Now we consider variational problem in 1-D of finding the curve \( y = f(x) \) on \( a \leq x \leq b \) with \( y(a) \) and \( y(b) \) given that minimize the functional

\[
I(y) = \int_a^b F(x, y, y') \, dx.
\]  

(1)

We will derive a necessary condition that \( y = y^*(x) \) is a minimizer (or at least a stationary point) of \( I(y) \).

We put \( y = y^x + \varepsilon w \) into (1) and linear in \( \varepsilon \). We obtain from Taylor series that

\[
J(\varepsilon) = I(y^x + \varepsilon w) = \int_a^b F(x, y^x + \varepsilon w, y^x + \varepsilon w') \, dx = \int_a^b F(x, y^x, y^x') \, dx + \varepsilon \int_a^b \left[ \frac{\partial F}{\partial y} (x, y^x, y^x') w + \frac{\partial F}{\partial y'} (x, y^x, y^x') w' \right] \, dx + O(\varepsilon^2).
\]

Thus

\[
J(\varepsilon) = I(y^x + \varepsilon w) = I(y^x) + \varepsilon \int_a^b \left[ \frac{\partial F}{\partial y} (x, y^x, y^x') w + \frac{\partial F}{\partial y'} (x, y^x, y^x') w' \right] \, dx + O(\varepsilon^2),
\]

where \( \frac{\partial F}{\partial y} = F_y (x, y^x, y^x') \). Now a necessary condition for \( y^x \) to be a stationary point of \( I(y) \) is that

\[
J'(\varepsilon)|_{\varepsilon = 0} = 0 \quad \text{so that}
\]

\[
\int_a^b \left( F_y y' + F_{yy'} y'^2 \right) \, dx = 0 \quad \forall w \in C^2[a,b].
\]  

(2)

Now we write \( \frac{d}{dx} (F_y y') = \frac{d}{dx} F_y y' + F_{yy'} y'^2 \) by product rule so that (2) becomes

\[
0 = \int_a^b \left[ \left( F_y y' - \frac{d}{dx} (F_y y') \right) + \frac{d}{dx} \left( F_y y' \right) \right] \, dx = \int_a^b \left( F_y - \frac{d}{dx} F_y \right) y' \, dx + F_{yy'} y'^2 \bigg|_a^b.
\]  

(3)

Now since \( y(a), y(b) \) are specified, i.e. \( y(a) = y_a, y(b) = y_b \) we have \( y^*(a) = y_a, y^*(b) = y_b \) and so \( w(a) = w(b) = 0 \). Thus in (3), the boundary-term vanishes and \( y^x \) is a minimizer (or stationary point) if and only if

\[
\int_a^b \left( F_y - \frac{d}{dx} F_y \right) y' \, dx = 0 \quad \forall w \in W \equiv \left\{ w \mid w \in C^2[0,1], w(a) = w(b) = 0 \right\}.
\]

By the fundamental lemma of variational calculus we must have that \( y^x \) satisfies Euler-Lagrange equation;

\[
F_y - \frac{d}{dx} F_y = 0 \quad \text{on} \quad a < x < b; \quad y^*(a) = y_a, y^*(b) = y_b
\]  

(4)

which is a second order nonlinear ODE boundary value problem.

We have seen the fundamental lemma before in our study of eigenvalue problems.
**Lemma (Fundamental Lemma)** If \( g(x) \) is a continuous function on \( \Omega \) and if
\[
\int_{\Omega} g(x) \, w(x) \, dx = 0 \quad \text{for all} \quad w \quad \text{in} \quad \Omega
\]
then \( g(x) \equiv 0 \) in \( \Omega \).

**Proof** Suppose for some \( x_0 \in \Omega \) that \( g(x_0) \neq 0 \). Since \( g \) is continuous, in some ball \( B \) around \( x_0 \) we have either \( g(x) > 0 \) or \( g(x) < 0 \). So let \( w(x) \) be a little bump centered at \( x_0 \) with \( w(x_0) = \text{sign}(g(x_0)) \) and \( w > 0 \) in \( B \), \( w = 0 \) outside \( B \). Then \( \int_{\Omega} g(x) \, w(x) \, dx > 0 \).

In summary, consider the variational problem
\[
\min_{y \in H} I(y) \quad \text{with} \quad I(y) = \int_a^b F(x, y, y') \, dx \quad \text{and} \quad H = \left\{ y \in C^2[a, b] \right\} \quad \text{with} \quad y(a), y(b) \quad \text{fixed}
\]
then with \( y = y^* + w \) with \( w \in C^2[a, b] \) and \( w(b) = 0 \) we have that the optimizer must satisfy the E-L equation
\[
F_y - \frac{d}{dx} F_{y'} = 0 \quad \text{with} \quad y(a), y(b) \quad \text{given}.
\]

Now suppose that \( y(a) \) is given but that no condition for \( y \) at \( x=b \) is given. Then from (13) we have that the optimizer \( y \) satisfies the E-L equation
\[
F_y - \frac{d}{dx} F_{y'} = 0 \quad \text{on} \quad a < x < b
\]
\( y(a) \) specified and \( F_{y'} = 0 \) at \( x=b \) (natural BC).

**Remark** In (13), the competitor \( w \) satisfies \( w(a) = 0 \). We must have that \( \forall w \in H = \left\{ w \in C^2[a, b] \right\} \), \( w(a) = 0 \)
\[
\int_a^b w [ F_{y'} - \frac{d}{dx} F_{y'} ] \, dx + F_{y'} w \bigg|_a^b = 0.
\]

For competitors with \( w(b) = 0 \) we must have \( F_{y'} - \frac{d}{dx} F_{y'} = 0 \). Then for competitors with \( \neq 0 \) we must have \( F_{y'} = 0 \).

Now we will consider an important special case.

**Lemma** Suppose that \( F \) is independent of \( x \), i.e. \( F = F(y, y') \), then

the E-L equation given by
\[
F_y - \frac{d}{dx} F_{y'} = 0
\]
has the first integral \( y' F_{y'} - F = c \quad \forall x \) where \( c \) is a constant.

**Proof** Multiply by \( y' \):
\[
y' F_{y'} - y' \frac{d}{dx} F_{y'} = 0.
\]
Now add and subtract \( y'' F_{y'} \) so that
\[
y'' F_{y'} + y' F_y - y' \left( \frac{d}{dx} F_{y'} \right) - y'' F_{y'} = 0 \rightarrow \frac{d}{dx} \left[ -y' F_{y'} + F \right] = 0
\]
then \( F - y' F_{y'} = c \).
EXAMPLE (BRACHISTOCHRON)

We wish to find the path connecting two points in \((x, y)\) plane that gives smallest travel time for a particle falling under gravity, released from rest.

Now the potential energy is \(-mgy\) and the speed of particle at a point is \(\frac{1}{2} mv^2\).

Thus \(\frac{1}{2} mv^2 = mgy\) so \(v = \sqrt{2gy}\).

Now the arclength element \(ds\) is \(ds = \sqrt{1 + y'^2} \, dx\).

The total travel time \(T\) is \(T = \int_{x_0}^{x_1} \frac{dx}{\sqrt{2gy}} = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}} \, dx\).

Let's try to find the optimal path that satisfies \(y(0) = 0\) and \(y(L) = L\). We might expect that the optimal path is a quarter-circle, but this is incorrect as we now show. Our path \(y(x)\) is to minimize

\[T = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}} \, dx\] with \(y(0) = 0\) and \(y(L) = L\).

Here \(F = \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}\) is independent of \(x\) so that by Lemma 1, the Euler-Lagrange equation has the first integral \(\int (y'F - F_y) = C\).

Now \(F(y, y') = -\frac{y'}{2g} \sqrt{1 + y'^2}\). Now \(F_y = \frac{y'}{2g} (1 + y'^2)^{1/2}\) so that

\[y'^{1/2} \sqrt{1 + y'^2} - \frac{y'}{2g} y'^{1/2} = C \Rightarrow y'^{1/2} (1 + y'^2 - y'^2) = C\sqrt{1 + y'^2}\]

This becomes \(y'^{1/2} = C (1 + y'^2)^{1/2}\). This gives \(y^2 = C^2 (1 + y'^2)\).

We solve to obtain \(y' = \sqrt{\frac{y - \mu}{y}}\) with \(\mu = 1/C^2 > 0\).

Now let's solve this first order ODE geometrically. Let \(q\) be the angle between the normal and the tangent line as shown.

\[\frac{dx}{dy} = \tan q\] so that \(\frac{1}{\cot q} = \sqrt{\frac{y - \mu}{y}}\).

