PROBLEM 1  CONSIDER THE TIME-DEPENDENT PROBLEM

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial u}{\partial t} &= \frac{1}{\varepsilon} \left[ D(x, x/\varepsilon) \frac{\partial u}{\partial x} \right] - f(x, x/\varepsilon) \quad 0 < x < 1, \ t > 0 \\
u(0, t) &= 0, \quad u(1, t) = 1, \quad u(x, 0) = g(x, x/\varepsilon)
\end{array} \right.
\end{align*}
\]

WITH \( g(0, 0) = 0, g(1, 1/\varepsilon) = 1. \)

ASSUME THAT \( f(x, y), g(x, y), D(x, y) \) ARE ALL PERIODIC IN THE \( y \)-VARIABLE WITH PERIOD \( 2\pi \).

(i) CALCULATE THE EFFECTIVE STEADY-STATE EQUATION FROM HOMOGENIZATION THEORY.

(ii) PLOT THE EXACT STEADY-STATE SOLUTION AND THE HOMOGENIZED STEADY-STATE SOLUTION FOR THE SPECIAL CASE

\[
D(x, x/\varepsilon) = \frac{1}{1 + B \cos(x/\varepsilon)} \quad f(x, x/\varepsilon) = 1 + \cos(x/\varepsilon)
\]

WITH \( B = 0.95 \) AND \( \varepsilon = 0.01 \).

WHAT HAPPENS IF ONE MISTAKENLY USES THE AVERAGE

\[
D_{\text{avg}}(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{dy}{1 + B \cos y}
\]

IN HOMOGENIZED EQUATION RATHER THAN THE HARMONIC MEAN? (WITH \( B = 0.95 \))

(iii) FOR THE TIME-DEPENDENT PROBLEM WHAT IS THE HOMOGENIZED EQUATION ON THE \( x, t \) SCALES?

(iv) DISCUSS ANY TRANSIENT-TIME DYNAMICS FOR (\( x \)). IN OTHER WORDS, WHAT IS THE PROBLEM TO BE SOLVED ON SUITABLY SHORT TIME SCALES? DERIVE FROM IT THE INITIAL CONDITION TO BE USED FOR YOUR HOMOGENIZED PROBLEM IN (iii).
PROBLEM 2

IN A TWO-DIMENSIONAL HALF-SPACE LET \( u(x,z) \) SOLVE

\[
\nabla \cdot [\sigma \nabla u] - \alpha u = f(x,z) \quad \text{in} \quad z > h(x/\varepsilon) \quad \nabla z \cdot (\partial / \partial x, \partial / \partial z) \nabla = 0 \quad \text{on} \quad z = h(x/\varepsilon) \quad \text{where} \quad \hat{n} = \text{unit normal to} \ z = h(x/\varepsilon)
\]

HERE \( \alpha > 0, \ \sigma = \sigma_0 > 0 \) constants and we assume that

\( h(y+1) = h(y) \quad \forall y \)

with \( h_{\min} = -A \varepsilon_0 \) and \( h_{\max} = 0 \) as shown.

(i) Find, by a multi-scale expansion, the homogenized limiting problem valid for \( \varepsilon \to 0 \). In your derivation assume \( A = O(1) \) as \( \varepsilon \to 0 \).

(ii) With the insight gained from (i), write down the homogenized limiting problem to be solved for the eigenvalues of

\[
\begin{array}{c}
\Delta u + \lambda u = 0 \\
\quad u = 0 \\
\end{array}
\]

THE DOMAIN IS

\[
\begin{array}{c}
0 < x < 1 \\
0 < z < b \\
\text{on} \ z = h(x/\varepsilon) \\
\hat{n} \ n = 0 \quad \text{on} \ z = h(x/\varepsilon)
\end{array}
\]

\( A = O(1) \), \( \text{as} \ \varepsilon \to 0 \).

Make sure to give all boundary and continuity conditions in your specification of the homogenized problem. (Hint: You do not need to derive the limiting problem again: use part (i)).

(iii) Suppose that the amplitude of the roughness is \( O(\delta) \) for some \( \delta \ll 1 \), i.e. \( A = -\delta \). Briefly indicate how one might use a regular perturbation scheme to approximately solve the problem in (ii). What is the unperturbed problem? What is the order in \( \delta \) of the correction to the eigenvalues of the unperturbed problem?
Problem 1

We write the steady-state problem as

\[ \frac{d}{dx} \left[ D(x, y/x) \frac{du}{dx} \right] = f(x, y/x) \]

\[ D(x, y+2\pi) = D(x, y) \]

\[ f(x, y+2\pi) = f(x, y) \]

\[ u = 0 \text{ at } x = 0, \quad u = 1 \text{ at } x = 1. \]

We look for a solution in the form \( u = u(x, y) \) with \( y = x/\varepsilon \).

