A Unified Hybrid Asymptotic-Numerical Approach for Low Reynolds Number Flow Calculations

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In this paper....

1. Introduction

In this paper we consider several variants of the classical problem of slow, steady, two-dimensional flow of a viscous incompressible fluid around an infinitely long straight cylinder. By slow we mean that the Reynolds number $\varepsilon \equiv U_\infty L/\nu$ is small, where U_∞ is the velocity of the fluid at infinity, ν is the kinematic viscosity, and $2L$ is the diameter of the cross-section of the cylinder.

For a circular cylinder with no slip boundary condition, and in the limit $\varepsilon \to 0$, the method of matched asymptotic expansions was used systematically in Kaplun (1957) and in Proudman & Pearson (1957) to resolve the well-known Stokes paradox, and to calculate asymptotically the stream function in both the Stokes region, which is near the body, and in the Oseen region, which is far from the body. These pioneering studies showed that, for $\varepsilon \to 0$, the asymptotic expansion for the drag coefficient C_D of a circular cylindrical body starts with $C_D \sim 4\pi\varepsilon^{-1}\mathcal{F}(\varepsilon)$, where $\mathcal{F}(\varepsilon)$ is an infinite series in powers of $1/\log\varepsilon$. The coefficients in this series are determined in terms of the solutions to certain forced Oseen problems.

In an effort to determine C_D quantitatively, analytical formulae for the first three coefficients in $\mathcal{F}(\varepsilon)$ were derived in Kaplun (1957). However, as a result of the slow decay of $1/\log\varepsilon$ with decreasing values of ε, the resulting three-term truncated series for C_D agrees rather poorly with the experimental results of Tritton (1959) unless ε is very small. Owing to the complexity of the calculations required, it is impractical to obtain a closer quantitative determination of the drag coefficient by calculating further coefficients in $\mathcal{F}(\varepsilon)$ analytically. As a result of these fundamental long-standing difficulties, the problem of slow viscous flow around a cylinder has served as a paradigm for problems where matched asymptotic analysis fails to be of much practical use, unless ε is very small.

In Kropinski *et al.* (1995) this problem was re-examined and a hybrid asymptotic-numerical method was formulated and implemented to effectively sum the infinite logarithmic expansions that arise from the singular perturbation analysis of slow viscous flow around a cylinder that has a cross-section that is symmetric with respect to the free stream. This approach differed from the hybrid method employed in Lee & Leal (1986) in which numerical methods are used within the framework of the method of matched asymptotic expansions to calculate the first few coefficients in the logarithmic

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expansions of the flow field and the drag coefficient. Instead, it was shown in Kropinski *et al.* (1995) that these entire infinite logarithmic series are contained in the solution to a certain related problem that does not involve the cross-sectional shape of the cylinder. The methodology of Kropinski *et al.* (1995) to treat infinite logarithmic expansions was extended in Titcombe *et al.* (2000) to allow for cylinders of arbitrary cross-section.

Infinite logarithmic expansions also arise in the analysis of singularly perturbed eigenvalue problems, in steady-state heat transfer, in low Peclet number convection, and in nonlinear biharmonic problems of MEMS (cf. Kropinski *et al.* (2011)). A unified approach to treat such problems is surveyed in Ward & Kropinski (2010). A comprehensive recent survey on asymptotic and renormalization group methods applied to slow viscous flow problems is given in Veysey II & Goldenfeld (2007).

In this paper, we use the hybrid asymptotic approach to ....

2. The Hybrid Formulation and an Exactly Solvable Model Problem

We first outline the conventional singular perturbation analysis of (2.1) for $\varepsilon \to 0$ (cf. Kaplun (1957) and Proudman & Pearson (1957)) for slow, steady, incompressible, viscous flow around a circular cylindrical body with a uniform stream of speed U_∞ in the x direction at large distances from the body. We then formulate the hybrid method of Kropinski *et al.* (1995) for summing the infinite-order logarithmic expansions that arise from the analysis.

