PROBLEM 1: In a 3-D domain the splitting probability $U(x)$ is defined as the probability of reaching a specific target trap $\Omega_{x_j}$ from the initial source point $x$ before reaching any of the other surrounding traps $\Omega_{x_j}$ for $j = 2, \ldots, N$. It is well-known that $U(x)$ satisfies

$$
\begin{align*}
\Delta U &= 0, \quad x \in \Omega \\
\sum_{j=1}^{N} \mathcal{P}_{x_j} &\subset \Omega_{x_j} \\
\frac{\partial U}{\partial n} &= 0, \quad x \text{ on } \partial \Omega \\
U &= 1, \quad x \text{ on } \partial \Omega_{x_j} \\
U &= 0, \quad x \text{ on } \cup_{j=2}^{N} \partial \Omega_{x_j},
\end{align*}
$$

Assume for simplicity that $\Omega_{x_j}$ is a small sphere of radius $\epsilon a_j$ centered at $x_j$ in $\Omega$ where $a_j > 0$.

(i) Show for $\epsilon \rightarrow 0$ that a two term expansion for $U(x)$ in the outer region has the form

$$
U \sim \frac{C_i}{N \epsilon} + 4\pi \epsilon C_i \left[ G(x; x_1) - \frac{1}{N \epsilon} \sum_{j=1}^{N} C_j G(x; x_j) \right] + \epsilon^2 C_i + O(\epsilon^2),
$$

where $C = \frac{1}{N} (C_1 + \cdots + C_N)$ and $C_j$ is capacitance of $j^{th}$ trap, while $G(x; x_s)$ is the unique Neumann g-function satisfying

$$
\Delta g = \delta(x - x_s), \quad x \in \Omega \\
\frac{\partial g}{\partial n} = 0, \quad x \text{ on } \partial \Omega \\
g(x; x_s) = -\frac{1}{4\pi |x - x_s|} + R(x_0; x_s + \Omega) \text{ at } x \rightarrow x_0,
$$

(iii) Determine $x_1$ in (i) explicitly by going to one higher order in the expansion.
**Problem 2:** Consider two rooms $\Omega_1$ and $\Omega_2$ in 3-D, modeled by cubes of volume $|\Omega_1|$ and $|\Omega_2|$ attached by a narrow opening of circular shape of radius $\varepsilon$. The side view is

![Side view of two rooms](image)

Consider the time dependent heat equation

$$u_t = \Delta u \quad x \text{ in } \Omega_1 \cup \Omega_2$$

$$\partial_\nu u = 0 \quad \text{on } \partial \Omega_1 \cup \partial \Omega_2 \setminus \partial \Omega_\varepsilon$$

$$u(x, 0) = f(x) \quad x \text{ in } \Omega_1 \cup \Omega_2$$

where $\partial \Omega_\varepsilon = \{ (x_1, x_2, x_3) \mid x_1 = 0 \text{ and } x_2^2 + x_3^2 \leq \varepsilon^2 \}$ is the hole between the two rooms.

Find, for $\varepsilon \ll 1$, an approximation to the solution valid for $t \gg 1$ that shows how $u$ tends to the steady-state limit.
PROBLEM 3 (THE CLAUSIUS-MOSER FORMULA)

We want to find the effective conductivity of a periodic array of spheres in the low-volume-fraction limit. The perturbed problem is

\[ \nabla \cdot \left[ D(y/\epsilon) \nabla u \right] = f(x) \text{ in } \mathbb{R}^N \]

where \( D(y) \) is 1-periodic with respect to the unit cell \( \Gamma \)

\[ \Gamma = \left\{ (y_1, y_2, y_3) \mid -1/2 < y_j < 1/2, \ j = 1, 2, 3 \right\} \]

in the unit cell \( \Gamma \), we assume that \( D \) is a periodic function within \( \Gamma \) and \( \partial D \) on \( \partial \Gamma \).

\[ D = \begin{cases} 
D_1 & \text{in } \Gamma \setminus B_3 \\
D_2 & \text{in } B_3
\end{cases} \]

where \( B_3 = \left\{ y \mid |y| < \delta \right\} \) with \( 0 < \delta < 1 \).

Here \( D_1 > 0, D_2 > 0 \) are constants. Across each sphere, we have \( u \) is continuous and - \( D_1 \nabla u \cdot \hat{n} \big|_{\text{out}} = D_2 \nabla u \cdot \hat{n} \big|_{\text{in}} \).

(i) For \( \epsilon \to 0 \) draw the leading order problem for \( u \) to be solved on the macroscale \( x \). Determine the "cell problem" that needs to be solved.

(ii) In the low-volume-fraction limit \( \delta \ll 1 \) solve the unit cell problem asymptotically using techniques from strong localized perturbation theory. In particular, if \( F = 4\pi \delta^3/3 \ll 1 \) is the volume fraction, show that

\[ \text{Deff} = D_1 + O(F) \]

and calculate the coefficient of the \( O(F) \) term explicitly in terms of \( D_1 \) and \( D_2 \).

(Hint: Please read and study notes for 2-D problem online)