Then (1) is

\[ \left( \frac{\partial^2}{\partial y^2} + \varepsilon \frac{\partial}{\partial x} \right) \left[ D \left( \frac{\partial}{\partial y} + \varepsilon \frac{\partial}{\partial x} \right) u \right] = \varepsilon^2 f. \]

We expand \( u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \) to obtain

\[ \frac{\partial}{\partial y} \left[ D \frac{\partial}{\partial y} u_0 \right] = 0 \]

\[ \frac{\partial}{\partial y} \left[ D \frac{\partial}{\partial y} u_1 \right] = -\frac{\partial}{\partial x} \left[ D \frac{\partial}{\partial y} u_0 \right] - \frac{\partial}{\partial y} \left[ D \frac{\partial}{\partial x} u_0 \right] \]

\[ \frac{\partial}{\partial y} \left[ D \frac{\partial}{\partial y} u_2 \right] = -\frac{\partial}{\partial x} \left[ D \frac{\partial}{\partial y} u_1 \right] - \frac{\partial}{\partial x} \left[ D \frac{\partial}{\partial x} u_0 \right] + f. \]

The only solution to (2) that is \( 2\pi \) periodic in \( y \) is

\[ u_0 = u_0(x). \]

Then (3) becomes (since \( \frac{\partial}{\partial y} u_0 = 0 \))

\[ \frac{\partial}{\partial y} \left[ D \frac{\partial}{\partial y} u_1 \right] = -\frac{\partial}{\partial x} u_0 \left[ \frac{\partial}{\partial y} D \right] \]

Now the solution is \( u_1 = -y u_{0x} + u_{1x} \) where \( \frac{\partial}{\partial y} \left[ D \frac{\partial}{\partial y} u_{1x} \right] = 0 \).

This yields that

\[ u_1 = -y u_{0x} + b_1(x) + b_0(x) \int_0^y ds \frac{d}{dx} \left[ \int_0^s ds' D(x, s') \right] \]

Now we want a solution that is \( 2\pi \) periodic in \( y \), i.e.

\[ u_1(x, y+2\pi) = u_1(x, y). \]

This yields that

\[ -2\pi u_{0x} + b_0(x) \int_0^{2\pi} ds \frac{d}{dx} \left[ \int_0^s ds' D(x, s') \right] = 0. \]
Thus
\[ u_{0x} = b_0(x) < D^{-1}>_{2\pi} \]
where
\[ < D^{-1}>_{2\pi} = \frac{1}{2\pi} \int_0^{2\pi} \frac{ds}{D(x, s)} \]  \( \text{(7)} \)

Now we calculate from (6) that
\[ u_{1y} = - u_{0x} + \frac{b_0}{D} \]

We substitute this into (4) to obtain
\[ \partial_y \left[ D \partial_y u_2 \right] = - \partial_y \left[ D \partial_x u_1 \right] - \partial_x \left[ D \left( - u_{0x} + \frac{b_0}{D} \right) \right] - \partial_x \left[ D \partial_x u_0 \right] + f \]

This yields that
\[ \partial_y \left[ D \partial_y u_2 \right] = - \partial_y \left[ D \partial_x u_1 \right] - b_0'(x) + f(x, y) \]

Now integrate after first writing
\[ \partial_y \left[ D \partial_y u_2 + D \partial_x u_1 \right] = - b_0'(x) + f(x, y) \]

Integrate
\[ 0 = \int_0^{2\pi} \partial_y \left[ D \partial_y u_2 + D \partial_x u_1 \right] dy = - 2\pi b_0'(x) + \int_0^{2\pi} f(x, y) dy \]

Periodicity

Thus
\[ b_0'(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x, y) dy = \frac{f_{\text{ave}}(x)}{2\pi} \]

Recall that, from (7),
\[ b_0(x) = \frac{1}{< D^{-1}>_{2\pi}} u_{0x} \]

Hence the homogenized steady-state equation is
\[ \left\{ \begin{array}{ll}
\frac{d}{dx} \left[ \overline{D} \frac{d u_0}{dx} \right] = f_{\text{ave}}(x) & f_{\text{ave}}(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x, y) dy \\
 u_0(0) = 0, & \text{HARMONIC MEAN (D) = } \overline{D} = \overline{D}(x) = \frac{1}{< D^{-1}>_{2\pi}} = \frac{2\pi}{\int_0^{2\pi} ds/D(x, s)}
\end{array} \right. \quad (8) \]
Now suppose that
\[ D(x, y) = \frac{1}{1 + B \cos(y)}, \quad f(x, y) = 1 + \cos(y). \]
We need \( 0 < B < 1 \).