In terms of polar coordinates centred inside the body, it is well-known that the dimensionless stream function ψ satisfies

$$\Delta_r^2 \psi = -\varepsilon\, J_r\left[\psi, \Delta_r \psi\right] , \quad \text{for} \quad r > 1 , \tag{2.1a}$$

$$\psi = \partial_r \psi = 0 , \quad \text{on} \quad r = 1 , \tag{2.1b}$$

$$\psi \sim r \sin\theta , \quad \text{as} \quad r \to \infty . \tag{2.1c}$$

Here $\varepsilon \equiv U_\infty L/\nu \ll 1$ is the Reynolds number based on the radius L of the cylinder, ν is the kinematic viscosity, Δ_r and Δ_r^2 denote the Laplacian and Biharmonic operators in terms of the Stokes variable r, respectively, and $J_r\left[a, b\right] \equiv r^{-1}\left(\partial_r a\, \partial_\theta b - \partial_\theta a\, \partial_r b\right)$ is the Jacobian.

In the Stokes, or inner, region where $r = \mathcal{O}(1)$, the stream function has an infinite logarithmic expansion, referred to as the Stokes expansion, of the form

$$\psi_s(r,\theta) = \sum_{j=1}^{\infty} \nu^j \psi_j(r,\theta) + \cdots , \tag{2.2}$$

where $\nu = \nu(\varepsilon) \equiv -1/\log\left(\varepsilon e^{1/2}\right)$. Upon substituting (2.2) into (2.1a), we obtain that $\psi_j = a_j \psi_c$, where the a_j for $j \geqslant 1$ are undetermined constants and $\psi_c \equiv \psi_c(r,\theta)$ is the unique solution to the canonical Stokes problem

$$\Delta_r^2 \psi_c = 0 , \quad \text{for} \quad r > 1 , \tag{2.3a}$$

$$\psi_c = \psi_{cr} = 0 , \quad \text{on} \quad r = 1 , \tag{2.3b}$$

$$\psi_c \sim r \log r \sin\theta , \quad \text{as} \quad r \to \infty , \tag{2.3c}$$

which has the solution

$$\psi_c = \left(r \log r - \frac{r}{2} + \frac{1}{2r} \right) \sin\theta . \tag{2.3d}$$

Upon substituting $\psi_j = a_j \psi_c$ and (2.3d) into (2.2), the far-field behaviour of the Stokes

expansion is

$$\psi_s(r,\theta) \sim \sum_{j=1}^{\infty} \nu^j \, a_j \left(\log r - \log \left[e^{1/2} \right] \right) r \sin\theta \,, \quad \text{as} \quad r \to \infty \,. \tag{2.4}$$

In terms of the Oseen, or outer, length-scale ρ, defined by $\rho = \varepsilon r$, (2.4) becomes

$$\psi_s \sim \frac{1}{\varepsilon} \left(a_1 \rho \sin\theta + \sum_{j=1}^{\infty} \nu^j \, \left[a_j \rho \log \rho + a_{j+1} \rho \right] \sin\theta \right) . \tag{2.5}$$

This expression provides a singularity structure for the Oseen, or outer, solution as $\rho \to 0$.

The behaviour (2.5) suggests that in the Oseen region, where $\rho = \mathcal{O}(1)$, we introduce the new variable Ψ by $\Psi(\rho,\theta) = \varepsilon \psi(\varepsilon^{-1}\rho, \theta)$, and that we expand Ψ as

$$\Psi(\rho,\theta) = \rho \sin\theta + \nu \Psi_1(\rho,\theta) + \sum_{j=2}^{\infty} \nu^j \Psi_j(\rho,\theta) + \cdots \,, \tag{2.6}$$

in order to satisfy the free-stream condition as $\rho \to \infty$ in (2.1c). Upon substituting (2.6) into (2.1a), and matching Ψ as $\rho \to 0$ to the required singular behaviour (2.5), we find that $a_1 = 1$ and that Ψ_j, for $j \geqslant 1$, satisfy the following forced Oseen problems on $0 < \rho < \infty$:

$$L_{0s} \Psi_1 \equiv \Delta_\rho^2 \Psi_1 + \left(\rho^{-1} \sin\theta \, \partial_\theta - \cos\theta \, \partial_\rho \right) \Delta_\rho \Psi_1 = 0 \,, \tag{2.7a}$$

$$\Psi_1 \sim (\log \rho + a_2) \, \rho \sin\theta \,, \quad \text{as} \quad \rho \to 0 \,; \quad \partial_\rho \Psi_1 \to 0 \,, \quad \text{as} \quad \rho \to \infty \,, \tag{2.7b}$$

$$L_{0s} \Psi_j = - \sum_{k=1}^{j-1} J_\rho \left[\Psi_k, \Delta_\rho \Psi_{j-k} \right] \,, \tag{2.7c}$$

$$\Psi_j \sim (a_j \log \rho + a_{j+1}) \, \rho \sin\theta \,, \quad \text{as} \quad \rho \to 0 \,; \quad \partial_\rho \Psi_j \to 0 \,, \quad \text{as} \quad \rho \to \infty \,. \tag{2.7d}$$

Here L_{0s} is the linearized Oseen operator and Ψ_1 is the linearized Oseen solution.

