**MATH 551 HW #2**

**Problem 1**
Assume that a plane wave $u_I = e^{-iwx}$ is incident on the parabola $x = \frac{1}{2}y^2 - \frac{a}{2}$ (with $a > 0$) from the right, and that the total field $u = u_I + u_S$ vanishes on the parabola. Find an asymptotic expansion of the scattered wave $u_S$ valid for $k \gg 1$. Does the ray field admit a focal point?

**Problem 2**
Assume that a plane wave $u_I = e^{iwx}$ is incident on a circular cylinder of radius $q$ and that the total field $u = u_S + u_I$ vanishes on the circular cylinder $\Gamma: x^2 + y^2 = q^2$. Let $u_J$ be the scattered field with $u_J$ outgoing at $\omega$, and $u_I$ defined outside the cylinder.

\[ \Gamma: x^2 + y^2 = q^2 \]

(i) For $k \gg 1$, find and plot the rays $x = x(j, \tau), \ y = y(j, \tau)$ where $\Gamma$ is parametrized by $x = q \cos \tau, \ y = q \sin \tau$ in the range $\pi/2 < \tau < 3\pi/2$. What are the rays for $-\pi/2 < \tau < \pi/2$?

(ii) Show that the caustic of these rays lies inside the circle but that the caustic intersects the circle at the point $x = 0$ and $y = \pm q$. Plot the caustic.

(iii) Show that Snell's law holds for the incident and reflected waves at their intersection with the boundary $\Gamma$.

(iv) Calculate the leading order amplitude $v_0(j, \tau)$. (You will not be able to calculate $v_0(x, y)$ explicitly.)
Show that in the dark region the total field $U \equiv 0$ whereas in the illuminated region the total field $U$ is

$$U = e^{i \chi} - \sqrt{\frac{a/2 \cos \tau}{q/2 \cos \tau - 2s}} \exp \left[ i \chi (2s + a \cos \tau) \right], \quad \frac{\pi}{2} < \tau < \frac{3\pi}{2}.$$  

**Problem 3** Consider the reaction-diffusion system

$$U_t = D_U \nabla^2 U + K \Phi (U, V) \quad \text{on} \quad 0 < x < L, \quad t > 0$$

$$V_t = D_V \nabla^2 V + N (U, V) \quad \text{on} \quad 0 < x < L, \quad t > 0$$

with insulating boundary conditions $U_x = V_x = 0$ on $x = 0, L$.

(i) By scaling $x = \chi L$ and $t$, and assuming $L$ is short, derive a dimensionless PDE system of the form

$$\begin{cases}
U_t = \frac{1}{\varepsilon} \nabla^2 U + f(U, V) \quad \text{on} \quad 0 < \chi L, \quad t > 0 \\
V_t = \frac{1}{\varepsilon} \nabla^2 V + g(U, V) \quad \text{on} \quad 0 < \chi L, \quad t > 0
\end{cases}$$

with $U_x = V_x = 0$ on $x = 0, L$ for some $\varepsilon > 0$ to be found.

(ii) Assuming $\varepsilon = 0$ and $\varepsilon \to 0^+$ derive a leading-order approximate solution to $\Phi$. (This is called the "well-mixed" regime).

(iii) If $U(x, 0) = \alpha(x)$ and $V(x, 0) = B(x)$ are the initial function for $\Phi$, derive the initial condition for the solution in (ii). (Hint: Let $t = \chi \varepsilon$, analyze the transient regime, and then match to solution in (iii).)