Then \( F_{\text{ave}} = 1 \) and \( \bar{D} = \frac{2\pi}{\int_0^{2\pi} (1 + B \cos y) \, dy} = 1. \)

Thus the homogenized equation is, from (8),
\[ \frac{d}{dx} \left[ \frac{du_0}{dx} \right] = 1, \quad u_0(0) = 0, \quad u_0(1) = 1 \rightarrow u_0 = \frac{x^2}{2} + \frac{x}{2}. \]

We label \( u_{0, \text{homog}} = \frac{x^2}{2} + \frac{x}{2} \), (independent of \( B \)). (9)

Now consider if we incorrectly used
\[ D_{\text{ave}} = \frac{1}{2\pi} \int_0^{2\pi} D(x, y) \, dy = \frac{1}{2\pi} \int_0^{2\pi} \frac{dy}{1 + B \cos y} = \frac{1}{\sqrt{1 - B^2}} \] (rederive calculus)

\[ F_{\text{ave}} = 1. \]

Thus \( \frac{d}{dx} \left[ \frac{D_{\text{ave}} du_0}{dx} \right] = F_{\text{ave}} = 1. \)

Thus \( u_{\text{ave}} = \frac{x^2}{2D_{\text{ave}}} + \left( 1 - \frac{1}{2D_{\text{ave}}} \right) x \) \( (10). \)

with \( D_{\text{ave}} = \frac{1}{\sqrt{1 - B^2}} \)

Notice that (10) depends on \( B \).

Now calculate the exact solution, we label this \( u_E(x) \).

Then,
\[ \frac{d}{dx} \left[ \frac{1}{1 + B \cos \left( \frac{x}{\varepsilon} \right)} \frac{du_E}{dx} \right] = 1 + \cos \left( \frac{x}{\varepsilon} \right) \]

\[ u_E(0) = 0, \quad u_E(1) = 1. \]
We integrate once to obtain

\[ \frac{1}{1 + B \cos(x/E)} \frac{dU_E}{dx} = x + \varepsilon \sin(x/E) + A_0. \quad A_0: \text{arbitrary.} \]

so

\[ \frac{dU_E}{dx} = \left(1 + B \cos(x/E)\right) \left(x + \varepsilon \sin(x/E)\right) + A_0 \left(1 + B \cos(x/E)\right) \]

\[ \frac{dU_E}{dx} = x + \varepsilon \sin(x/E) + B x \cos(x/E) + \varepsilon B \cos(x/E) \sin(x/E) \]

\[ + A_0 + A_0 B \cos(x/E) \]

This yields that

\[ \frac{dU_E}{dx} = x + \varepsilon \sin(x/E) + \varepsilon B \cos(x/E) + \frac{\varepsilon B}{2} \sin(2x/E) + A_0 \left(1 + B \cos(x/E)\right) \]

Integrating again we obtain

\[ U_E(x) = \frac{x^2}{2} - \varepsilon^2 \cos(x/E) + B \left[\varepsilon x \sin(x/E) + \varepsilon^2 \cos(x/E)\right] \]

\[ - \frac{B \varepsilon^2}{4} \cos(2x/E) + A_0 \left[x + \varepsilon B \sin(x/E)\right] + A_1 \]  \hspace{1cm} (11)

Now impose \( U_E(0) = 0 \) and \( U_E(1) = 1 \) to find \( A_0 \) and \( A_1 \).

