Math 401: Quiz 1

Instructions: Open book and open notes. No collaboration or discussion of the problems with others. No posting of questions related to this quiz on Piazza or Chegg.

1. (15 points) Consider the following boundary value problem for $u = u(x)$:

$$Lu \equiv xu'' + 2u' - \frac{2}{x}u = f(x), \quad 1 \leq x \leq 2; \quad u'(1) = 0, \quad u(2) = 0. \quad (1)$$

(a) (2 points) Convert this problem to the Sturm Liouville form $\tilde{L}u = \tilde{f}$, where the Sturm Liouville operator $\tilde{L}$ and $\tilde{f}$ are to be identified.

(b) (5 points) Calculate explicitly the Green’s function $G(t; x)$ for $\tilde{L}$.

(c) (2 points) From your solution in part (ii), write down a formula for the solution $u(x)$ to (1) at $x = 1$.

(d) (3 points) Write down but DO NOT SOLVE the problem for the adjoint Green’s function $v(t, x)$ for the operator $L$ in (1). Make sure to specify the adjoint boundary conditions for $v$ and write down the jump conditions across $t = x$ for $v$.

(e) (3 points) Consider (1) but where the boundary condition $u(2) = 0$ is now replaced with the modified condition $u'(2) = 7u(2)/17$. Show for this modified problem that there is a condition on $f(x)$ needed for a solution $u$ to exist. Determine this solvability condition explicitly.

2. (5 points) Consider the following boundary value problem for $u(x)$:

$$u'' - u = \delta(\sin(\pi x)), \quad -L \leq x \leq L; \quad u(L) = 0, \quad u(-L) = 0, \quad (13)$$

where $L = 41/2$. Determine a solution representation for $u(x)$ in the form

$$u(x) = \sum_{m=N}^{m=N} A_m G(x; x_m),$$

where you are to identify $N$, $x_m$, and the non-zero coefficients $A_m$. Here $G(x; \xi)$, with $|\xi| < L$, is the Green’s function satisfying

$$G_{xx} - G = \delta(x - \xi), \quad |x| \leq L; \quad G = 0 \quad \text{at} \quad x = \pm L. \quad (14)$$

You do not need to solve for $G$ explicitly as a very similar problem was done in class. (Hint: first write $\delta(\sin(\pi x))$ as a sum of Delta functions on $|x| \leq L$ and then use superposition).