Then \(\tan^2 q = \frac{y}{\mu - y}\) so that \(y [1 + \tan^2 q] = \mu \tan^2 q\)

\(q\) = Angle between tangent to curve and the vertical direction.
WE CONCLUDE THAT \( y = \frac{n}{2} \sin^2 \phi \).

Now \( \frac{dy}{dx} = \sqrt{\frac{y}{\frac{n}{2}} = \frac{dy}{du}/(du/dq)} \), so that \( \sqrt{\frac{\frac{n}{2}}{\sin^2 \phi}} \frac{2 \frac{n}{2} \sin \phi \cos \phi}{(du/dq)} \).

**NOW SOLVING:** \( \frac{dx}{du} = \frac{2 \frac{n}{2} \sin \phi \cos \phi}{\cot \phi} \).

**INTEGRATING ONCE** we obtain \( x = \frac{n}{2} \left( \phi - \frac{1}{2} \sin(2\phi) \right) \).

**y = \frac{n}{2} \left( 1 - \cos(2\phi) \right) \).**

**THIS PATH WHICH GOES THROUGH (0,0) IS THE PATH THAT GIVES SHORTEST TRAVEL TIME. THE PATH IS A CYCLOID, WHICH IS THE PATH OF A PARTICLE ON A BICYCLE.**

Finally, we must find \( n \) so that \( y(L) = L \).

**THE PARAMETRIC FORM OF THE SOLUTION, GIVEN IN \( x \), SPECIFIES THE OPTIMAL PATH IN TERMS OF THE ANGLE FROM THE VERTICAL TO TANGENT LINE.**

We put \( x = L \) and \( y = L \) in \( x \) to find \( n \) and end angle \( \phi_0 \)

Thus \( L = \frac{n}{2} \sin^2 \phi_0 \)

\( L = \frac{n}{2} \left( \phi_0 - \frac{1}{2} \sin(2\phi_0) \right) \).

**WE ELIMINATE \( L \) TO OBTAIN THAT \( \phi_0 \) SATISFIES \( \sin^2 \phi_0 = \phi_0 - \frac{1}{2} \sin(2\phi_0) \).**

**A PLOT OF THE INTERSECTION POINTS IN (x,y) YIELDS**

\( \begin{align*}
\sin^2 \phi_0 &= \phi_0 - \frac{1}{2} \sin(2\phi_0) \\
\phi_0 &= 1.206 \\
\phi_0 &= 68.8^\circ.
\end{align*} \)

**EXAMPLE (HAMILTON'S PRINCIPLE)**

Let \( T(x,x') \) be the kinetic energy of a particle and \( V(x,t) \) be the potential energy. The Lagrangian \( L(x,x',t) \) is defined by

\( L(x,x',t) = T(x,x') - V(x,t) \).

Hamilton's least action principle is that the particle path \( x(t) \) is to minimize the action \( I = \int_{t_1}^{t_2} L(x,x',t) \, dt \). In other words, \( x(t) \) is to satisfy the E-L equation

\( \frac{d}{dt} \frac{\partial L}{\partial \dot{x}'} - \frac{\partial L}{\partial x} = 0 \).
As an example consider the nonlinear pendulum

The kinetic energy is $\frac{1}{2} m v^2$ with $v = \dot{\varphi} \to T = \frac{1}{2} m (\dot{\varphi}^2)$

Now $V = mg \ell (1 - \cos \varphi)$ so that $V = 0 \to \varphi = 0$ (bottom)

Thus $L = \frac{1}{2} m \ell^2 \dot{\varphi}^2 - mg \ell (1 - \cos \varphi)$ and $T = \int_0^{\frac{1}{2}} L (\varphi, \dot{\varphi}) \, dt$.

Thus $L_q = -mg \ell \sin \varphi$ and $L_{\dot{\varphi}} = m \ell^2 \ddot{\varphi}$

The EL equation is $L_q - \frac{d}{dt} L_{\dot{\varphi}} = 0 \to -mg \ell \sin \varphi - m \ell^2 \ddot{\varphi} = 0$.

Thus $\ddot{\varphi} + \frac{g}{\ell} \sin \varphi = 0$.

Example (catenary)

Let $y = y(x) > 0$ and suppose $-a < x < a$. We want to find the curve $y(x)$ that minimizes the surface obtained by revolving $y = y(x)$ around $x$-axis.

\[ ds = \sqrt{1 + y'^2} \, dx \]

Now the surface area element is $dA = 2\pi y \sqrt{1 + y'^2} \, dx$.

The minimization problem is to minimize $S = 2\pi \int_a^b \sqrt{1 + y'^2} \, dx$ with $y(\pm a)$ prescribed. Now $F(y, y') = y \sqrt{1 + y'^2}$.

From Lemma 1, since $F$ is independent of $x$, we have the first integral for the EL equation given by

$$F - y' F_{y'} = C.$$

Then $F = y(1 + y'^2)^{1/2}$, $F_{y'} = y y' (1 + y'^2)^{-1/2}$. Thus,

$$y(1 + y'^2)^{1/2} - y y' (1 + y'^2)^{-1/2} = C$$

Then $$(1 + y'^2)^{1/2} \left[ y - y y' (1 + y'^2)^{-1} \right] = C$$

This gives $(1 + y'^2)^{1/2} y \left[ 1 - \frac{y'^2}{1 + y'^2} \right] = C \quad \text{so} \quad \frac{y}{\sqrt{1 + y'^2}} = C$.

This yields that $1 + y'^2 = \frac{y^2}{C^2}$ or $y'^2 = \frac{y^2}{C^2} = -1$.

Now we can proceed by solving for $y'$ and integrating or else more quickly observe a family of solutions, recalling $\cosh^2 x - \sinh^2 y = 1$. 
\[ y = C \cosh \left[ \frac{1}{C} (x - x_0) \right]. \quad \text{Then} \quad y' = \sinh \left[ \frac{1}{C} (x - x_0) \right]. \]

And so \[ y'' = \frac{y'}{C} = -1, \quad \text{as required.} \]

Hence solutions to the EL equation are

\[ y = C \cosh \left[ \frac{1}{C} (x - x_0) \right], \quad \text{on} \quad -a < x < a. \quad \text{This is a catenary curve.} \]

Now we will consider \( y(a) = 1 \) and \( y(-a) = 1 \), so that by symmetry \( x_0 = 0 \).

Thus, we find \( C \) so that

\[ y = C \cosh \left( \frac{a}{C} \right). \]

Let \( Z = \frac{a}{C} \) so \( C = \frac{a}{Z} \). Then

\[ \frac{a}{Z} = \frac{a}{Z} \cosh(Z). \]

As such we must look for intersection points to \( Z \left( \frac{a}{Z} \right) = \cosh(Z) \), in \( Z > 0 \).

We plot as follows:

\[ \frac{a}{Z} \quad \text{small} \]

\[ \frac{a}{Z} \quad \text{large} \]

We conclude that if \( \frac{a}{Z} > \text{threshold} \), then \( \exists \) two roots \( Z_1, Z_2 \).

No roots if \( \frac{a}{Z} < \text{threshold} \).

To find the threshold we set the tangency condition

solve \( Z \left( \frac{a}{Z} \right) = \cosh(Z) \) together with \( \frac{a}{Z} = \sinh(Z) \).

This gives \( Z = \coth(Z) \rightarrow Z \approx 1.19 \)

Hence the critical value of \( \frac{a}{Z} \) is \( \sinh(1.19) \approx 1.49 \).

We conclude that if \( \frac{a}{Z} > 1.49 \) then are two stationary points for the functional and none if \( \frac{a}{Z} < 1.49 \). The non-existence when \( a \) is large suggests that perhaps we should look for a non-smooth extremal consisting of two portions of spheroids.

In the 1-D problem this means \( y = 0 \) in \( -b < x < b \)

and is non-smooth at \( x = \pm b \).
EXAMPLE (GEODESICS)

Given any two points on the surface of the unit sphere, characterize geometrically the path that has shortest distance. We will show that the shortest path is a great circle, so that the initial point, final point, and center of the sphere lie on a common plane that contains the optimal path, i.e., the geodesic.

We first calculate the distance element on the sphere.

Recall in spherical coordinates with a unit sphere,

\[
\begin{align*}
x &= \cos \phi \sin \theta \\
y &= \sin \phi \sin \theta \\
z &= \cos \theta
\end{align*}
\]

of \( \phi \in [-\pi, 0) \)

of \( \theta < 2\pi \)

Plane containing initial point, final point, and center of sphere.