The forced Oseen problems in (2.7) recursively determine the coefficients a_j for $j \geqslant 2$. The first two coefficients are well-known, and are given by (cf. Kaplun (1957), Proudman & Pearson (1957))

$$a_2 = \gamma_e - \log 4 - 1 \approx -1.8091 \,, \tag{2.8a}$$

$$a_3 - a_2^2 = - \int_0^\infty \left[r^{-1} I_1(2r) + 1 - 4 K_1(r) I_1(r) \right] K_0(r) K_1(r) \, dr \approx -0.8669 \,. \tag{2.8b}$$

Here K_1, K_0, I_0 and I_1 are the usual modified Bessel functions, and γ_e is Euler's constant. The expression for a_2 was first obtained in Proudman & Pearson (1957), while the expression for a_3 was given in Kaplun (1957). Explicit analytical formulae for a_j when $j \geqslant 4$ are not available.

In terms of the constants a_j for $j \geqslant 2$, the well-known drag coefficient C_D is given by (cf. Kaplun (1957))

$$C_D \sim 4\pi \varepsilon^{-1} \nu \left(1 + \sum_{j=2}^{\infty} a_j \nu^{j-1} + \cdots \right) , \qquad \nu \equiv - \frac{1}{\log \left[\varepsilon e^{1/2} \right]} \,. \tag{2.9}$$

However, as discussed in Van Dyke (1975) (see also Veysey II & Goldenfeld (2007)), the the truncated three-term expansion for C_D agrees rather poorly with the experimentally measured drag coefficient of Tritton (1959) unless ε is very small (cf. Van Dyke (1975)).

In order to obtain a higher order approximation to the drag coefficient one can, in

principle, compute numerically further coefficients a_j, for $j \geqslant 4$, from the the infinite sequence of PDE's (2.7c) with singularity structures (2.7d). This recursive determination of the coefficients would still require truncating the series (2.9) at some finite j. As an alternative to series truncation, in Kropinski *et al.* (1995) a hybrid asymptotic-numerical method was proposed that has the effect of summing all the terms on the right-hand side of (2.9), but which avoids computing the coefficients a_j for $j \geqslant 1$ individually.

In this hybrid approach of Kropinski *et al.* (1995), the Oseen solution is no longer expanded in powers of ν as in (2.6). Instead, the full problem (2.1a) and (2.1c) for $\rho > 0$ is solved subject to a parameter-dependent singularity structure, which is to hold as $\rho \to 0$. In this way, $\Psi_H \equiv \Psi_H(\rho, \theta; S)$ is defined to be the solution to

$$\Delta_\rho^2 \Psi_H = -J_\rho \left[\Psi_H, \Delta_\rho \Psi_H\right] , \quad \rho > 0 , \tag{2.10a}$$

$$\Psi_H \sim \rho \sin\theta , \quad \text{as} \quad \rho \to \infty , \tag{2.10b}$$

$$\Psi_H \sim S\rho \log\rho \sin\theta , \quad \text{as} \quad \rho \to 0 . \tag{2.10c}$$

In Kropinski *et al.* (1995), this parameter-dependent problem is solved numerically for a range of S values, and in terms of this solution we identify the regular part $R = R(S)$ of the singularity structure at the origin by the limiting process

$$\Psi_H - S\rho \log\rho \sin\theta = R(S)\rho \sin\theta + o(\rho) , \quad \text{as} \quad \rho \to 0 . \tag{2.10d}$$

Upon introducing the Stokes variables r and ψ defined by $r = \rho/\varepsilon$ and $\psi = \psi_H/\varepsilon$, (2.10$d$) becomes

$$\psi \sim \left[Sr \log r + (S \log\varepsilon + R)\, r\right] \sin\theta , \tag{2.11}$$

which determines the required far-field behaviour of the Stokes solution. The Stokes, or inner, solution that satisfies $\psi \sim Sr \log r \sin\theta$ as $r \to \infty$ is simply $\psi = S\psi_c$, where ψ_c is given in (2.3d). Finally, upon matching the $\mathcal{O}(r \sin\theta)$ terms in (2.11) and $S\psi_c$, we obtain that $S = S(\nu)$ satisfies the transcendental equation

$$\frac{S}{R(S)} = \nu \equiv -\frac{1}{\log\left[\varepsilon e^{1/2}\right]} . \tag{2.12}$$

In terms of $S = S(\nu)$, the drag coefficient accurate to all powers of ν is

$$C_D \sim 4\pi\varepsilon^{-1} S(\nu) = 4\pi\varepsilon^{-1} \nu R\left[S(\nu)\right] \tag{2.13}$$

This completes the summary of the hybrid method of Kropinski *et al.* (1995).