We obtain that

\[ U_0(0) = 0 \quad \rightarrow \quad A_1 = \varepsilon^2 - \varepsilon^2 B + \varepsilon^2 B/4 \quad \rightarrow \quad A_1 = \varepsilon^2 \left(1 - 3B/4\right) \]  \hspace{1cm} (12a)

Now \( U_0(1) = 1 \) yields

\[ \frac{1}{2} - \varepsilon^2 \cos(1/E) + B \left[\varepsilon \sin(1/E) + \varepsilon^2 \cos(1/E)\right] - \frac{B \varepsilon^2}{4} \cos(2/E) \]

\[ + A_0 \left[1 + \varepsilon B \sin(1/E)\right] + A_1 = 1. \]  \hspace{1cm} (12b)

Now (12a, b) determine \( A_0 \) and \( A_1 \). Then \( U_E(x) \) is given in (11).
\( \varepsilon = 0.01, \quad \theta = 0.95 \)

- heavy solid \( \rightarrow \) D\text{ave} (not even close)
- wiggly solid \( \rightarrow \) exact \( U_E \)
- dotted \( \rightarrow \) homog. (close to \( U_E \)).

Clearly, homog closely approximate exact.

**Remark**

Notice from (12a, b) that \( A_1 \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \) and \( A_0 \rightarrow \frac{1}{2} A_1 \) as \( \varepsilon \rightarrow 0 \). Hence (11) for \( U_E(x; \varepsilon) \)

\[ U_E(x) \sim x^2 + x + O(\varepsilon), \text{ which reproduces the } 0(1) \text{ term in } U_{\text{homog}}. \]
Consider the time-dependent problem:

$$U_t = \frac{\partial}{\partial x} \left[ D \frac{\partial U}{\partial x} \right] - F(x, y) x / \varepsilon.)$$

We look for a solution in the form:

$$U = U(x, y, t) \quad y = x / \varepsilon.$$

We derive equations (2) - (14) on (P.1), with (14) replaced by

$$\frac{\partial}{\partial y} \left[ D \frac{\partial U_2}{\partial y} \right] = \frac{\partial}{\partial y} \left[ D \frac{\partial U_1}{\partial y} \right] - \frac{\partial}{\partial x} \left[ D \frac{\partial U_0}{\partial x} \right] + \frac{F + U_{0t}}{D}.$$

We have that $U_0 = U(x, t)$ and that (see equation (16) on (P.1)).

$$U_1 = -y U_{0x} + b_x(x, t) + b_o(x, t) \int_0^y \frac{ds}{D(x, y)}$$

We substitute into (13) to obtain, with

$$U_{1y} = -U_{0x} + \frac{b_o}{D}$$

that

$$\frac{\partial}{\partial y} \left[ D \frac{\partial U_2}{\partial y} \right] = \frac{\partial}{\partial y} \left[ D \frac{\partial U_1}{\partial y} \right] - \frac{\partial}{\partial x} b_o + F + U_{0t}.$$

Now integrate over $0 < y < 2\pi$ and use periodicity in $y$ to obtain

$$0 = \int_0^{2\pi} \left[ \frac{\partial}{\partial y} \left( D \frac{\partial U_2}{\partial y} + D \frac{\partial U_1}{\partial y} \right) \right] dy = \int_0^{2\pi} \left[ -b_{0x} + F + U_{0t} \right] dy$$

(by periodicity).

Thus,

$$U_{0t}(2\pi) + \int_0^{2\pi} F(x, y) dy = b_{0x}(2\pi) = 0.$$

Thus

$$U_{0t} + F_{ave}(x) = b_{0x} \quad (14)$$
The condition that \( u, \) in (14) is \( 2 \pi \) periodic in \( y \) is that
\[
-2\pi \ u_0(x) + b_0 \int_0^{2\pi} \frac{ds}{D(x, s)} = 0
\]

Thus
\[
b_0 = \frac{2\pi}{\int_0^{2\pi} \frac{ds}{D(x, s)}} \ u_0(x)
\]  
(15)

Combining (14) and (15) we obtain
\[
\begin{cases}
\ u_0_t = \frac{\partial}{\partial x} \left[ D(x) \ u_0_x \right] - F_{\text{ave}} (x) \\
\ u_0(0, t) = 0, \quad u_0(1, t) = 1
\end{cases}
\]  
(16)

\[
D(x) = \frac{2\pi}{\int_0^{2\pi} \frac{ds}{D(x, s)}} \quad F_{\text{ave}} (x) = \frac{1}{2\pi} \int_0^{2\pi} F(x, s) ds
\]

The PDE (16) is the averaged or homogenized version of

The PDE
\[
\begin{cases}
\ u_t = \frac{\partial}{\partial x} \left[ D(x, x/\varepsilon) \ u_x \right] - F(x, x/\varepsilon)
\end{cases}
\]  
(17)

(iV) A key point is that the initial condition for (16) can only be a function of the macroscale \( x \) (since \( \ u_0 = u_0(x, t) \)).

