( z = ed \) and \( \gamma \) is Euler's constant. Also \( \lambda^* \rightarrow \mu_0^2 \) where \( J_0(\mu_0) = 0 \) is the first root of \( J_0(x) = 0 \).
Example: Find an asymptotic solution in 3-D to

\[ \Delta u = M(x) \quad \text{in} \quad \Omega \setminus \bigcup_{j=1}^{N} \Omega_{\varepsilon_j} \quad \Omega \quad \text{in} \quad \mathbb{R}^3. \]

\[ u = d_j \quad \text{on} \quad \partial \Omega_{\varepsilon_j}, \quad j = 1, \ldots, N \]

\[ u = 0 \quad \text{on} \quad \partial \Omega \]

Each \( \Omega_{\varepsilon_j} \) is a hole of "radius \( \varepsilon \)" with \( \Omega_{\varepsilon_j} \to B(x_j, \varepsilon) \) as \( \varepsilon \to 0 \). For instance, if \( \Omega_{\varepsilon_j} = \{ x - x_j \leq \varepsilon \} \) then we have a sphere of radius \( \varepsilon \) centered at \( x = x_j \).

In the outer region we expand

\[ u = u_0 + \varepsilon u_1 + \ldots \]

We have

\[ \begin{cases} \Delta u_0 = M(x) \quad \text{in} \quad \Omega, \\ u_0 = 0 \quad \text{on} \quad \partial \Omega \end{cases} \]

Unperturbed problem

And that \( u_1 \) satisfies

\[ \begin{cases} \Delta u_1 = 0 \quad \text{in} \quad \Omega \setminus \{ x_1, \ldots, x_N \}, \\ u_1 = 0 \quad \text{on} \quad \partial \Omega, \\ u_1 \text{ singular as } x \to x_j, \quad j = 1, \ldots, N. \end{cases} \]

Now in the inner region near \( x = x_j \) we write \( \nu = \varepsilon^{-1}(x - x_j) \)

And \( \nu(x; \varepsilon) = u(x_j + \varepsilon y, \varepsilon) = v_0(\nu) + \ldots \). The matching condition yields that \( v_0 \to u_0(x_j) \) as \( |\nu| \to \infty \) so that

\[ \begin{cases} \Delta y v_0 = 0 \quad \text{outside } \Omega_j = \Omega_{\varepsilon_j}/\varepsilon, \\ v_0 = d_j \quad \text{on} \quad \partial \Omega_j, \\ v_0 \to u_0(x_j) \quad \text{as} \quad |\nu| \to \infty \end{cases} \]

The solution is simply

\[ v = u_0(x_j) + (d_j - u_0(x_j)) v_0(\nu) \]
WHERE $V_C(y)$ SATISFIES

\[ \Delta y V_C = 0, \quad y \text{ outside } \Omega_j \]

\[ V_C = 1, \quad y \text{ on } \Omega_j \]

\[ V_C \sim C_j / |y| \quad \text{as } |y| \to \infty. \quad C_j \text{ capacitance of the } j \text{th hole.} \]

THIS YIELDS THAT

\[ V_0 \sim u_0(x_j) + (d_j - u_0(x_j)) C_j / |y| \quad \text{as } |y| \to \infty. \]

NOW THE MATCHING CONDITION IS SIMPLY

\[ u_0(x_j) + \varepsilon u_1 \sim u_0(x_j) + (d_j - u_0(x_j)) \frac{C_j}{1|x-x_j|} \quad x \to x_j \]

THIS YIELDS THAT

\[ u_1 \sim (d_j - u_0(x_j)) C_j / |x-x_j| \quad \text{as } x \to x_j. \]

THEREFORE, THE PROBLEM FOR $u_1$ IS SIMPLY

\[ \Delta u_1 = -4\pi \sum_{j=1}^{N} (d_j - u_0(x_j)) C_j \delta(x-x_j) \quad \text{in } \Omega. \]

\[ u_1 = 0 \quad \text{on } \partial \Omega. \]

THE SOLUTION IS SIMPLY

\[ u_1 = -4\pi \sum_{j=1}^{N} (d_j - u_0(x_j)) C_j G(x; x_j) \]

WHERE $G(x; x_j)$ IS THE GREEN'S FUNCTION SATISFYING

\[ \Delta G = \delta(x-x_j) \quad \text{in } \Omega \]

\[ G = 0 \quad \text{on } \partial \Omega. \]
Example: Now consider the corresponding Neumann problem

\[ \Delta u = M(x) \text{ in } \Omega \setminus \bigcup_{j=1}^{N} \Omega_{ej} \subseteq \mathbb{R}^3, \]

\[ \partial_\nu u = 0 \text{ on } \partial \Omega, \]

\[ u = \alpha_j \text{ on } \partial \Omega_{ej}, \quad j = 1, \ldots, N. \]

We assume for simplicity that \( \Omega_{ej} : |x - x_j| = \varepsilon \Gamma_j \)
so that we have \( N \)-small spheres of radius \( \varepsilon \Gamma_j \).

