Problem 1

Consider the time-dependent problem

\[ \begin{align*}
    u(t) &= \frac{d}{dx} \left[ D(x, x/\varepsilon) \frac{du}{dx} \right] - F(x, x/\varepsilon) \quad 0 < x < 1, \quad t > 0 \\
    u(0, t) &= 0, \quad u(1, t) = 1, \quad u(x, 0) = g(x, x/\varepsilon)
\end{align*} \]

with \( g(0, 0) = 0 \), \( g(1, 1/\varepsilon) = 1 \).

Assume that \( F(x, y), g(x, y), D(x, y) \) are all periodic in the \( y \)-variable with period \( 2\pi \).

(i) Calculate the effective steady-state equation from homogenization theory.

(ii) Plot the exact steady-state solution and the homogenized steady-state solution for the special case:

\[ D(x, x/\varepsilon) = \frac{1}{1 + B \cos(x/\varepsilon)}, \quad F(x, x/\varepsilon) = 1 + \cos(x/\varepsilon) \]

with \( B = 0.95 \) and \( \varepsilon = 0.01 \).

What happens if one mistakenly uses the average

\[ D_{avg}(x) = \frac{1}{2\pi} \int_0^{2\pi} dy \frac{1}{1 + B \cos y} \]

in homogenized equation rather than the harmonic mean?

(with \( B = 0.95 \))

(iii) For the time-dependent problem, what is the homogenized equation on the \( x, t \) scales?

(iv) Discuss any transient-time dynamics for (\( \star \)). In other words, what is the problem to be solved on suitably short time scales? Derive from it the initial condition to be used for your homogenized problem in (iii).
PROBLEM 2 (THE CLAUSINS-MOSSEHTTI FORMULA)

We want to find the effective conductivity of a periodic array of spheres in the low-volume-fraction limit. The perturbed problem is

\[ \nabla \cdot [D(x) \nabla u] = f(x) \text{ in } \mathbb{R}^n \]

where \( D(y) \) is 1-periodic with respect to the unit cell \( \mathcal{Y} \)

\[ \mathcal{Y} = \left\{ (y_1, \ldots, y_3) \mid - \frac{1}{2} < y_j < \frac{1}{2}, \ j=1, \ldots, 3 \right\} \]

In the unit cell \( \mathcal{Y} \) we assume that

\[ D = \begin{cases} D_1 \text{ in } \mathcal{Y} \setminus B_8 & \frac{1}{2} \leq y \leq 1 \\ D_2 \text{ in } B_8 & -\frac{1}{2} \leq y < \frac{1}{2} \end{cases} \]

where \( B_8 = \left\{ y \mid 1 - 8 \leq y \leq 1 \right\} \)

Here \( D_1 > 0, D_2 > 0 \) are constants. Across each sphere we have

\[ u \text{ is continuous and } D_1 \nabla u \cdot \hat{n} \bigg|_{\text{out}} = D_2 \nabla u \cdot \hat{n} \bigg|_{\text{in}} \]

(i) For \( \varepsilon \to 0 \) derive the leading order problem for \( u \) to be solved on the macroscale \( x \). Determine the "cell problem" that needs to be solved.

(ii) In the low-volume-fraction limit \( \varepsilon \ll 1 \) solve the unit cell problem asymptotically using techniques from strong localized perturbation theory. In particular if \( F = 4\pi \delta^3 / 3 \ll 1 \) is the volume fraction show that

\[ \text{Deff} = D_1 + O(F) + \ldots \]

and calculate the coefficient of the \( O(F) \) term explicitly in terms of \( D_1 \) and \( D_2 \).

(Hint: Please read and study note for 2-D problem online)