PROBLEM 1: (15 Points) For $\lambda \geq 0$, consider the following differential equation for $u(x)$:

$$u'' + \lambda u = f(x), \quad 0 < x < L; \quad u'(0) = 0, \quad u'(L) = 0.$$  

(i) For what values of $\lambda$ is a condition on $f(x)$ required for there to be a solution? Find this solvability condition.

(ii) For $\lambda$ not one of the values in (i), calculate explicitly the required Green’s function and find an integral representation for the solution $u(x)$.

(iii) What is the problem for the generalized Green’s function when $\lambda = 0$? (do not solve for it).

PROBLEM 2: (15 Points) Suppose that in a 3-D half-space $u(x, y, z)$ satisfies

$$u_{xx} + u_{yy} + u_{zz} = 0, \quad -\infty < x < \infty, \quad -\infty < y < \infty, \quad z > 0,$$

$$u_z(x, y, 0) = f(x, y); \quad u \sim C \frac{1}{|x|} \text{ as } |x| = (x^2 + y^2 + z^2)^{1/2} \rightarrow \infty.$$

Assume that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy$ is finite.

(i) Find the Green’s function relevant to this problem.

(ii) Find an explicit representation for $u$ in terms of this Green’s function.

(iii) For $|x| = (x^2 + y^2 + z^2)^{1/2} \rightarrow \infty$, find an approximation for $u$ in the form

$$u \sim \frac{C}{|x|} + \frac{p \cdot x}{|x|^3} + \cdots, \quad \text{as } |x| \rightarrow \infty,$$

where the scalar $C$ and the vector $p$ are to be found.
PROBLEM 3: (10 Points) Between two infinite parallel plates in 3-D separated by a distance \( \pi \) suppose that \( u(x, y, z) \) satisfies
\[
u_{xx} + u_{yy} + u_{zz} = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \quad \text{in} \quad 0 \leq z \leq \pi, \quad -\infty < x < \infty, \quad -\infty < y < \infty,
\]
\[
u(x, y, 0) = 0, \quad u(x, y, \pi) = 0; \quad u \to 0 \quad \text{as} \quad (x^2 + y^2 + z^2)^{1/2} \to \infty,
\]
where \( 0 < z_0 < \pi \).

(i) Write an eigenfunction representation for \( u \) in terms of the eigenfunctions in the \( z \)-direction, and derive PDE problems for the coefficients \( c_n(x, y) \) with \( n \geq 1 \) in this expansion.

(ii) For \( n \geq 1 \), determine the \( c_n(x, y) \) explicitly in (i) in terms of an appropriate special function.

PROBLEM 4: (10 Points) (Quick Response Questions):

(i) Write \( \delta(x^2 - 5x + 4) \) as a linear combination of two Delta functions \( \sum_{i=1}^2 a_i \delta(x - x_i) \) for some \( a_i \) and \( x_i \) to be found.

(ii) Interpret the limit of the delta sequence
\[
\lim_{\sigma \to 0} \frac{\sigma}{(x - 1)^2 + \sigma^2}
\]
as a generalized function. (Hint: \( d/dz \arctan(z) = 1/(z^2 + 1) \))

(iii) By multiplying both sides by some appropriate function \( p(r) \) put the following problem for \( u(r) \) in self-adjoint form
\[
u'' + \frac{2}{r}u' - u = f(r), \quad \text{in} \quad 1 < r < 2; \quad u(1) = 0, \quad u(2) = 0.
\]

(iv) Let \( \mathbf{p} \) be a constant vector in 2-D, and consider the 2-D Laplacian with a dipole singularity:
\[
\Delta u = \mathbf{p} \cdot \nabla_x \delta(x - \mathbf{x}_0).
\]

Here \( \nabla_x \) is the gradient operator acting on \( x \) and \( \cdot \) is the dot product. Find an explicit expression for the singular behavior of \( u \) as \( x \to \mathbf{x}_0 \).