Remark: i) We cannot expand \( u = u_0 + \varepsilon u_1, \ldots \)

since \( \Delta u_0 = M(x) \text{ in } \Omega \)

\[ \partial_\nu u_0 = 0 \text{ on } \partial \Omega \]

has no solution in general unless \( \int_{\Omega} M(x) \, dx = 0. \)

(ii) Also recall that the eigenvalue problem

\[ \Delta \phi + \lambda \phi = 0 \text{ in } \Omega \setminus \bigcup_{j=1}^{N} \Omega_{ej}, \]

\[ \partial_\nu \phi = 0 \text{ on } \partial \Omega, \]

\[ \phi = 0 \text{ on } \partial \Omega_{ej}, \quad j = 1, \ldots, N. \]

has the principal eigenvalue (from page 7) that

\[ \lambda \sim \frac{4 \pi \varepsilon}{|\Omega|} \sum_{j=1}^{N} \Gamma_j. \]

Notice that in the \( j \)-th inner region

\[ \Delta y v_c = 0, \quad |y| > \Gamma_j, \]

\[ v_c = 1, \quad |y| = \Gamma_j, \quad v_c \sim \frac{C_j}{|y|} \text{ as } |y| \to \infty \]

\[ \to v_c = \frac{\Gamma_j}{|y|}. \quad \text{This implies that } \Gamma_j \]

This eigenvalue with \( \lambda = O(\varepsilon) \) suggests that the expansion for \( u \) should be

\[ u = u_0 / \varepsilon + u_1 + \varepsilon u_2 + \ldots \]
(iii) Consider the special case

\[ \Delta u = M \text{ in } \varepsilon < r < 1 \text{ with } M = \text{constant} \]
\[ u = 0 \text{ on } r = 1 \]
\[ u = 1 \text{ on } r = \varepsilon \]

Then we calculate \( U = \frac{M r^2}{6} + \frac{A}{r} \).

Now \( u = 0 \text{ on } r = 1 \rightarrow \frac{M}{3} - A = 0 \) so \( A = \frac{M}{3} \).

\[ u = 1 \text{ on } r = \varepsilon \rightarrow \frac{M \varepsilon^2}{6} + \frac{M}{3 \varepsilon} + B = 1 \]

This yields that \( B = 1 - \frac{M}{3 \varepsilon} - \frac{M \varepsilon^2}{6} \).

Then \( U = \frac{M}{6} (r^2 - \varepsilon^2) + \frac{M}{3} \left( \frac{1}{r} - \frac{1}{\varepsilon} \right) + 1 \).

Notice that in the outer region

\[ \hat{U} = \hat{u}_0 / \varepsilon + \hat{u}_1 + \ldots \]

and in the inner region \( \sqrt{v} = v_0 / \varepsilon + \sqrt{v}_1 + \ldots \).

We return to the general problem (ix)

In the outer region we expand

\[ \hat{U} = \hat{u}_0 / \varepsilon + \hat{u}_1 + \varepsilon \hat{u}_2 + \ldots \]

We obtain that \( \Delta \hat{u}_0 = 0 \) and so \( \hat{u}_0 = \mu \) where \( \mu \) is a constant.