Now

\[
\begin{align*}
dx &= -\sin \theta \sin \phi \, d\phi + \cos \theta \cos \phi \, d\theta \\
dy &= \cos \theta \sin \phi \, d\phi + \sin \theta \cos \phi \, d\theta \\
dz &= -\sin \phi \, d\phi
\end{align*}
\]

Now

\[
(dx)^2 + (dy)^2 = (\sin^2 \theta \sin^2 \phi + \cos^2 \theta \sin^2 \phi)^2 + (\cos^2 \theta \cos^2 \phi + \sin^2 \theta \cos^2 \phi)^2
\]

Since \( d\phi \, d\theta \) term cancels, this gives

\[
(dx)^2 + (dy)^2 = \sin^2 \theta (d\phi)^2 + \cos^2 \phi (d\theta)^2.
\]

Now

\[
(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = \sin^2 \theta (d\phi)^2 + (\cos^2 \phi + \sin^2 \theta)^2 (d\phi)^2
\]

Thus

\[
(ds)^2 = \sin^2 \phi (d\phi)^2 + (d\theta)^2.
\]

Now we think of the path with \( \theta = \theta(\phi) \). Hence

\[
ds = \sqrt{1 + \sin^2 \phi (d\phi)^2}
\]

Then the distance is simply

\[
D = \int_{\phi_0}^{\phi_1} \sqrt{1 + \sin^2 \phi (d\phi/\phi)^2} \, d\phi.
\]

Now with

\[
F(\phi, \theta, \phi') = (1 + (\sin^2 \theta) \phi'^2)^{1/2}
\]

The Eichten is

\[
F_\phi - \frac{d}{d\theta} F_\theta = 0 \quad \text{but} \quad F_\phi = 0 \quad \text{so} \quad \frac{d}{d\theta} F_\phi = 0
\]

implies that

\[
F_\phi = C \quad \text{but} \quad F_\phi = \frac{\phi' \sin^2 \phi}{\sqrt{1 + (\sin^2 \phi) \phi'^2}}.
\]

Observe that \( C \leq 1 \), must hold.

This gives

\[
\phi'^2 \sin^4 \phi = C^2 \left[ 1 + \sin^2 \phi \phi'^2 \right]
\]

\[
\phi'^2 \sin^4 \phi \left( \sin^2 \phi - C^2 \right) = C^2.
\]
We conclude that
\[ q^1 = \frac{C}{\sin \phi (\sin^2 \phi - C^2)^{1/2}} = \frac{C}{\sin^2 \phi \left[ 1 - C^2 \csc^2 \phi \right]^{1/2}} \quad \text{since} \quad 0 < \phi < \phi_0, \quad \sin \phi > 0. \]

Now put \( u = \cot \phi \) as a substitution. Then
\[ du = -C \csc^2 \phi \, d\phi \]

Then
\[ q = -C \left/ \sqrt{1 - C^2 \csc^2 \phi} \right. \, d\phi = -C \left/ \sqrt{1 - C^2 \left( 1 + \cot^2 \phi \right)} \right. \, d\phi = -C \left/ \sqrt{[1 - C^2 (1 + u^2)]^{1/2}} \right. \]

Hence
\[ q = -C \left/ \sqrt{1 - C^2 - C^2 u^2} \right. = -C \left/ C \right. \left/ \sqrt{(1 - C^2) - u^2} \right. = -\int \frac{du}{\sqrt{1 - C^2 - u^2}}. \]

Now let \( u = \sqrt{\frac{1 - C^2}{C^2}} \sin t \) as a further substitution.

Then
\[ q = -\int \frac{\cot dt}{\sqrt{1 - C^2 + C^2 \sin^2 t}} = -\int dt. \]

Thus \( q - q_0 = -t \) with \( u = a \sin t \) and \( a = \pm \sqrt{\frac{1 - C^2}{C^2}} \).

Therefore we have \( u = a \sin \left[ -(q - q_0) \right] = -a \sin \left( q - q_0 \right) \). But \( u = \cot \phi \)

so that
\[ \cot \phi = -a \sin \left( q - q_0 \right) \quad \text{where} \quad q = \sqrt{\frac{1 - C^2}{C^2}}. \]

To visualize the solution we put \( \cot \phi = \frac{\cos \phi}{\sin \phi} \) and multiply by \( \sin \phi \)

this gives
\[ \cos \phi = -a \sin \phi \sin \left( q - q_0 \right) = -a \sin \phi \left( \sin q \cos q_0 - \cos q \sin q_0 \right). \]

Now recalling that \( z = \cos \phi \), \( x = \cos q \sin \phi \), \( y = \sin q \sin \phi \) we obtain
\[ z = -a \cos q_0 \, y + a \sin q_0 \, x. \]

This is a plane through the origin. This plane intersects the sphere in a great circle. We conclude that the optimal path on the sphere, called the geodesic, lies on a great circle. Given an initial point \((\phi_A, q_A)\) and a final point \((\phi_B, q_B)\) we can solve for \( q \) and \( q_0 \) in
\[ \cot \phi = -a \sin \left( q - q_0 \right). \]
Now we study variational problems for PDEs. We will focus on scalar problems in 2-D and assume that minimizers and stationary points are $C^2$ smooth.

Consider the problem of finding stationary points or a minimizer for the functional

$$ I(u) = \int_{\Omega} F(x,y,u,u_x,u_y) \, dx \, dy, \quad \text{with } u=g \text{ on } \partial \Omega \text{ specified}. $$

We now derive a necessary condition to make $u^\varepsilon$ a minimizer (or stationary point) of $I(u)$ with $u \in C^2(\Omega)$.

We let $u = u^\varepsilon + \varepsilon w$ with $u^\varepsilon=g$ on $\partial \Omega$ and $w=0$ on $\partial \Omega$.

We define

$$ J(\varepsilon) = I(u^\varepsilon + \varepsilon w) = \int_{\Omega} F(x,y,u^\varepsilon + \varepsilon w, u^\varepsilon + \varepsilon w_x, u^\varepsilon + \varepsilon w_y) \, dx \, dy. $$

Now Taylor expanding as $\varepsilon \to 0$:

$$ F(x,y,u^\varepsilon + \varepsilon w, u^\varepsilon + \varepsilon w_x, u^\varepsilon + \varepsilon w_y) = F(x,y,u^\varepsilon, u^\varepsilon_x, u^\varepsilon_y) + \varepsilon \left[ F_{u}^\varepsilon w + F_{u_x}^\varepsilon w_x + F_{u_y}^\varepsilon w_y \right] + \ldots $$

where $(\varepsilon)$ indicates evaluation at $u=u^\varepsilon$. Thus substituting gives

$$ J(\varepsilon) = I(u^\varepsilon) + \varepsilon \int_{\Omega} \left( F_{u}^\varepsilon w + F_{u_x}^\varepsilon w_x + F_{u_y}^\varepsilon w_y \right) \, dx \, dy + \ldots $$

Now since $u^\varepsilon$ is a minimizer (or stationary point), we must have $J'(0)=0$, i.e., that

$$ \int_{\Omega} \left( F_{u}^\varepsilon w + F_{u_x}^\varepsilon w_x + F_{u_y}^\varepsilon w_y \right) \, dx \, dy = 0 \quad \forall \, w \in C^2(\Omega), \, w=0 \text{ on } \partial \Omega. \quad (V2) $$

Now by using

$$ \frac{d}{dx} (F_{u_x}^\varepsilon w) = F_{u_x}^\varepsilon w_x + \frac{d}{dx} (F_{u_x}^\varepsilon) \, w $$

$$ \frac{d}{dy} (F_{u_y}^\varepsilon w) = F_{u_y}^\varepsilon w_y + \frac{d}{dx} (F_{u_y}^\varepsilon) \, w $$

Thus $(V2)$ yields

$$ \int_{\Omega} \left( F_{u}^\varepsilon w + \frac{d}{dx} \left( F_{u_x}^\varepsilon w \right) + \frac{d}{dy} (F_{u_y}^\varepsilon w) - \frac{d}{dx} (F_{u_x}^\varepsilon) \, w - \frac{d}{dy} (F_{u_y}^\varepsilon) \, w \right) \, dx \, dy = 0. $$

Using the divergence theorem we get:

$$ \int_{\Omega} \left( F_{u}^\varepsilon - \frac{d}{dx} (F_{u_x}^\varepsilon) - \frac{d}{dy} (F_{u_y}^\varepsilon) \right) \, w \, dx \, dy + \int_{\partial \Omega} (F_{u_x}^\varepsilon, F_{u_y}^\varepsilon) \cdot \hat{n} \, w \, ds = 0. \quad (V3) $$

Now since $w=0$ on $\partial \Omega$, the boundary term vanishes and we conclude that

$$ \int_{\Omega} \left( F_{u}^\varepsilon - \frac{d}{dx} (F_{u_x}^\varepsilon) - \frac{d}{dy} (F_{u_y}^\varepsilon) \right) \, w \, dx \, dy = 0 \quad \forall \, w \in C^2(\Omega). $$

We conclude that $u^\varepsilon$ must satisfy the E-L equation

$$ F_{u}^\varepsilon - \frac{d}{dx} (F_{u_x}^\varepsilon) - \frac{d}{dy} (F_{u_y}^\varepsilon) = 0 \quad \text{in } \Omega. \quad (V4) $$

with $u^\varepsilon=0$ on $\partial \Omega$. 