In the next section, we need the following higher order result for the local behaviour of ψ_H as $\rho \to 0$:

Lemma 1. *The solution to (2.10) has the following asymptotic behaviour as $\rho \to 0$:*

$$\begin{aligned} \Psi_H \sim{}& \left[S\rho \log\rho + R\rho + o(\rho)\right] \sin\theta \\ &+ \left[\frac{S^2}{16}\left(\rho \log\rho\right)^2 + \frac{1}{8}\left(SR + \frac{S^2}{4}\right)\rho^2 \log\rho + \mathcal{C}_2 \rho^2\right] \sin(2\theta) + \cdots . \end{aligned} \tag{2.14}$$

Here $R = R(S)$ and $\mathcal{C}_2 = \mathcal{C}_2(S)$ must be computed from (2.10). To leading order as $S \to 0$, we have

$$R = 1 + a_2 S + \mathcal{O}(S^2) , \qquad \mathcal{C}_2 = \frac{S}{8}\left(a_2 + \frac{1}{2}\right) + \mathcal{O}(S^2) , \tag{2.15}$$

where a_2 is defined in (2.8). At one higher order, the curve $R = R(S)$ for $S \to 0$ is given

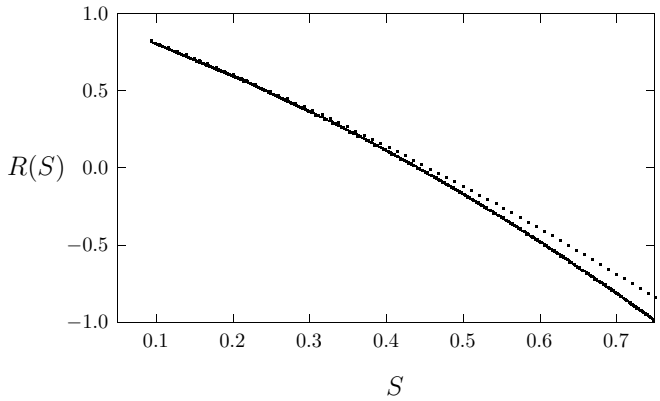

FIGURE 1. Plot of $R = R(S)$ computed numerically from the hybrid formulation (2.10) (heavy solid curve). The three-term approximation (2.17) is the dotted curve.

parametrically in terms of $\delta \ll 1$ *by*

$$S = \delta + a_2\delta^2 + \cdots\,, \qquad R \sim 1 + a_2\delta + a_3\delta^2 + \cdots\,, \tag{2.16}$$

where a_2 *and* a_3 *are defined in (2.8). This yields the three-term approximation for* $R(S)$

$$R(S) \sim 1 + a_2 S + \left(a_3 - a_2^2\right) S^2 + \mathcal{O}(S^3)\,, \quad \textit{for} \quad S \ll 1\,. \tag{2.17}$$

A plot of the numerically computed function $R = R(S)$, together with its three-term approximation from (2.17), is shown in Fig. 1. The numerical method used to calculate $R(S)$ is described in Kropinski *et al.* (1995) (see also Keller & Ward (1996)).

2.1. *An Exactly Solvable Model Problem*

In this subsection we illustrate the approach for treating problems with infinite logarithmic expansions by considering the following simple perturbed problem in an annulus:

$$\Delta_\rho^2 u = 0\,, \quad \varepsilon < \rho < 1\,, \tag{2.18a}$$

$$u = \sin\theta\,, \quad u_\rho = 0\,, \quad \text{on} \quad \rho = 1\,, \tag{2.18b}$$

$$u = u_\rho = 0\,, \quad \text{on} \quad \rho = \varepsilon\,. \tag{2.18c}$$

We first determine the exact solution of (2.18) and then expand it for $\varepsilon \to 0$. Since the solutions to (2.18a) proportional to $\sin\theta$ are linear combinations of $\{\rho^3, \rho\log\rho, \rho, \rho^{-1}\}\sin\theta$, the solution to (2.18$a$), which satisfies (2.18$b$), is given in terms of two constants A and B by

$$u = \left(A\rho^3 + B\rho\log\rho + \left(-2A + \frac{1}{2} - \frac{B}{2}\right)\rho + \left(\frac{1}{2} + A + \frac{B}{2}\right)\frac{1}{\rho}\right)\sin\theta\,. \tag{2.19}$$