The problems for \( \hat{u}_1 \) and \( \hat{u}_2 \) are

\[ \Delta \hat{u}_1 = M(x) \text{ in } \Omega \setminus \{x_1, \ldots, x_N\} \]
\[ \partial_n \hat{u}_1 = 0 \text{ on } \partial \Omega \]
\[ \hat{u}_1 \text{ singular at } x \rightarrow x_i \]
\[ \begin{align*}
\text{Then } U_2 \text{ satisfies } & \quad \Delta U_2 = 0 \quad \text{in } \Omega \setminus \{x_1, \ldots, x_M\} \\
& \quad \partial_\Omega U_2 = 0 \quad \text{on } \partial \Omega \\
U_2 \text{ singular at } & \quad x \rightarrow x_j. \\
\text{Now in the inner region we let } & \quad y = \varepsilon^{-1}(x - x_j) \text{ and we expand } \\
& \quad V = \frac{V_0}{\varepsilon} + V_1 + \varepsilon V_2 + \ldots \\
\text{We then obtain that, upon using matching condition } & \quad V_0 \rightarrow U_0 \text{ at } \infty \\
\Delta y V_0 & = 0 \quad \text{for } |y| > \Gamma_j \\
V_0 & = 0 \quad \text{for } |y| = \Gamma_j \\
V_0 & \rightarrow \mu A_j \quad |y| \rightarrow \infty \\
\text{The solution is written as } & \quad V_0 = \mu \left(1 - V_C\right) \\
\text{where } V_C(y) \text{ satisfies } & \quad \Delta y V_C = 0, \quad |y| > \Gamma_j \\
V_C & = 1, \quad |y| = \Gamma_j \\
V_C & \sim \frac{C_j}{|y|} \quad \text{as } |y| \rightarrow \infty. \text{ We get } V_C = \frac{\Gamma_j}{|y|} \text{ so } C_j = \Gamma_j. \\
\text{Therefore, the matching condition becomes near } & \quad x = x_j: \\
\frac{U_j}{\varepsilon} + U_j + \varepsilon U_2 + \ldots & \sim \frac{V_0}{\varepsilon} + V_1 + \ldots = \frac{\mu}{\varepsilon} \left(1 - \frac{C_j}{|x - x_j|}\right) + V_1 \\
& \quad x \rightarrow x_j, \quad y \rightarrow \infty \\
\text{Therefore, we obtain } & \quad U_j \rightarrow -\frac{\mu}{|x - x_j|} C_j \quad \text{as } x \rightarrow x_j.
\end{align*} \]
THE PROBLEM FOR \( U_i \) IS SIMPLY:
\[
\Delta U_i = M(x) \quad \text{in } \Omega \setminus \{x_1, \ldots, x_N\}
\]
\[
\partial_n U_i = 0 \quad \text{on } \partial \Omega
\]
\[
U_i \sim \frac{-\mu c_j}{|x - x_j|} \quad x \to x_j, \quad j = 1, \ldots, N
\]

THIS PROBLEM IS EQUIVALENT TO
\[
\left\{ \begin{array}{l}
\Delta U_i = M(x) + 4\pi \mu \sum_{j=1}^{N} c_j \delta(x - x_j) \quad \text{in } \Omega \\
\partial_n U_i = 0 \quad \text{on } \partial \Omega 
\end{array} \right.
\]

THEN USING THE DIVERGENCE THEOREM
\[
\int_{\Omega} M(x) \, dx + 4\pi \mu \sum_{j=1}^{N} c_j = 0
\]

THUS YIELDS
\[
\mu = -\frac{1}{4\pi} \frac{\int_{\Omega} M(x) \, dx}{\sum_{j=1}^{N} c_j}
\]

WHICH DETERMINES THE LEADING ORDER OUTER SOLUTION
\[
U_i \sim \frac{-\mu}{\varepsilon}
\]

NOW WE PROCEED TO SECOND ORDER. WE NEXT SOLVE FOR \( U_i \) EXPLICITLY. WE INTRODUCE THE NEUMANN GREEN'S FUNCTION \( G(x; x_j) \) DEFINED BY THE SOLUTION TO
\[
\left\{ \begin{array}{l}
\Delta G = \frac{1}{|x - x_j|} - \delta(x - x_j), \quad \text{in } \Omega \\
\partial_n G = 0 \quad \text{on } \partial \Omega \\
\int_{\Omega} G \, dx = 0
\end{array} \right.
\]

NOTICE THAT \( G(x; x_j) \) IS UNIQUE AND EXISTS SINCE \( \int_{\Omega} \left( \frac{1}{|x - x_j|} - \delta(x - x_j) \right) \, dx = 0 \)
THEN WE HAVE
\[ G(x; x_j) = \frac{1}{4\pi |x - x_j|} + R(x; x_j) \quad R = \text{regular part of Neumann Green's function.} \]

Therefore as \( x \to x_j \) we obtain
\[ G(x; x_j) \sim \frac{1}{4\pi |x - x_j|} + R_j, \quad R_j = R(x_j; x_j). \]

Now we write the problem for \( u_j \), \( \forall \)
\[ \Delta u_j = (M(x) - \frac{1}{\Omega_1} \int_{\Omega_1} M dx) + \left( \frac{1}{\Omega_1} \int_{\Omega_1} M dx \right) + 4\pi \mu \sum_{j=1}^{N} C_j \delta(x - x_j) \quad \text{in } \Omega \]
\[ \partial_n u_j = 0 \quad \text{on } \partial \Omega. \]