Thus if the initial data is "rough", i.e. dependent on \( x/\varepsilon \), we must have a transient in time for

\[
\begin{cases}
\ u_t = \frac{\partial}{\partial x} \left[ D(x, x/\varepsilon) \ u_x \right] - F(x, x/\varepsilon)
\end{cases}
\]  
(18)

\[
\ u(x, 0) = g(x, x/\varepsilon) \quad u(0, t) = 0, \quad u(1, t) = 1
\]

That smooth out the roughness in \( g \) quickly in time.
WE NOW INTRODUCE AN INITIAL TIME LAYER. 

WE DEFINE $\tau = t/\varepsilon^2$ AND WE LOOK FOR A SOLUTION TO (17) 

IN THE FORM 

$$U = \nu(x, y, \tau) = \nu_0(x, y, \tau) + \varepsilon \nu_1(x, y, \tau) + \ldots$$

WE SUBSTITUTE INTO (17) TO OBTAIN THAT $\nu_0$ SOLVES 

$$\begin{cases} 
    \nu_0(x, y, \tau) = \frac{\partial \nu}{\partial \tau} [D \frac{\partial \nu_0}{\partial y}] , & 0 < y < 2\pi, \\
    \nu_0(x, y, 0) = g(x, y) 
\end{cases}$$

SINCE $\exists$ A ZERO EIGENVALUE 

(19) 

$\nu_0$ IS $2\pi$ PERIODIC IN $y$

WE NOTICE: 

$$\nu_0 = A_0 + \sum_{j=1}^{\infty} A_j e^{-A_j \tau}$$

WE WANT TO CALCULATE $\lim_{\tau \to \infty} \nu_0(x, y, \tau)$. WITH $A_j = A_j(x, y)$, $j \geq 1$ AND $A_0 = A_0(x)$

TO DO SO, WE INTEGRATE OVER $0 < y < 2\pi$ AND USE PERIODICITY TO OBTAIN, 

$$\int_0^{2\pi} \nu_0 \frac{\partial \nu_0}{\partial y} dy = \int_0^{2\pi} \frac{\partial \nu_0}{\partial \tau} [D \frac{\partial \nu_0}{\partial y}] dy = D \frac{\partial \nu_0}{\partial y} \big|_0^{2\pi} = 0.$$ 

THUS, 

$$\int_0^{2\pi} \nu_0 dy = 0 \quad \Rightarrow \quad \int_0^{2\pi} \nu_0 dy = \text{constant in } \tau.$$ 

HENCE, 

$$\int_0^{2\pi} \nu_0(x, y, \tau) dy = \int_0^{2\pi} g(x, y) dy.$$ 

NOW AS $\tau \to \infty$, $\nu_0(x, y, \tau) \to \nu_0(x)$ WHERE 

$$\int_0^{2\pi} \nu_0(x) dx = \int_0^{2\pi} g(x, y) dy.$$ 

WE CONCLUDE THAT 

$$\nu_0(x) = \frac{1}{2\pi} \int_0^{2\pi} g(x, y) dy = \text{ave}(g).$$ 

FINALLY, BY MATCHING $\tau \to \infty$ CORRESPONDING TO $t \to 0$, THE HOMOGENIZED 

PDE (16) IS TO BE SOLVED WITH INITIAL DATA $\nu_0(x, 0) = \text{ave}(g).$
**Problem 2.** We write

\[
\nabla \cdot \left[ \sigma \nabla u \right] - \alpha u = f(x, z) \quad \text{in } z > h(x/\varepsilon)
\]

\[
\hat{n} \cdot \nabla u = 0 \quad \text{on } z = h(x/\varepsilon)
\]

We have \( \hat{n} = (\varepsilon^2 h_y, -1) \). Now let \( u = u(x, y, z) \).