Upon imposing that $u = u_\rho = 0$ on $\rho = \varepsilon$, we obtain that A and B satisfy

$$A\varepsilon^3 + B\varepsilon\log\varepsilon + \left(-2A + \frac{1}{2} - \frac{B}{2}\right)\varepsilon + \kappa\varepsilon = 0\,, \tag{2.20a}$$

$$3A\varepsilon^3 + B\varepsilon + B\varepsilon\log\varepsilon + \left(-2A + \frac{1}{2} - \frac{B}{2}\right)\varepsilon - \kappa\varepsilon = 0\,, \tag{2.20b}$$

where $\kappa = \mathcal{O}(1)$ is defined by

$$\kappa\varepsilon^2 = \frac{1}{2} + A + \frac{B}{2}\,. \tag{2.21}$$

We add the two equations in (2.20) to eliminate κ, and neglect the higher order $A\varepsilon^3$ terms. In addition, since $A \sim -(1+B)/2$ from (2.21), we obtain the approximate system

$$B + 2B\log\varepsilon \sim 4A - 1 + B\,, \qquad A \sim -(1+B)/2\,, \tag{2.22}$$

for A and B, which has the solution

$$B \sim \frac{3\nu}{2-\nu}\,, \qquad A = 1 - \frac{3}{2-\nu}\,, \qquad \text{where} \qquad \nu \equiv \frac{-1}{\log\left[\varepsilon e^{1/2}\right]}\,. \tag{2.23}$$

From (2.23) and (2.19), we obtain that the outer solution, valid for $\rho \gg \mathcal{O}(\varepsilon)$, is

$$u \sim \left(B\rho\log\rho + \frac{B}{\nu}\rho - \frac{1}{2}(1+B)\rho^3 + \mathcal{O}(\varepsilon^2)\right)\sin\theta\,, \quad \rho \gg \mathcal{O}(\varepsilon)\,. \tag{2.24}$$

where B and ν are defined in (2.23).

We remark that (2.24) contains all of the logarithmic terms in the expansion of the exact solution. However, it does not contain transcendentally small terms of algebraic order in ε as $\varepsilon \to 0$.

Next, we show how to derive (2.24) by formulating an appropriate singularity structure for the outer problem in terms of a free parameter S, representing the strength of the singularity. Then, S is determined from matching the outer solution to an appropriate inner solution. To this end, we look for an outer solution $u_H = u_H(\rho,\theta;S)$ that satisfies

$$\Delta_\rho^2 u_H = 0\,, \quad 0 < \rho < 1\,, \tag{2.25a}$$

$$u_H = \sin\theta\,, \quad \partial_\rho u_H = 0\,, \quad \text{on} \quad \rho = 1\,, \tag{2.25b}$$

$$u_H \sim S\rho\log\rho\sin\theta\,, \quad \text{as} \quad \rho \to 0\,. \tag{2.25c}$$

This problem has a unique solution since the singularity condition (2.25c) eliminates the more singular term of order $\mathcal{O}(\rho^{-1})$ as $\rho \to 0$.

In terms of the unique solution to (2.25), we define the regular part R of the singularity structure by

$$\psi_H \sim (S\rho\log\rho + R\rho + o(1))\sin\theta\,, \quad \text{as} \quad \rho \to 0\,. \tag{2.26}$$

The exact solution to (2.25) is readily calculated as

$$u_H = \left(S\rho\log\rho + R\rho + \alpha\rho^3\right)\sin\theta\,, \tag{2.27a}$$

where R and α are determined explicitly in terms of S as

$$R = \frac{S}{2} + \frac{3}{2}\,, \qquad \alpha = -\frac{S}{2} - \frac{1}{2}\,. \tag{2.27b}$$

In terms of the inner variable $r = \rho/\varepsilon$, the near-field behaviour of (2.27a) for $\rho \to 0$ is

$$u_H \sim \varepsilon\left[Sr\log r + r\left(S\log\varepsilon + R\right) + \mathcal{O}(\varepsilon^2)\right]\sin\theta\,. \tag{2.28}$$

The requirement that the far-field behaviour of the inner solution match with (2.28) is the condition that determines S.