Now we write the solution as
\[ u_j = u_{ip} - 4\pi \mu \sum_{i=1}^{N} C_i G(x; x_i) + \bar{u}_i \]
where \( \bar{u}_i \) is a constant and \( u_{ip} \) satisfies
\[ \Delta u_{ip} = (M(x) - \frac{1}{\Omega_1} \int_{\Omega_1} M dx) \quad \text{in } \Omega \]
\[ \partial_n u_{ip} = 0 \quad \text{on } \partial \Omega \]
\[ \int_{\Omega} u_{ip} dx = 0 \]

Notice that \( u_{ip} \) is uniquely determined. Now since \( \int_{\Omega} u_{ip} dx = 0 \) and \( \int_{\Omega} G(x; x_j) dx = 0 \) then \( \int_{\Omega} u_i dx = \bar{u}_i |\Omega_1| \), and so \( \bar{u}_i = \frac{1}{|\Omega_1|} \int_{\Omega} u_i dx \).

Now we expansion \( (\ast) \) as \( x \to x_j \) for each \( j = 1, \ldots, N \) to obtain
\[ U_i \sim u_{ip} (x_j) - 4\pi \mu \left( \sum_{i \neq j}^{N} C_i G(x_j; x_i) + C_j \left( \frac{1}{4\pi |x - x_j|} + R_j \right) \right) + \bar{u}_i. \]

We write \( U_i \sim B_j + \bar{u}_j - \frac{\mu}{|x - x_j|} \quad \text{as } x \to x_j \).
WHERE \[ B_j = U_{j \rho} (x_j) - 4 \pi \mu \left( C_j R_j + \sum_{i \neq j}^N C_i G(x_j; x_i) \right) \]

THEN THE MATCHING CONDITION IS
\[ \frac{U}{\varepsilon} + U_1 + e U_2 + \cdots \sim \frac{V_0}{\varepsilon} + V_1 + \cdots \]

WRITING THIS OUT WE GET
\[ \frac{U}{\varepsilon} + \bar{U}_1 + B_j - \frac{U}{\varepsilon} \frac{C_j}{|x - x_j|} + e U_2 \sim \frac{U}{\varepsilon} \left( 1 - \frac{C_j}{|x - x_j|} \right) + V_1 \]

THIS IMPLIES THAT \( V_1 \) SATISFIES, FOR EACH \( j = 1, \ldots, N \),
\[ \Delta_y V_1 = 0, \quad |y| \geq \Gamma_j \]
\[ V_1 = d_j, \quad |y| = \Gamma_j \]
\[ V_1 \sim \bar{U}_1 + B_j \quad A_j, \quad |y| \to \infty \]

THEN WE HAVE
\[ V_1 = (\bar{U}_1 + B_j) - \left[ (\bar{U}_1 + B_j) - d_j \right] C_j \frac{1}{|y|} \]

WHERE
\[ \Delta_y V_c = 0, \quad |y| \geq \Gamma_j \]
\[ V_c = 1, \quad |y| = \Gamma_j \]
\[ V_c \sim \frac{C_j}{|y|} \quad A_j, \quad |y| \to \infty \]

THEREFORE,
\[ V_1 \sim (\bar{U}_1 + B_j) - \left[ (\bar{U}_1 + B_j) - d_j \right] \frac{C_j}{|y|} \quad \text{as} \quad |y| \to \infty \]

This implies that \( U_2 \) SATISFIES
\[ \Delta U_2 = 0 \quad \text{in} \quad \Omega \setminus \{x_1, \ldots, x_N\} \]
\[ \partial_n U_2 = 0 \quad \text{on} \quad \partial \Omega \]
\[ U_2 \sim \left[ d_j - (\bar{U}_1 + B_j) \right] C_j \frac{1}{|x - x_j|} \quad A_j, \quad x \to x_j \]

THEREFORE,
\[ \Delta U_2 = -4 \pi \sum_{j=1}^N \left[ d_j - (\bar{U}_1 + B_j) \right] C_j \frac{\delta (x - x_j)}{|x - x_j|} \]
Finally, we determine \( \bar{u}_j \) by divergence theorem,

\[
\sum_{j=1}^{N} \left[ d_j - \left( \bar{u}_j + B_j \right) \right] C_j = 0
\]

Thus yield \( \bar{u}_j \sum_{j=1}^{N} C_j = \sum_{j=1}^{N} \left( d_j - B_j \right) C_j \).