Now suppose that no condition for \( u^x \) on \( \partial \Omega \) is specified. Then we require that for any \( w \in C^1(\Omega) \), we have that \( \nabla \cdot (F_u^x) - \frac{\partial}{\partial y} (F_y^x) = 0 \).

While for function \( w \) with no condition on \( \partial \Omega \), we must have \( (F_{ux}^x, F_{uy}^x) \cdot \hat{n} = 0 \) on \( \partial \Omega \).

In summary, a necessary condition that \( u^x \) is a minimizer of \( I(u) \) with no condition for \( u \) specified on \( \partial \Omega \), we must have that \( u^x \) satisfy
\[
F_u^x - \frac{\partial}{\partial x} F_{ux}^x - \frac{\partial}{\partial y} F_{uy}^x = 0 \quad \text{in} \; \Omega
\]
subject to the natural boundary conditions \( (F_{ux}^x, F_{uy}^x) \cdot \hat{n} = 0 \) on \( \partial \Omega \).

Now consider a class of variational problems with surface energy. Minimize
\[
I(u) = \int_{\Omega} F(x, u, u_x, u_y) \, dx \, dy + \int_{\partial \Omega} G(x, u) \, ds \quad \text{(V6)}
\]
with \( u \) unspecified on \( \partial \Omega \), as such we will want to find natural BC for \( u \).

We let \( u^x \) be a minimizer and write \( u^x = u^x + \epsilon w \). Define \( J(\epsilon) = I(u^x + \epsilon w) \).

We calculate from \( J'(0) = 0 \) that
\[
\int_{\Omega} (F_u^x - \frac{\partial}{\partial x} F_{ux}^x - \frac{\partial}{\partial y} F_{uy}^x) w \, dx \, dy + \int_{\partial \Omega} \left[ (F_{ux}^x, F_{uy}^x) \cdot \hat{n} + G_x^x \right] w \, ds = 0
\]
\( \forall w \in C^2(\Omega) \). Now if no condition is specified for \( w \) on \( \partial \Omega \), we have that \( u^x \) must satisfy
\[
F_u^x - \frac{\partial}{\partial x} F_{ux}^x - \frac{\partial}{\partial y} F_{uy}^x = 0 \quad \text{in} \; \Omega
\]
subject to the natural boundary conditions
\[
(F_{ux}^x, F_{uy}^x) \cdot \hat{n} + G_u^x = 0 \quad \text{on} \; \partial \Omega.
\]

We now consider a few examples of the theory.
Example Among all functions \( u(x,y) \in C^2(\Omega) \) find \( u \) that minimizes in a bounded domain \( \Omega \):

\[
I(u) = \int_{\Omega} \left[ \frac{1}{2} (u_x^2 + u_y^2) - p u \right] \, dx \, dy
\]

(i) For \( u = g \) on \( \partial \Omega \).

(ii) No condition for \( u \) on \( \partial \Omega \) is specified.

Solution

(i) Write \( u = u^x + v \) with \( u^x = g \) on \( \partial \Omega \) and \( v = 0 \) on \( \partial \Omega \). We obtain from (V4) that \( u^x \) satisfies

\[
F_{u^x} - \frac{1}{\partial x} F_{u_{x}^x} - \frac{1}{\partial y} F_{u_{y}^x} = 0 \quad \text{in} \quad \Omega \quad \text{with} \quad u^x = g \quad \text{on} \quad \partial \Omega.
\]

We have \( F = \frac{1}{2} (u_x^2 + u_y^2) - p u \). Now \( F_{u^x} = u_x, F_{u_{y}^x} = u_y \) and \( F_u = p \).

We get that \(-p - \frac{1}{\partial x} (u_x) - \frac{1}{\partial y} (u_y) = 0 \rightarrow \Delta u = -p \text{ in } \Omega \)

\( u = g \text{ on } \partial \Omega \).

(ii) Now for (ii) we have that \( u \) satisfies the equation subject to the natural boundary conditions given in (V5).

As such, with \((F_{u^x}, F_{u_y}) = (2u_x, 2u_y)\) we obtain that

\[
\Delta u = p \text{ in } \Omega\]

\[\nabla u \cdot \hat{n} = 0 \text{ on } \partial \Omega \rightarrow \text{natural BC}.\]

Now we show that \( u^x \) is actually a minimizer of \( I(u) \) and not just a stationary point. We calculate for \( u = u^x + v \) with \( v \) a competitor that

\[
I(u^x + v) = I(u^x) + \int_{\Omega} \left[ \frac{1}{2} (2u_x^2 + 2u_y^2 v_x + 2u_y^2 v_y) - p v \right] \, dx + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx
\]

\[
= I(u^x) + \int_{\Omega} \left[ \nabla u^x \cdot \nabla v - p v \right] \, dx + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx
\]

Now use \( \nabla \cdot (v \nabla u^x) = v \Delta u^x + \nabla \cdot \nabla u^x \). We get

\[
I(u^x + v) = I(u^x) + \int_{\Omega} \left[ (\nabla \cdot (v \nabla u^x) - v \Delta u^x - p v) \right] \, dx + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx
\]

\[
I(u^x + v) = I(u^x) + \int_{\Omega} \left[ -\Delta u^x - p \right] v \, dx + \int_{\partial \Omega} v \nabla u^x \cdot \hat{n} \, ds + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx
\]

Now since \( \Delta u^x = -p \text{ in } \Omega \), we obtain

\[
I(u^x + v) = I(u^x) + \int_{\partial \Omega} v \nabla u^x \cdot \hat{n} \, ds + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx
\]
Now if \( u^x = 0 \) on \( \partial \Omega \), if no condition on \( \partial \Omega \) is specified then the natural boundary condition \( \partial \Omega u^x = 0 \) on \( \partial \Omega \) hold. In either case, \( \int_{\Omega} \nabla u^x \cdot \nabla p \, ds = 0 \). Thus, we conclude that

\[
I(u^x + v) = I(u^x) + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx.
\]

We conclude that \( I(u^x + v) > I(u^x) \), \( \forall v \).

Thus, we conclude that \( u^x \) is a minimizer of \( I(u) \) and not just a stationary point.

**Example 2** Suppose that \( I(u) = \int_{\Omega} \frac{1}{2} p (u^2 + u_y^2) \, dx \, dy + \int_{\partial \Omega} \frac{1}{2} \sigma u^2 \, ds \).

Here \( p = p(x,y) \) and \( \sigma \) is a constant. Find the E-L equation for a minimizer.

No condition on \( \partial \Omega \) is assumed.

**Solution** We use (27), with \( F = \frac{1}{2} p (u^2 + u_y^2) \), \( G = \frac{1}{2} \sigma u^2 \). Thus \( F_u = p u_x, F_y = p u_y \).

Now

\[
F_u - \frac{d}{dx} F_u_x - \frac{d}{dy} F_y = 0 \implies \frac{d}{dx} \left[ p \frac{du}{dx} \right] + \frac{d}{dy} \left[ p \frac{du}{dy} \right] = 0 \implies \nabla \cdot (p \nabla u) = 0
\]

And on boundary we have \( (F_u, F_y), \nabla + \sigma = 0 \) on \( \partial \Omega \).

Thus gives, \( p \nabla u \cdot \nabla + \sigma u = 0 \) on \( \partial \Omega \).

Hence, \( \nabla \cdot (p \nabla u) = 0 \) in \( \Omega \)

And \( p \nabla u + \sigma u = 0 \) on \( \partial \Omega \).

Is the problem that any minimizer must satisfy.