We obtain

\[
\sigma \left[ \varepsilon^{-1} u_{yy} + \varepsilon \varepsilon' u_{xy} + \varepsilon' u_{yx} + u_{xx} + u_{zz} \right] - \alpha u = f \quad \text{in } z > h(y)
\]

\[
\varepsilon^{-2} h_y u_{yy} + \varepsilon' h_y u_{xy} + u_{xx} = u_{zz} \quad \text{on } z = h(y)
\]

We now expand \( u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots \) and collect powers of \( \varepsilon \) to get

\[
\begin{align*}
\text{(1)} & \quad u_{0yy} = 0 \quad \text{in } z > h(y) \\
& \quad u_{0y} = 0 \quad \text{on } z = h(y)
\end{align*}
\]

\[
\begin{align*}
\text{(2)} & \quad u_{1yy} = -u_{0xy} - u_{0yx} \quad \text{in } z > h(y) \\
& \quad u_{1y} = -u_{0x} \quad \text{on } z = h(y)
\end{align*}
\]

\[
\begin{align*}
\text{(3)} & \quad u_{2yy} + \sigma u_{1xy} = -\sigma u_{1yx} - \sigma u_{0xx} - \sigma u_{0zz} + \alpha u_0 + f \\
& \quad h_y u_{2y} + h_y u_{1x} = u_{0z} \quad \text{on } z = h(y)
\end{align*}
\]

We want \( u_0, u_1, u_2 \) periodic in \( y \) with period 1.

The solution to (1) is

\[
u_0 = u_0(x, z)\]

At next order, (1) becomes

\[
\begin{align*}
\text{(2')} & \quad u_{1yy} = 0 \quad \text{in } z > h(y) \\
& \quad u_{1y} = -u_{0x} \quad \text{on } z = h(y)
\end{align*}
\]

We let \( \nabla \) be defined by \( u_1 = \nabla u_0 - y u_0x \). We substitute to get

\[
\begin{align*}
\text{(2'')} & \quad \nabla \cdot u_1 = 0 \quad \text{in } z > h(y) \\
& \quad \nabla \cdot u_1 = 0 \quad \text{on } z = h(y)
\end{align*}
\]

The solution is

\[
\nabla = \nabla(x, z).
\]
Now we integrate the $u_2$ equation from $y_i$ to $y_1$ to get
\[
\left( \sigma_{+} u_{2y} + \sigma_{+} u_{1x} \right)_{y_i}^{y_1} = -\sigma_{+} u_{1x} \bigg|_{y_i}^{y_1} - \sigma_{+} (y_1 - y_i) u_{0xx} - \sigma_{+} (y_1 - y_i) u_{0zz} + a u_0 (y_1 - y_i) + \int_{y_i}^{y_1} f(x, z) \, dy.
\]

Now the boundary condition gives
\[
\sigma_{+} u_{0y} \left( \frac{1}{h_{f}(y_i)} - \frac{1}{h_{f}(y_1)} \right) = -\sigma_{+} \left[ \nu u_{0xx} - \gamma u_{0xx} \right]_{y_i}^{y_1} - \sigma_{+} (y_1 - y_i) u_{0xx} - \sigma_{+} (y_1 - y_i) u_{0zz} + a u_0 (y_1 - y_i) + (y_1 - y_i) f
\]

Now this gives
\[
\sigma_{+} u_{0} \left( \frac{1}{h_{f}(y_i)} - \frac{1}{h_{f}(y_1)} \right) + \sigma_{+} (y_1 - y_i) u_{0zz} = a u_0 (y_1 - y_i) + (y_1 - y_i) f
\]

Since $v = v(x, z)$, next since $z = h(y)$, we have $y = h(y) z$. Thus gives,
\[
\sigma_{+} u_{0z} \left( y_1' - y_i' \right) + \sigma_{+} (y_1 - y_i) u_{0zz} = a u_0 (y_1 - y_i) + (y_1 - y_i) f
\]

Now define
\[
< \sigma > = \sigma_{+} (y_1 - y_i) + \sigma
\]

Then
\[
\frac{d}{dz} \left[ < \sigma > u_{0z} \right] - a u_0 (y_1 - y_i) = (y_1 - y_i) f \quad \text{in} \quad -A \leq z \leq 0.
\]

Calculate the B.C. for $(x)$

Now when $z = 0$ we have $< \sigma > = \sigma_{+}$

$z = -A$ we have $< \sigma > = 0$ $\rightarrow$ $(x)$ has a singularity.

Therefore at $z = 0$ we impose that
\[
\begin{bmatrix} u_0 \\ u_{0z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{at} \quad z = 0
\]

ID at $z = -A$ we have $< \sigma > u_{0z} = 0$ to remove singularity, in the region with $z > 0$ we solve
\[
\nabla \cdot [ \sigma \nabla u] - \sigma u = f(x, z) \quad \text{in} \quad z > 0
\]

$\rightarrow \sigma (u_{xx} + u_{zz}) - \sigma u = f(x, z) \quad \text{in} \quad z > 0.$
Suppose we had a problem as follows:

\[ \Delta u + \lambda u = 0 \quad \text{in} \quad 0 < x < 1, \]
\[ h(\gamma/\varepsilon) < z < b \]
\[ \delta_0 u = 0 \quad \text{on} \quad x = 0, \ x = 1 \]
\[ z = b \quad z = b \left( \frac{x}{\varepsilon} \right) \]

Assume \( h(y + 1) = h(y) \).