This yield that \( \bar{u}_j = \frac{\sum_{j=1}^{N} \left( d_j - B_j \right) C_j}{\sum_{j=1}^{N} C_j} \).

In summary, a two-term outer expansion is given by

\[
\bar{u} = \frac{\mu}{\varepsilon} + \bar{u}_j + \ldots
\]

with \( \mu = -\frac{1}{4\pi} \int_{\Omega} \frac{M(x)}{r} \frac{1}{C_j} d\mathbf{x} \).

while \( \bar{u}_j = \bar{u}_{ip}(x) + \bar{u}_j - 4\pi \mu \sum_{j=1}^{N} C_j G(x; x_j) \).

with \( \bar{u}_j \) is given above

and \( B_j = \bar{u}_{ip}(x_j) - 4\pi \mu \left( C_j R_j + \sum_{i \neq j}^{N} C_j G(x_i; x_j) \right) \).

Now consider the special case with \( M \) constant and

\( \Delta \bar{u} = M \) in \( \varepsilon < r < 2 \)

exact solution

\( \bar{u}_{ip} = 0 \) on \( r = 1 \)

\( \bar{u} = 1 \) on \( r = \varepsilon \)

Then \( j = 1, \ x_i = 0, \ C_i = 1, \ d_i = 1 \) and \( \bar{u}_{ip} = 0 \) since \( M \) constant.

Now \( \Delta \bar{G} = 0 \) in \( \Omega \setminus \{0\} \)

\( \bar{G} = 0 \) on \( r = 1 \)

\( \bar{G} \sim + \frac{1}{4\pi r} + \mathcal{R}, \ \text{as} \ r \to 0 \)

\( \int_{\Omega} \bar{G} d\mathbf{x} = 0 \)

we calculate

\( \bar{G} = D_0 + \frac{D_1}{r} + \frac{r^2}{16\pi} \).
Now \( D_0 = \frac{1}{4\pi} \), \( |\Omega| = \frac{4\pi}{3} \).

Then \( G = D_0 + \frac{1}{4\pi \Gamma} + \frac{r^2}{8\pi} \).

Notice that \( G = 0 \) on \( \Gamma = 1 \). Now \( \int_0^1 r^2 G \, dr = 0 \) so that

\[
\int_0^1 \left( D_0 r^2 + \frac{r}{4\pi} + \frac{r^4}{8\pi} \right) \, dr = 0 \rightarrow \frac{D_0}{3} + \frac{1}{8\pi} + \frac{1}{40\pi} = 0
\]

Thus,

\[
\frac{D_0}{3} + \frac{-9}{40\pi} = 0 \quad \Rightarrow \quad D_0 = -\frac{9}{20\pi}.
\]

Then \( G = -\frac{9}{20\pi} + \frac{1}{4\pi \Gamma} + \frac{r^2}{8\pi} \).

Now as \( \Gamma \to 0 \), \( G \approx \frac{1}{4\pi \Gamma} + R_1 \), with \( R_1 = -\frac{q}{20\pi} \).

Now we calculate \( J = -\frac{1}{4\pi c_1} \int_\Omega M \, dx = -\frac{M}{4\pi (l_1)} \left( \frac{4\pi}{3} \right) = -\frac{M}{3} \).

Now we calculate that

\( B_1 = -4\pi \mu c_1, R_1 = -4\pi \left( -\frac{M}{3} \right) \left( -\frac{9\pi}{20\pi} \right) = -\frac{3M}{5\pi} \).

Then \( U_1 = (d_1 - B_1) c_1 = (1 + \frac{3M}{5}) \).

Then \( U = U_{JP} + U_1 - 4\pi \mu c_1 G = 1 + \frac{3M}{5} - 4\pi \left( -\frac{M}{3} \right) \left[ -\frac{9}{20\pi} + \frac{1}{4\pi \Gamma} + \frac{r^2}{8\pi} \right] \)

\( U_1 = 1 + \frac{4\pi}{3} M \left( \frac{1}{4\pi \Gamma} + \frac{r^2}{8\pi} \right) = 1 + \frac{M}{3\Gamma} + \frac{M r^2}{6} \).

Therefore to two terms \( U \sim \frac{M}{\varepsilon} + U_1 + ... \)

with \( U \sim 1 + \frac{M}{3} \left( \frac{1}{r} - \frac{1}{\varepsilon} \right) + \frac{M r^2}{6} \)

which agrees with the exact solution up to term of \( O(\varepsilon^3) \).