**Rayleigh-Ritz Method**

The Rayleigh-Ritz method is a classical but simple method for approximating solutions to variational problems. Suppose we seek a minimum to

\[
\min_{u \in H} I(u)
\]

Where \( I \) is a functional on \( H \) and \( H \) is a class of functions \( \{u \} \) (with some BC's). Consider a class of functions \( V_1, \ldots, V_n \), with \( V_j \in H \) and let

\[
u^*(x) = \sum_{j=1}^n c_j V_j(x)
\]

so that \( u \in H \).
EXAMPLE 3 IN 1-D CONSIDER THE FUNCTIONAL

$$I(y) = \int_0^1 \left( y'^2 + (1+x) y^2 + 2y \right) dx$$  with  $$y(0) = y(1) = 0. $$

THE EQUATION WITH  $$F = (1+x) y^2 + 2y + y'^2 \Rightarrow F_y - \frac{d}{dx} F_y' = 0 \rightarrow 2(1+x)y + 2 = 2y'$$

SO THAT  $$y'' - (1+x) y = 0,$$  WITH  $$y(0) = y(1) = 0.$$ 

NATURAL CHOICE FOR TRIAL FUNCTION ARE  $$y_0(x) = x^0(1-x)$$  AND  $$y_0(x) = \sin (\pi x).$$

NOW Suppose we try  $$y = c_1 y_1(x)$$  with  $$y_1 = x(1-x).$$

WE CALCULATE

$$I(c,y) = c_1 \int_0^1 \left[ \left( y', y_1 \right) + (1+x) y_1^2 \right] dx + 2c_1 \int_0^1 y_1(x) dx.$$

WE CALCULATE

$$\int_0^1 \left( y'^2 + (1+x) y_1^2 \right) dx = \frac{23}{60} \text{ AND } 2\int_0^1 y_1(x) dx = \frac{1}{3}.$$

THUS WE HAVE

$$J(c) = I(c,y_1) = \frac{23}{60} c_1^2 + \frac{1}{3} c_1.$$

NOW  $$J'(c) = 0 \rightarrow \frac{46}{60} c_1 = -\frac{1}{3} \Rightarrow c_1 = c_1^* = \frac{-10}{23} \text{ thus } c_1^* = \frac{-10}{23} x(1-x)$$  is THE OPTIMUM FOR THIS TRIAL FUNCTION.  THEN  $$J(c, x^*) = \frac{23}{60} \left( \frac{10}{23} \right)^2 + \frac{1}{3} \left( -\frac{10}{23} \right) \left. = \frac{27}{140} \right. c_1 = c_1^* = \frac{-10}{23} x(1-x)$$

NOW WE TRY TO IMPROVE THE APPROXIMATION.

IF WE PUT  $$y = c_1 x(1-x) + c_2 x^2(1-x)$$  we obtain  $$J(c_1, c_2).$$

Setting  $$J(c_1) = 0 \text{ AND } J(c_2) = 0$$  we obtain

$$27/140 c_1 + 27 / 140 c_2 = -1/6 \rightarrow c_1 = c_1^* = \frac{-10}{23}, c_2 = c_2^* = \frac{-1}{12} \Rightarrow$$

$$c_1 = c_1^* = 0.435 \text{ and } c_2 = c_2^* = 0.111 \Rightarrow$$

so  $$y = c_1^* x(1-x) + c_2^* x^2 (1-x).$$

NOW IN GENERAL SUPPOSE THAT

$$y = \bar{y} \bar{c} \text{ WITH } \bar{c} = \left( \begin{array}{c} c_1 \\ c_0 \end{array} \right), \bar{y} = \left( \begin{array}{c} y_1 \\ y_0 \end{array} \right) \text{ AND } y = x^\top (1-x).$$

THEN IF  $$I(y) = \int_0^1 \left( p(x) y'^2 + q(x) y^2 + 2r(x) y \right) dx \text{ WITH } y = 0 \text{ AT } x = 0, 1,$$

WE OBTAIN

$$I(y) = \int_0^1 \left( p(\bar{y}^\top \bar{c}) \bar{c}^\top \bar{c} + q \bar{y}^\top \bar{y} \bar{c} + 2r \bar{y}^\top \bar{c} \bar{c}^\top \bar{c} \right) dx \Rightarrow$$

$$J(c) = I(y) = c^\top A \bar{c} + c^\top B \bar{c} + 2c^\top \Gamma \bar{c} \text{ WHERE } A = \int_0^1 p \bar{y}^\top \bar{y} \bar{c}^\top \bar{c} \text{ MATRICE}$$

NOW TO OBTAIN THE OPTIMAL POINT WE CALCULATE THE JACOBIAN OF  $$J$$  to obtain that the optimal  $$\dot{c} = c^*$$

$$(A + B) \dot{c} = -\Gamma$$

THIS IS A LINEAR ALGEBRAIC SYSTEM FOR  $$c^*.$$.  

Example 4: Let $\Omega$ be unit square $0 < x < 1, 0 < y < 1$. We want to minimize
$$I(u) = \int_{\Omega} \left( |\nabla u|^2 + 2u f \right) \, dx$$
over $u \in H = \{ u \mid u \in C^2(\Omega), u = 0 \text{ on } \partial \Omega \}$.

We take a trial function of the form
$$u(x, y) = \sum_{m=1}^{N} \sum_{n=1}^{N} \tilde{a}_{mn} \sin(m\pi x) \sin(n\pi y)$$
where $\tilde{w}_{mn}(x, y) = \sin(m\pi x) \sin(n\pi y)$.

Noticing that $\tilde{w}_{mn} \in H$ and we want to find coefficients $\tilde{a}_{mn}$ that minimize $I(u)$,

we now have
$$u_x = \sum_{m=1}^{N} \sum_{n=1}^{N} m \tilde{a}_{mn} \cos(m\pi x) \sin(n\pi y), \quad u_y = \sum_{m=1}^{N} \sum_{n=1}^{N} n \tilde{a}_{mn} \sin(m\pi x) \cos(n\pi y).$$

This gives us
$$\int_{\Omega} \left( |u_x|^2 + |u_y|^2 \right) \, dx \, dy = \sum_{m=1}^{N} \sum_{n=1}^{N} \sum_{m'=1}^{N} \sum_{n'=1}^{N} \frac{m^2 + n^2}{4} \tilde{a}_{mn} \tilde{a}_{m'n'} \int_{0}^{1} \cos(m\pi x) \cos(m'\pi x) \, dx \int_{0}^{1} \sin(n\pi y) \sin(n'\pi y) \, dy.$$  

But
$$\int_{0}^{1} \cos(m\pi x) \cos(m'\pi x) \, dx = \begin{cases} 1/2 & \text{if } m = m', \\ 0 & \text{if } m \neq m'. \end{cases}$$
$$\int_{0}^{1} \sin(n\pi y) \sin(n'\pi y) \, dy = \begin{cases} 1/2 & \text{if } n = n', \\ 0 & \text{if } n \neq n'. \end{cases}$$

Thus, we have
$$\int_{\Omega} \left( |u_x|^2 + |u_y|^2 \right) \, dx \, dy = \frac{\pi^2}{4} \sum_{m=1}^{N} \sum_{n=1}^{N} (m^2 + n^2) \tilde{a}_{mn}^2.$$  

Now we have
$$f = \sum_{m=1}^{N} \sum_{n=1}^{N} \tilde{b}_{mn} \sin(m\pi x) \sin(n\pi y), \quad \text{where } \tilde{b}_{mn} = 4 \int_{0}^{1} f \sin(m\pi x) \sin(n\pi y) \, dx \, dy.$$  

We calculate
$$\int_{\Omega} f \, dx = 2 \sum_{m=1}^{N} \sum_{n=1}^{N} \tilde{a}_{mn} \int_{0}^{1} f \sin(m\pi x) \sin(n\pi y) \, dx \, dy = 2 \sum_{m=1}^{N} \sum_{n=1}^{N} \tilde{a}_{mn} \tilde{b}_{mn}.$$  

This yields that
$$I(u) = \int_{\Omega} \left( |\nabla u|^2 + 2u f \right) \, dx = \frac{\pi^2}{4} \sum_{m=1}^{N} \sum_{n=1}^{N} (m^2 + n^2) \tilde{a}_{mn}^2 + \frac{1}{2} \sum_{m=1}^{N} \sum_{n=1}^{N} \tilde{a}_{mn} \tilde{b}_{mn} = G.$$  

Here we have 2N unknowns $a_{11}, a_{12}, \ldots, a_{1N}, \ldots, a_{NN}$, i.e., $a = (a_{i1}, \ldots, a_{iN})^T$.

$$G(a) = \frac{\pi^2}{4} \sum_{m=1}^{N} \sum_{n=1}^{N} (m^2 + n^2) \tilde{a}_{mn}^2 + \frac{1}{2} \sum_{m=1}^{N} \sum_{n=1}^{N} \tilde{a}_{mn} \tilde{b}_{mn}.$$  

Finite minimization problem.