Now this is replaced by the homogenized problem:

We must solve

\[ \begin{align*}
\Delta u + \lambda u &= 0 \quad \text{in} \quad 0 < z < b, \ 0 < x < 1 \\
\frac{d}{dz} \left[ \sigma U_z \right] + \lambda U \left( \gamma_z(z) - \gamma_z(z) \right) &= 0 \quad -A < z < 0 \\
[U] &= [U_z] = 0 \quad \text{on} \quad z = 0 \\
<\sigma>U_z &= 0 \quad \text{on} \quad z = -A \\
\end{align*} \]

Here \( <\sigma> = (\gamma_z - \gamma_1) \)

This gives a coupled problem to solve.
Suppose \( A = 8 \) so that we have a high frequency but small amplitude boundary roughness.

We have two regions:
- Region in \( 0 < z < b, \quad 0 < x < 1 \)
- Region in \( -\delta < z < 0, \quad 0 < x < 1 \).

Now we have

\[
\Delta u_t + \Lambda u_t = 0 \quad \text{in} \quad 0 < z < b, \quad 0 < x < 1
\]

\[
\frac{d}{dz} \left[ \langle \sigma \rangle u_z \right] + \Lambda u_t \langle \sigma \rangle = 0 \quad \text{in} \quad -\delta < z < 0.
\]

With \( \langle \sigma \rangle = (\gamma_2(z) - \gamma_1(z)) \), now if \( A = -\delta \) then \( \gamma_2(z) - \gamma_1(z) = SF(\hat{z}/\delta) \)

where \( F(\hat{z}) = \gamma_2(\hat{z}) - \gamma_1(\hat{z}) \), with \( \hat{z} = z/\delta \).

Thus

\[
\Delta u_t + \Lambda u_t = 0 \quad \text{in} \quad 0 < z < b, \quad 0 < x < 1
\]

\( u_t = 0 \) on top, and sides

\[
\frac{1}{\delta^2} \frac{d}{dz} \left[ \int u_z \right] + \Lambda u_t F = 0 \quad \text{in} \quad -1 < z < 0.
\]

We let \( u_{t0}, \Lambda_0 \) denote the solution to unperturbed problem

\[
\begin{align*}
\Delta u_{t0} + \Lambda_0 u_{t0} &= 0 \quad \text{in} \quad 0 < z < b, \quad 0 < x < 1 \\
\left\{ \begin{array}{l}
\quad u_{t0} = 0 \quad \text{on} \quad 0 < x < 1 \quad \text{with} \quad z = b, \\
\quad u_{t0} = 0 \quad \text{on} \quad x = 0, \quad 0 < z < b, \quad x = 1, \quad 0 < z < b
\end{array} \right.
\end{align*}
\]

And impose \( u_{t0} = 0 \) on \( 0 < x < 1, \quad z = 0 \).

We then have, as \( z \to 0 \) that

\[
|u_{t0} - u_{t0} \bigg|_{z = 0} + \frac{z^2}{2} \left. u_{t0zzz} \right|_{z = 0} + \ldots
\]

Writing in terms of \( \hat{z} = z/\delta \), we have

\[
|u_{t0} - u_{t0} \bigg| + \delta^3 \hat{z}^2 \left. u_{t0zz} \right| + \ldots
\]
IN THE OUTER REGION \(0 < Z < b, 0 < X < 1\) WE EXPAND

\[ U_+ = U_{+0} + \delta^p U_{+1} + \ldots \quad \Lambda = \Lambda_0 + \Lambda_1 \delta + \ldots \]

THE POWER \(p\) IS FOUND BELOW.

WE OBTAIN THAT

\[ \Lambda U_{+0} + \Lambda_0 U_{+0} = -\Lambda U_{+0} \quad \text{IN} \quad 0 < Z < b, 0 < X < 1 \]

(2) \[ U_{+0} = 0 \quad \text{ON} \quad 0 < X < 1 \quad \text{WITH} \quad Z = b \]

\[ U_{+0} = 0 \quad \text{ON} \quad X = 0, 0 < Z < b; X = 1, 0 < Z < b \]

WE MUST DETERMINE THE BOUNDARY CONDITION FOR (2) \(\Lambda Z \to 0^+\).

