**Problem 1**

In a cylinder of revolution \(0 < z < L\) with \(0 < R < R_0\) \(F(z/L)\), consider the axially symmetric Green's function satisfying

\[
G_{RR} + \frac{1}{R} G_R + G_{zz} = \frac{\delta(z-z_0)}{2\pi R} \quad \text{in} \quad 0 < R < R_0, F(z/L),
\]

\(0 < z < L\). Assume \(F > 0\) smooth.

Assume that \(\nabla G \cdot \hat{n} = 0\) on \(R = R_0, F(z/L)\); \(G\) bounded as \(R \to 0\) for \(z \neq z_0\). And \(G = 0\) on \(z = 0, L\).

Assume that \(z_0\) satisfies \(0 < z_0 < L\), and for convenience \(F(z_0/L) = 1\).

Find an asymptotic approximation for \(G\) in the long thin domain limit \(R_0/L \ll 1\). In your analysis find the outer solution away from the Dirac source and the inner solution (Hint: Bessel function) are needed here for inner solution. A leading-order theory is sufficient.

**Problem 2**

In a 3-D domain, the splitting probability \(U(x)\) is defined as the probability of reaching a specific target trap \(\sum_{i}^{N} \Omega_{x_i}\), from an initial starting point \(x \in \Omega \setminus \bigcup_{j=1}^{N} \Omega_{x_j}\), before reaching any of the other remaining traps \(\Omega_{x_j}\) for \(j = 2, \ldots, N\). It is well known that \(U(x)\) satisfies

\[
\Delta U = 0, \quad \text{for} \ x \in \Omega \setminus \bigcup_{j=1}^{N} \Omega_{x_j}, \quad \partial_{n} U = 0 \quad \text{on} \ \partial \Omega
\]

\[U = 1 \quad \text{for} \ x \in \Omega_{x_1}, \quad U = 0 \quad \text{for} \ x \in \Omega_{x_j}, \ j = 2, \ldots, N.\]

Assume for simplicity that each trap \(\Omega_{x_j}\) is a sphere of radius \(\epsilon a_j\) centered at \(x = x_j\) with \(a_j > 0\) and \(a_j = O(1), \ \epsilon \ll 1\).

I.e. \(\Omega_{x_j} = \{ x \mid |x - x_j| < a_j \epsilon \}\).
(i) Show that a two-term approximation for $u(x)$ in the outer region has the form for $\varepsilon \ll 1$ that

$$u(x) \sim \frac{C}{\varepsilon^2} + \frac{4\pi \varepsilon}{C} \sum_{j=1}^{N} C_j G(x; x_j) + \varepsilon \chi_1 + O(\varepsilon^2) \quad (8)$$

Here $C = \frac{1}{N} \left( C_1 + \cdots + C_N \right)$ where $C_j = a_j$ is the capacitance of the $j^{th}$ trap, and $G(x; x_j)$ is the Neumann Green's function satisfying

$$\Delta G = \frac{1}{r} - \frac{\delta(x - x_j)}{\varepsilon}
, \quad x \in \Omega; \quad \partial_n G = 0 \text{ on } \partial \Omega$$

$$G = \frac{1}{4\pi|x - x_j|} + R(x) + O(\varepsilon) \quad \text{as} \quad x \to x_j \quad \int_{\Omega} G(x; x_j) dx = 0$$

Here $R(x)$ is the "regular" part of $G(x; x_j)$ at $x = x_j$.

(ii) Determine explicitly the constant $\chi_1$ in (8). \text{Hint: you will need to use the divergence theorem on the problem at } O(\varepsilon^2).

(iii) Interpret the leading order term in the expansion (8). Is it more or less likely to reach a small target sphere or a large target sphere? Explain qualitatively what information on the splitting probability is contained in the $O(\varepsilon)$ term in (8).

\textbf{Problem 3} Let $\Sigma = \{ x | |x| \leq 1 \}$ be the unit sphere in 3-D and introduce the usual spherical coordinates $x = (r \cos \phi \sin \theta, r \sin \phi \sin \theta, r \cos \theta)^T$ on $0 < r < 1$ where $0 \leq \phi < \pi$ is polar angle and $0 \leq \theta < 2\pi$. Suppose that $\Sigma$ contains four small spherically-shaped non-overlapping holes $\Sigma_{x_j}$ of a common radius $\varepsilon \ll 1$ centered at some $x_j$ for $j = 1, \ldots, 4$. For an arbitrary function $f(x)$ and constants $a_j$ for $j = 1, \ldots, 4$ consider the PDE

$$\Delta u = 0 \quad x \in \Sigma \setminus \bigcup_{j=1}^{4} \Sigma_{x_j}$$

$$u = f(x) \quad \text{on } |x| = 1$$

$$u = a_j \quad \text{on } \partial \Sigma_{x_j}, \quad j = 1, \ldots, 4$$

(i) Derive a leading-order asymptotic expansion for the solution to this problem for $\varepsilon \ll 1$.

(Note: your solution will involve a particular solution and the Green's function for the Laplacian in a sphere.)
(iii) Let \( f(q) = \sin^2 q \) and \( \alpha_j = 1 \) for \( j = 1, \ldots, 4 \).

Suppose that the centers of the holes of radius \( \varepsilon \)
are placed symmetrically at \( \left( 0, \pm \frac{1}{2}, \pm \frac{1}{2} \right) \) in plane \( x = 0 \).

Derive a formula for \( u(0) \) versus \( \varepsilon \) and plot it numerically.

(Hint: the particular solution is easy to find in terms of Legendre polynomial. The Green's function in part (i) can be found by method of image).