Now set $\partial G / \partial a_{mn} = 0 \rightarrow \frac{\pi^2}{2} (m^2 + n^2) \tilde{a}_{mn} + \tilde{b}_{mn} / 2 = 0 \rightarrow \tilde{a}_{mn} = \frac{-\tilde{b}_{mn}}{\pi^2 (m^2 + n^2)}.$

We conclude that the optimizer is simply
$$u = -\sum_{m=1}^{N} \sum_{n=1}^{N} \tilde{b}_{mn} \sin(m\pi x) \sin(n\pi y)$$

is the eigenfunction expansion of the Euler Lagrange equation $\delta u = f$ in $\Omega$, $u = 0$ on $\partial \Omega$. 
NOW KANTOROVICH'S METHOD IS A MINOR VARIANT OF THE RAYLEIGH-RITZ PROCEDURE.

IF WE SEEK MIN I(u) IN 2-D DOMAIN Ω, WE CONSIDER A TRIAL FUNCTION

\[ u \in \mathcal{U} \]

WHERE \( u \) IS SPECIFIED EXPLICITLY IN ONE DIRECTION BUT ALLOWED TO BE ARBITRARY IN SECOND DIRECTION. THE OPTIMUM OF THIS TRIAL FUNCTION IS FOUND BY A VARIATIONAL PROBLEM IN \( \mathcal{U} \). WE CONSIDER TWO EXAMPLES.

**Example 5**

Let \( \Omega = \{(x,y) \mid a < x < b, -b < y < b\} \), we aim to \( \text{MIN } I(u) \)

WHERE \( \mathcal{U} = \{u \mid u(0,y) = 0, u(x,2b) = 0, \text{no condition on } x : a \} \) AND \( I(u) = \int_0^a \int_{-b}^b (u_x^2 + u_y^2 + 2u) \, dx \, dy \).

WE WILL USE A TRIAL FUNCTION OF THE FORM \( u_0 = (b^2 - y^2) v(x) \).

WE MUST HAVE \( v(0) = 0 \), AND THEN \( u_0 \in \mathcal{U} \).

NOW \( I(u_0) = \int_0^a \int_{-b}^b \left[ (b^2 - y^2) v_x^2 + 4y^2 v_x^2 - 2(b^2 - y^2) v \right] \, dx \, dy \)

\[ = \int_0^a \left[ 2v_x^2 \int_{-b}^b (b^2 - y^2) \, dy + 4v_x^2 \int_{-b}^b \frac{y^2}{3} \, dy - 4v \int_{-b}^b (b^2 - y^2) \, dy \right] \, dx \]

\[ = \int_0^a \left[ \frac{16}{15} b^5 v_x^2 + 8 b^3 v^2 - \frac{8}{3} b^3 v \right] \, dx \]

Thus \( I(u_0) = \int_0^a F(x,v,v') \, dx \) WITH \( F(x,v,v') = \frac{16}{15} b^5 v_x^2 + \frac{8}{3} b^3 v^2 - \frac{8}{3} b^3 v \)

WE WILL MINIMIZE \( I(u_0) \) WITH \( v(0) = 0 \), AND \( v(a) \) UNSPECIFIED.

THE EL EQUATION IS \( F_v - \frac{d}{dx} F_{v'} = 0 \) WITH \( v(0) = 0 \).

WITH THE NATURAL BOUNDARY CONDITION \( F_{v'} = 0 \) AT \( x = a \).

NOW \( F_v = \frac{16}{3} b^3 v - \frac{8}{3} b^3 \), \( F_{v'} = \frac{32}{15} b^5 v^2 \).

Thus \( F_v - \frac{d}{dx} F_{v'} = 0 \implies \frac{32}{15} b^5 v_x^2 - \frac{16}{3} b^3 v + \frac{8}{3} b^3 v = 0 \), \( 0 < x < a \).

With \( v(0) = 0 \) AND \( v(a) = 0 \).

This yields that \( v'' - \frac{5}{2b^3} \frac{2b^3}{4b^5} = 0 \), \( 0 < x < a \) WITH \( v(0) = 0 \) AND \( v'(a) = 0 \).

THE SOLUTION IS \( v(x) = \frac{1}{2} + A \cosh \left[ \sqrt{\frac{5}{2b^3}} (x-a) \right] \).

NOW \( v(0) = 0 \implies A = -\frac{1}{2} \sech \left[ \sqrt{\frac{5}{2b^3}} \right] \).

We get \( v = \frac{1}{2} \left[ 1 - \frac{\cosh \left[ \sqrt{\frac{5}{2b^3}} (x-a) \right]}{\cosh \left[ \sqrt{\frac{5}{2b^3}} a \right]} \right] \).

With \( \lambda = (\sqrt{2b^3})^{1/2} \), so that \( u_0 = \frac{1}{2} (b^2 - y^2) \left( 1 - \frac{\cosh \left[ \sqrt{\frac{5}{2b^3}} (x-a) \right]}{\cosh \left[ \sqrt{\frac{5}{2b^3}} a \right]} \right) \).
EXAMPLE 6

**STANDARD APPROXIMATE MINIMA**

\[ I(u) \text{ with } I(u) = \int_{\Omega} \left( (\nabla u)^2 + 2u \right) \, dx \text{ and } \Omega = \text{ THE TRIANGLE} \]

\[ \Omega = \{(x,y) \mid 1 \leq x \leq b, 0 \leq y \leq \sqrt{a^2 - x^2} \} \]

WE ASSUME \( u \in C^1(\Omega) \). NOTicing THAT THE EL EQUATION \( u_{x} = 1 \) IN \( \Omega \) WITH \( u = 0 \) ON \( \partial \Omega \).

NOW WE TAKE AS A TRIAL FUNCTION

\[ u(x,y) = \int_{0}^{y} \left( y^2 - (x/\sqrt{3})^2 \right) v(x) \]

WHERE \( v \in C^1(\Omega) \) AND \( v(b) = 0 \).

WE CALCULATE

\[ u_x = -\frac{2x}{3} v + \left[ y^2 - \frac{x^2}{3} \right] v' \]

\[ u_y = 2yv \]

SO THAT

\[ u_x^2 + u_y^2 = \frac{4x^2}{9} v^2 + \left[ y^2 - \frac{x^2}{3} \right]^2 v'^2 - \frac{4x}{3} v v' (y^2 - x^2/3) + 2y^2 v^2 \]

NOTicing THE \( y \mapsto -y \) SYMMETRY WE GET

\[ I(u_0) = 2 \int_{0}^{b} \int_{0}^{\sqrt{3}y/3} \left[ \frac{4x^2}{9} v^2 + \left( y^2 - \frac{x^2}{3} \right)^2 v'^2 - \frac{4x}{3} v v' (y^2 - x^2/3) + 2y^2 v^2 \right] \, dy \, dx \]

NOW LET \( y = \sqrt{3}x/3 \). SO THAT \( 0 < y < 1 \). THEN \( dy = x/\sqrt{3} \, ds \). SO

\[ I(u_0) = \frac{2}{\sqrt{3}} \int_{0}^{b} \int_{0}^{\sqrt{3}y/3} \left[ \frac{4x^2}{9} v^2 + \frac{x^2}{9} \left( x^2 - \frac{y^2}{3} \right)^2 v'^2 - \frac{4x}{3} v v' (x^2 - y^2/3) + 2y^2 v^2 \right] \, dv \, dx \]

NOW USING

\[ (x^2 - y^2/3)^2 \, dy = 9/15 \quad \text{AND} \quad (x^2 - y^2/3) \, dy = -2/3 \]

WE INTEGRATE OUT THE \( y \) VARIABLE:

\[ I(u_0) = \frac{2}{\sqrt{3}} \int_{0}^{b} \left[ \frac{4x^3}{9} v^2 + \frac{8}{15} x^5 v'^2 + \frac{8}{27} x^4 v v' + \frac{4}{9} x^3 v^2 + \frac{4}{9} x^3 v \right] \, dx \]

\[ I(u_0) = \frac{8}{9\sqrt{3}} \int_{0}^{b} \left[ F(x,v,v') \right] \, dx \]