THE INNER PROBLEM IS

\[ \frac{d}{dz} [ \int U_{-2} ] + \delta^2 \Lambda U_{-0} = 0 \quad \text{IN} \quad -1 < Z < 0 \]

\[ f U_{-2} = 0 \quad \text{AT} \quad Z = -1 \]

\[ U_0 = U_{+0} \bigg|_{Z=0} + \ldots \quad \text{AT} \quad Z = 0. \]

THE SOLUTION IS EXPAND AS \( U_0 = U_{-0} + \delta^2 U_{-1} + \ldots \quad \Lambda = \Lambda_0 + \ldots \)

WE OBTAIN THAT \( U_{-0} = U_{+0} \bigg|_{Z=0} \quad \text{IN} \quad -1 < Z < 0 \),

AND THAT

\[ \frac{d}{dz} [ \int U_{-2} ] + \Lambda_0 f U_{-0} = 0 \quad \text{IN} \quad -1 < Z < 0 . \]

WE INTEGRATE TO GET

\[ \int_{-1}^{0} \frac{d}{dz} [ \int U_{-2} ] dZ + \Lambda_0 U_{-0} \bigg|_{Z=0} \bigg|_{-1} f dZ = 0 . \]

SINCE \( f = 1 \) AT \( Z = 0 \) AND \( f U_{-2} = 0 \) AT \( Z = -1 \), WE GET

(3) \[ U_{-2} \bigg|_{Z=0} = -\Lambda_0 U_{-0} \bigg|_{Z=0} \bigg|_{-1} + \ldots \quad \text{NOW ON} \quad Z = 0 \quad \text{WE HAVE} \quad U_{-2} = \delta^2 U_{-2} \bigg|_{Z=0} + \ldots \]

THUS

\[ U_{+0} \bigg|_{Z=0} = \delta^2 U_{-2} \bigg|_{Z=0} \quad \rightarrow \quad U_{+0} \bigg|_{Z=0} = \delta U_{-2} \bigg|_{Z=0} . \]
This gives that \( p = 1 \) in (1.5) and

\[
(4) \quad U_{+1z} \bigg|_{z=0} = U_{-1z} \bigg|_{z=0} = -\Lambda_0 U_0 \int_0^1 f \, d\zeta
\]

is the missing boundary condition for (2).

Now use Green's identity on \( U_{+0} \) and \( U_{+1} \). Let \( B = [0, b] \times [0, 1] \)

\[
\int_B \left( U_{+0} \left( \Delta U_{+1} + \Lambda_0 U_{+1} \right) - U_{+1} \left( \Delta U_{+0} + \Lambda_0 U_{+0} \right) \right) \, dz \, dx = \int_0^1 U_{+0} (-\mathbf{n} \cdot \nabla U_{+1}) \bigg|_{z=0} \, dx
\]

so

\[
\int_0^1 \Lambda_0 U_{+0} \bigg|_{z=0} \left( \int_0^1 f \, d\zeta \right) \, dx = \int_0^1 \left( \int_0^1 f \, d\zeta \right) \, dx
\]

This gives

\[
(5) \quad \Lambda_1 = -\Lambda_0 \left( \int_0^1 f \, d\zeta \right) \int_0^1 \left( U_{+0} \bigg|_{z=0} \right)^2 \, dx
\]

\[
\int_B U_{+0}^2 \, dx \, dz
\]

We conclude that \( \Lambda = \Lambda_0 + \delta \Lambda_1 + \ldots \) with \( \Lambda_1 \) given above.

Now for \( B \) the unperturbed solution has form

\[
U_{0+} = \sin \left( \frac{n \pi x}{b} \right) \cos \left( \frac{2m+1}{2} \pi z}{b} \right) \quad \text{NOTICE} \quad U_{0+} = 0 \quad \text{on} \quad x=0, \ b
\]

and

\[
\Lambda_0 = \left[ \frac{n^2 \pi^2 + (2m+1)^2 \pi^2}{b^2} \right] \quad n=0, 1, \ldots
\]

and

\[
U_{0+} = 0 \quad \text{on} \quad z=b
\]

\[
U_{0+} = 0 \quad \text{on} \quad z=0
\]