The condition (2.28) yields that the inner solution v is $v(r,\theta) = \varepsilon^{-1}u(\varepsilon r,\theta)$, and must

satisfy

$$\Delta_r^2 v = 0\,, \quad r > 1\,, \tag{2.29a}$$

$$v = v_r = 0\,, \quad \text{on} \quad r = 1\,, \tag{2.29b}$$

$$v \sim \varepsilon S r \log r \sin\theta\,, \quad \text{as} \quad r \to \infty\,. \tag{2.29c}$$

The solution to (2.29c) is unique since we have eliminated the possibility of $\mathcal{O}(r^3)$ growth at infinity. The solution is

$$v = S\left[r\log r - \frac{r}{2} + \frac{1}{2r}\right]\sin\theta\,. \tag{2.30}$$

Upon requiring that the $\mathcal{O}(r)$ terms in (2.30) and εv agree, we obtain that S satisfies $S\log\varepsilon + R = -S/2$. Since $R = S/2 + 3/2$ from (2.27b), we can readily solve for S. In this way, we obtain that the outer solution u_H from (2.27a) is

$$u_H \sim \left(S\rho\log\rho + \frac{S}{\nu}\rho - \frac{1}{2}(S+1)\rho^3\right)\sin\theta\,, \tag{2.31a}$$

where

$$S = \frac{3\nu}{2-\nu}\,, \qquad R = \frac{S}{2} + \frac{3}{2} = \frac{S}{\nu}\,, \qquad \nu \equiv \frac{-1}{\log\left[\varepsilon e^{1/2}\right]}\,. \tag{2.31b}$$

This expression, referred to as the hybrid result, agrees with that obtained in (2.24) from an expansion of the exact solution.

The examination of this very simple model problem has shown that by formulating an outer problem with a parameter-dependent singularity structure, we are ultimately able to determine an approximate solution that contains all the logarithmic terms in powers of $-1/\log\varepsilon$, while avoiding calculating each individual term in an expansion of S in powers of ν as in

$$S \sim \frac{3\nu}{2\,(1-\nu/2)} = \frac{3\nu}{2}\sum_{j=0}^{\infty}\left(\frac{\nu}{2}\right)^j\,, \qquad \text{for} \quad \varepsilon < e^{-1} \approx 0.3679\,. \tag{2.32}$$

To highlight the performance of (2.31a) at finite ε, we define $f''(1)$ by $u_{\rho\rho} = f''(1)\sin\theta$ at $\rho = 1$ and in Fig. 2 we compare the exact, hybrid, and two-term result, for $f''(1)$ as a function of ε. The hybrid and two-term result for $f''(1)$, as obtained from (2.31) and (2.32), is

$$f''_H(1) = -2S - 3\,, \qquad f''_{2T}(1) = -3 - 3\nu\,. \tag{2.33}$$

The corresponding exact result is $f''_E(1) = 4A - 1$, where A satisfies (2.20).

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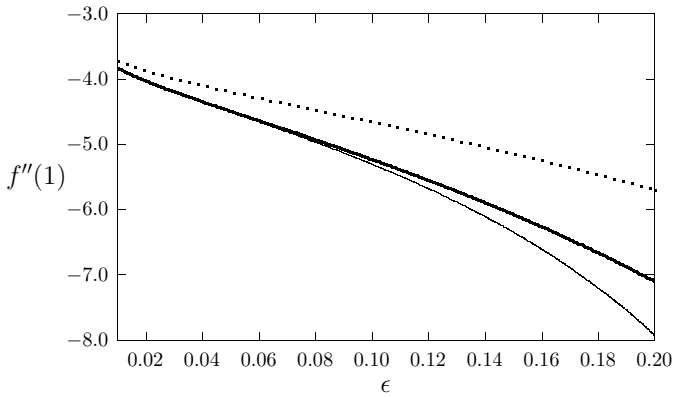

FIGURE 2. Plot of $f''(1)$ where $u = f(\rho)\sin\theta$ is either the exact or approximate solution to (2.18). The heavy solid curve is the exact result from (2.19) and (2.20), while the solid and dotted curves are the hybrid and 2-term results, respectively, as obtained from (2.33).

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