WHERE

\[ F(x,v,v') = x^3 v^2 + \frac{2}{15} x^5 v'^2 + \frac{2}{3} x^4 v v' + 2x^3 v^2 + x^3 v \]

WE SIMPLIFY TO GET

\[ F(x,v,v') = \frac{2}{15} x^5 v'^2 + \frac{2}{3} x^4 v v' + 2x^3 v^2 + x^3 v \]

NOW THE EL EQUATION IS

\[ F_v - \frac{1}{\partial x} F_v = 0 \]

WHERE

\[ F_v = \frac{2}{15} x^5 v'^2 + 4x^3 v + x^3 \]

AND

\[ F_{v'} = \frac{4}{15} x^5 v' + \frac{2}{3} x^4 v \]

THUS

\[ \left( \frac{4}{15} x^5 v'^2 + \frac{2}{3} x^4 v \right)^3 \left( \frac{4}{15} x^5 v' + \frac{2}{3} x^4 v \right) = \frac{2}{3} x^4 v'^2 + 4x^3 v + x^3 \]

THUS YIELDING

\[ \frac{4}{15} x^5 v'^2 + \frac{4}{3} x^4 v' + \frac{4}{3} x^3 v = x^3 \]

\[ \rightarrow x^2 v'^2 + 5xv' - 5v = \frac{15}{4}, \quad 0 < x < b \]

THIS IS EULER EQUATION. THE PARTICULAR SOLUTION IS \( v_p = -7/4 \) AND \( v_M = x^2 \rightarrow (x^2 - 5) + 5x - 5 = 0 \)

SO THAT

\[ 0^2 - 4 \cdot 5 = (x^2 - 5) - 5 = 0 \rightarrow x = 5, 1. \]

HENCE \( V = C_1 x^5 + C_2 x + 3/4 \).

WE WANT \( V \) BOUNDED AT \( x \rightarrow 0 \) AND \( v(b) = 0 \)

\[ \rightarrow \quad V = \frac{3}{4} \left( \frac{x}{b} - 1 \right) \left( y^2 - x^2/3 \right) \]

HENCE THE OPTIMUM \( u_0 \) IS

\[ u_0 = \frac{3}{4} \left( \frac{x}{b} - 1 \right) \left( y^2 - x^2/3 \right) \]
Next, we will consider variational problem with higher derivatives.

We seek to minimize $J(u) = \int_\Omega \left( F(u, \nabla u, \nabla^2 u, \nabla^3 u, \nabla^4 u, \nabla^5 u) \right) \, dx \, dy$ with $u \in C^5$ and $\Omega$ a bounded 2-D domain. We will consider both prescribed and natural boundary conditions for $u$.

We first derive a necessary condition for $u^* \in C^5$ to be a minimizer. Let $u = u^* + v$ and we calculate

$$J'(u) = \int_\Omega \left( F_u \, v + F_{uu} \, u_x + F_{uy} \, u_y + F_{ux} \, u_{xx} + F_{uyy} \, u_{xy} + F_{uyy} \, u_{yy} \right) \, dx \, dy + O(e^3).$$

Now set $J'(u)|_{v=0} = 0$ so that $u^*$ must satisfy

$$\int_\Omega \left( F_u \, w + F_{uu} \, u_x + F_{uy} \, u_y + F_{ux} \, u_{xx} + F_{uyy} \, u_{xy} + F_{uyy} \, u_{yy} \right) \, dx \, dy = 0 \quad (\forall b).$$

Now use

$$\begin{align*}
\partial_x \left( f_{ux} \right) &= \partial_x \left( f_{ux} \right) \, w_x + f_{uxx} \, u_{xx} \\
\partial_y \left( f_{uy} \right) &= \partial_y \left( f_{uy} \right) \, w_y + f_{uyy} \, u_{yy}.
\end{align*}$$

Also

$$F_{uyy} \, u_{xy} = \frac{1}{2} \left[ \partial_x \left( f_{uyy} \, w_y \right) + \partial_y \left( f_{uyy} \, w_x \right) - \partial_y \left( f_{uyy} \right) \, w_x - \partial_x \left( f_{uyy} \right) \, w_y \right]$$

Thus

$$F_{uxx} \, u_{xx} + F_{uyy} \, u_{xy} + F_{uyy} \, u_{yy} = \partial_x \left[ f_{uxx} \, w_x + \frac{1}{2} f_{uyy} \, w_y \right] + \partial_y \left[ f_{uyy} \, w_y + \frac{1}{2} f_{uyy} \, w_x \right]$$

$$- \partial_x \left( f_{uxx} \right) \, w_x - \partial_y \left( f_{uyy} \right) \, w_y - \frac{1}{2} \partial_x \left( f_{uyy} \right) \, w_x - \frac{1}{2} \partial_x \left( f_{uyy} \right) \, w_y.$$

Now putting this into (\forall b) we obtain:

$$\int_\Omega \left[ F_u \, w + (F_{ux} - \partial_x (f_{uxx})) - \frac{1}{2} \partial_y (f_{uyy}) \right] w_x + (F_{uy} - \partial_y (f_{uyy})) - \frac{1}{2} \partial_x (f_{uyy}) \right) w_y \right] \, dx \, dy$$

$$+ \int_\Omega \left[ \partial_x \left( f_{uxx} \, w_x + \frac{1}{2} f_{uyy} \, w_y \right) + \partial_y \left( f_{uyy} \, w_y + \frac{1}{2} f_{uyy} \, w_x \right) \right] \, dx \, dy = 0 \quad (\forall g).$$

Now define

$$K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}^T \quad \text{with} \quad k_1 = F_{uxx} \, w_x + \frac{1}{2} F_{uyy} \, w_y, \quad k_2 = F_{uyy} \, w_y + \frac{1}{2} F_{uyy} \, w_x$$

and

$$G = (G_1, G_2)^T \quad \text{with} \quad G_1 = F_{uxx} - \partial_x (f_{uxx}) - \frac{1}{2} \partial_y (f_{uyy}), \quad G_2 = F_{uyy} - \partial_y (f_{uyy}) - \frac{1}{2} \partial_x (f_{uyy}).$$

Then (\forall g) becomes

$$\int_\Omega \left[ F_u \, w + G_1 \, w_x + G_2 \, w_y + \nabla \cdot K \right] \, dx \, dy = 0. \quad (\forall g)$$

Now we use our familiar identity

$$G_1 \, w_x = \frac{\partial}{\partial x} (G_1 \, w), \quad G_2 \, w_y = \frac{\partial}{\partial y} (G_2 \, w).$$
Putting this into (VII) and using divergence theorem we obtain:

$$\int_{\Omega} \left[ \mathbf{F}_u \cdot \frac{\partial}{\partial x} \mathbf{G}_1 - \frac{\partial}{\partial y} \mathbf{G}_2 \right] \mathbf{w} \, dx + \int_{\partial \Omega} \left( \mathbf{G} \cdot \mathbf{n} \, \mathbf{w} + \mathbf{w} \cdot \mathbf{n} \right) \, ds = 0. \quad (VII)$$

Now we rewrite the boundary term involving \( \mathbf{w} \cdot \mathbf{n} \).

We have from (VI0) that with \( \mathbf{n} = (n_1, n_2) \)

$$\mathbf{w} \cdot \mathbf{n} = \left( F_{u,x} + \frac{1}{2} F_{u,y} \right) n_1 W_1 + \left( F_{u,y} + \frac{1}{2} F_{u,x} \right) W_2 n_2. \quad (VI3)$$

Now define

$$\mathbf{A} = \begin{pmatrix} F_{u,x} & \frac{1}{2} F_{u,y} \\ \frac{1}{2} F_{u,y} & F_{u,y} \end{pmatrix} \quad \text{and} \quad \mathbf{w}' = \begin{pmatrix} W_x \\ W_y \end{pmatrix}.$$

Then (VI3) is equivalent to

$$\mathbf{w} \cdot \mathbf{n} = \mathbf{A} \mathbf{w}' \cdot \mathbf{n} = \left( \mathbf{w}' \right)^T \mathbf{A} \mathbf{n} \quad \text{since} \quad \mathbf{A} = \mathbf{A}^T. \quad (\text{Here we use} \quad \mathbf{a} \cdot \mathbf{b} = \mathbf{b}^T \mathbf{a})$$

