Problem 1. Let Ω be a bounded 2-D domain. The MFPT, allowing for a variable diffusivity \( D(x) \) satisfies

\[
\begin{align*}
\Delta u &= \frac{1}{D(x)} \text{ in } \Omega \setminus \bigcup_{j=1}^{N} \Omega_{e_j} \\
d_n u &= 0 \text{ on } \partial \Omega \\
u &= 0 \text{ on } \partial \Omega_{e_j}, \quad j = 1, \ldots, N,
\end{align*}
\]

where we assume that the traps \( \Omega_{e_j} \) are circular disks of a common radius \( e \) centered at \( x_j \) in \( \Omega \) for \( j = 1, \ldots, N \).

(i) Derive a linear algebraic system to determine the average MFPT \( \bar{u} = \frac{1}{\left| \bigcup_{j=1}^{N} \Omega_{e_j} \right|} \int_{\bigcup_{j=1}^{N} \Omega_{e_j}} u \, dx \) asymptotically with an error smaller than any power of \( (\varepsilon \log \varepsilon)^{-1} \). (Your result should be in terms of the Neumann Green's function and a "particular" solution \( u_p \) satisfying \( \Delta u_p = F(x) \) with \( \int_{\Omega} F(x) \, dx = 0 \) and \( \int_{\Omega} u_p \, dx = 0 \) for some \( F(x) \) to be found.)

(ii) Solve the linear algebraic system determining \( \bar{u} \) when \( \Omega \) is the unit disk with \( D = \frac{1}{q_0 + 1/(2q_0^2)} \), \( q_0 = 1/2, q = 3/2 \)

and there are four traps of radius \( e \) centered at either

(A) \( (\pm 0.5/\sqrt{2}, \pm 0.5/\sqrt{2}) \)

or (B) \( (\pm 0.5, 0), (0, \pm 0.5) \)

Plot \( \bar{u} \) versus \( e \) on \( 0 < e < 0.035 \) for configuration (A) and (B) and also compare with result for the spatially uniform case where \( D = 1 \) in \( \Omega \).
Problem 2. Consider the non-dimensional Gray-Scott reaction-diffusion model written as:

\[
\begin{align*}
\dot{V} &= D_1 V_{xx} - V + AV^2 \\ 
\dot{U} &= D_2 U_{xx} + (1-U) - UV^2
\end{align*}
\]

(i) Suppose that \( 0 < T < 2 \). Find and classify the stability properties of the equilibrium point of the kinetics to spatially homogeneous perturbations. Show that these equilibria exhibit bistability for some range of \( A \). Plot a bifurcation diagram of \( V \) versus \( A \).

(ii) Suppose that \( 0 < T < 2 \). Determine any \textit{turing-type} instabilities of the equilibria found in (i). In particular, on the infinite line show that such an instability occurs for the equilibrium \((V_+, U_+)\) given by:

\[
\left( \frac{A + \sqrt{A_0^2 - 4}}{2}, \frac{1}{A V_+} \right)
\]

if and only if \( D_2/D_1 > 1 + V_+^2 \) and \( D_2/D_1 > \alpha \), where \( \alpha \) is the largest root of \((\alpha - (1 + V_+^2))^2 = 4 \alpha (V_+^2 - 1)\).

(iii) For the \((U_+, V_+)\) equilibrium in (ii), show that we can have a \textit{Hopf bifurcation} together with a \textit{turing-type} instability along some curve in the \( D_2/D_1 \) vs \( \gamma \) parameter plane. Such a \textit{co-dimension 2} instability is called a \textit{Turing-Hopf} instability. Plot this curve on \( 2 < A < 6 \).

What would you expect to see numerically for the solution to (\(\dot{\chi}\)) near this co-dimension two point?

(iv) Look up the Gray-Scott model online. Write a brief paragraph describing what it models physically.
Problem 3 The reaction-diffusion model of Bertozzi et al. of urban crime is to solve for the attractiveness $A$ and criminal density $P$ in

\[
\begin{aligned}
A_t &= \varepsilon^2 A_{xx} - A + PA + \alpha \\
\tau P_t &= D \left( P_x - \frac{2 P A_x}{A} \right)_x - \alpha P + (\gamma - \alpha) \\
\end{aligned}
\]

with $\alpha > 0$, $\gamma > 0$ (constant) and $\gamma - \alpha > 0$.

(i) On the infinite line show that the spatially homogeneous equilibrium solution $A_e, P_e$ is stable to spatially uniform perturbations for all $\tau > 0$.

(ii) For perturbations of the form $A = A_e + e^{\lambda t + imx}$, $P = P_e + e^{\lambda t + imx}$

show that $\lambda$ satisfies

\[
\left( \frac{\lambda + (A_e + Dm^2)}{\tau} \right) \left( \lambda + \varepsilon^2 m^2 + (1 - P_e) \right) + \frac{A_e}{\tau} \left( P_e - 2 D \frac{P_e m^2}{A_e} \right) = 0.
\]

(iii) Let $\varepsilon \to 0^+$. Show that if $\gamma > 3 \alpha/2$ there is an instability band $M_{low} < M < M_{up}$ within which $\lambda > 0$.

Here $M_{low} \sim \frac{\gamma}{\sqrt{D} (2 \gamma - 3 \alpha)^{1/2}} \varepsilon \to 0^+$

and $M_{up} \sim \frac{1}{\varepsilon} (2 \gamma - 3 \alpha)^{1/2} \lambda \varepsilon \to 0^+$

(iv) Show that the most unstable mode within the instability band in (iii) is given by

\[
M_{dominant} \sim \frac{1}{D^{1/4} \varepsilon^{1/2}, \gamma^{1/2}} X(\gamma, \alpha, \tau)
\]

where $X(\gamma, \alpha, \tau)$ is some function you are to find.

Show that the maximum corresponding growth rate is

\[
\lambda_{dominant} \sim 3 P_e - 1 + O(\varepsilon).
\]
Now consider \( x \) on a domain of length 1 with \( a = 1, \sigma = 2, \epsilon = 0.02, b = 1 \) and with initial condition

\[
A(x, 0) = \phi \left( 1 - 0.01 \cos(6\pi x) \right) \quad p(x, 0) = 1 - \frac{x}{\phi}
\]

Assume Neumann condition \( A_x = p_x = 0 \) at \( x = 0,1 \).

What does the theory predict about the instability band of wavenumber? Plot this instability band for the parameter \( \alpha \) and identify on this the most unstable mode.

Solve the PDE system numerically and plot \( A(x, t) \) versus \( x \) for different time up to \( 0 \leq t \leq 35 \).

What do you see? Explain qualitatively.