In this way \( \mathbf{u}^k \) is a minimizer or stationary point if

$$\int_{\Omega} \left[ \mathbf{F}_u \cdot \frac{\partial}{\partial x} \mathbf{G}_1 - \frac{\partial}{\partial y} \mathbf{G}_2 \right] \mathbf{w} \, dx + \int_{\partial \Omega} \left( \mathbf{G} \cdot \mathbf{n} \, \mathbf{w} + \left( \mathbf{w}' \right)^T \mathbf{A} \mathbf{n} \right) \, ds = 0, \quad \forall \mathbf{w}. \quad (VI4)$$

Here

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_1, \mathbf{G}_2 \end{pmatrix}^T \quad \text{with} \quad \mathbf{G}_1 = \mathbf{F}_{u,x} - \frac{\partial}{\partial x} \left( \mathbf{F}_{u,y} \right) - \frac{1}{2} \frac{\partial}{\partial y} \left( \mathbf{F}_{u,y} \right),$$

$$\mathbf{G}_2 = \mathbf{F}_{u,y} - \frac{\partial}{\partial y} \left( \mathbf{F}_{u,y} \right) - \frac{1}{2} \frac{\partial}{\partial x} \left( \mathbf{F}_{u,y} \right).$$

**Case I**

Suppose \( \mathbf{w} = \mathbf{0} \) on \( \partial \Omega \) and \( \partial_n \mathbf{w} = \mathbf{0} \) on \( \partial \Omega \), i.e., this means that in \( \mathcal{H} \) we have specified both \( \mathbf{u}^k \) and \( \partial_n \mathbf{u}^k = \mathbf{0} \) on \( \partial \Omega \).

Then \( \mathbf{w}' = \mathbf{0} \) and the boundary term in (VI4) vanish. We obtain from the fundamental lemma that \( \mathbf{u}^k \) satisfies the equation

$$\mathbf{F}_u - \frac{\partial}{\partial x} \left[ \mathbf{F}_{u,x} \right] + \frac{\partial^2}{\partial x^2} \left[ \mathbf{F}_{u,y} \right] + \frac{\partial}{\partial x} \frac{\partial}{\partial y} \left( \mathbf{F}_{u,y} \right) - \frac{\partial}{\partial y} \left( \mathbf{F}_{u,y} \right) + \frac{\partial^2}{\partial y^2} \left( \mathbf{F}_{u,y} \right) = 0, \quad (VI5)$$

as obtained by setting \( \mathbf{F}_u - \frac{\partial}{\partial x} \mathbf{G}_1 - \frac{\partial}{\partial y} \mathbf{G}_2 = \mathbf{0} \).

**Case II** (Natural BC).

Suppose that \( \mathbf{u}^k = \mathbf{g} \) on \( \partial \Omega \) but no other condition on \( \mathbf{u}^k \) specified. From (VI4) we obtain that \( \mathbf{u}^k \) satisfies (VI5) and the natural boundary condition \( \mathbf{A} \mathbf{n} = \mathbf{0} \) on \( \partial \Omega \) (VI6)

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{F}_{u,x} & \frac{1}{2} \mathbf{F}_{u,y} \\ \frac{1}{2} \mathbf{F}_{u,y} & \mathbf{F}_{u,y} \end{pmatrix} \quad \text{since} \quad \mathbf{w} = \mathbf{0} \text{ on } \partial \Omega \text{ we have } \mathbf{G} \cdot \mathbf{n} = \mathbf{0} \text{ on } \partial \Omega.$$
CASE III: If no BC for \( u^* \) are given on \( \partial \Omega \) then \( u^* \) satisfies (VI5) and the natural boundary conditions
\[
G \cdot \hat{n} = 0 \text{ on } \partial \Omega \quad \text{and} \quad A \cdot \hat{n} = 0 \text{ on } \partial \Omega.
\]

We now consider a simpler important special case of the formula.

**Special Case (F independent of \( u_{xy} \)).** Suppose that \( F = F(x, y, u, u_x, u_y, u_{xx}, u_{yy}) \) with \( u \) specified on \( \partial \Omega \). We then obtain from (VI5) and (VI6) that \( u^* \) satisfies
\[
\begin{align*}
F_{u} - \frac{\partial}{\partial x} (F_{u,x}) + \frac{\partial^2}{\partial x^2} (F_{u,xx}) - \frac{\partial}{\partial y} (F_{u,y}) + \frac{\partial^2}{\partial y^2} (F_{u,yy}) &= 0 \quad \text{in } \Omega \\
F_{u_x,x} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

With \( u^* \) specified on \( \partial \Omega \) and \( (F_{u_x}, F_{u_y}) \cdot \hat{n} = 0 \) on \( \partial \Omega \).

If in fact we have that \( F = F(x, u, u_x, u_{xx}) \), i.e., no dependence on \( y \), then (VI7) simplifies further to
\[
\begin{align*}
F_{u} - \frac{\partial}{\partial x} (F_{u,x}) + \frac{\partial^2}{\partial x^2} (F_{u,xx}) &= 0 \quad \text{in } \Omega \\
F_{u_x,x} &= 0 \quad \text{at } x = a, b
\end{align*}
\]

We now give a few examples:

**Example 7** Suppose that \( I(u) = \int_0^1 \frac{1}{2} u_{xx}^2 + V(u) \, dx \) with \( u(0) = u(1) = 0 \) but no other condition specified at \( x = 0, 1 \). Find the equation and boundary conditions.

We have \( F = F(x, u, u_{xx}) = \frac{1}{2} u_{xx}^2 + V(u) \) and \( F \) independent of \( u_x \).

From (VI8) we calculate \( F_u = V'(u) \), \( F_{u,xx} = u_{xx} \). Thus, \( u^* \) satisfies
\[
\begin{align*}
u_{xxxx} + V'(u) &= 0 \quad \text{in } 0 < x < 1 \quad \text{and} \quad u_x(0) = u_x(1) = 0
\end{align*}
\]

**Example 8** Consider a thin plate with a load \( p(x, y) \) that causes it to sag. The deflection \( u(x, y) \) of the plate is given by
\[
\min_{u \in H^1(\Omega)} \left\{ I(u) \right\} \quad \text{with} \quad I(u) = \frac{1}{2} \int_\Omega \left( u_{xx} + u_{yy} \right)^2 \, dx - \int_\Omega p u \, dx
\]

where \( u = \frac{1}{2} u \left| u \right| \in C^4(\Omega) \), with \( u = u_\partial = 0 \) on \( \partial \Omega \).
Now since \( F = F(x, y, u, u_x, u_y, u_{xx}, u_{yy}) \) the EL equation from (VI7) is obtained with

\[
E_d = -p, \quad E_{u_{xx}} = (u_{xx} + u_{yy}) = E_{u_{yy}}.
\]

Thus from (VI7), the EL equation is

\[-p + \frac{\partial}{\partial x} (u_{xx} + u_{yy}) + \frac{\partial}{\partial y} (u_{xx} + u_{yy}) = 0 \text{ in } \Omega
\]

\[u = 0 \text{ on } \partial \Omega, \quad u_0 = 0 \text{ on } \partial \Omega.
\]

Thus, we have

\[-p + (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}) u = 0
\]

on \( \Delta^2 u = p \) in \( \Omega \)

with \( u = u_0 = 0 \) on \( \partial \Omega \).

Now, we can also use a Rayleigh-Ritz procedure to estimate the minimum energy in the case of a constant load where \( p = \text{constant} \) (independent of \( x, y \)).

We consider a trial function

\[u_0 = c_0 \left[ 1 - \cos \left( \frac{2\pi x}{a} \right) \right] \left[ 1 - \cos \left( \frac{2\pi y}{b} \right) \right], \]

where we take \( \Omega = \{(x, y) | 0 \leq x \leq a, 0 \leq y \leq b\} \).

Notice that \( u_0 = 0 \) on \( x = 0, a \) and \( y = 0, b \). Also \( u_{0x} = 0 \) on \( x = 0, a \); \( u_{0y} = 0 \) on \( y = 0, b \).

Thus, \( u_0 \in H \). We seek to determine \( c_0 \) to minimize \( J(u_0) \).

We have

\[\text{Min } J(u) \leq \text{Min } J(u_0).
\]

We calculate

\[J(c) = I(u_0) = \frac{1}{2} \int_0^a \int_0^b \left[ (\Delta u_0)^2 - 2p u_0 \right] dx dy.
\]

Put in the form for \( u_0 \) and assume \( p = \text{constant} \). We calculate after some algebra that

\[J(c_0) = \frac{2\pi^4}{a^3 b^3} c_0^2 \left[ 3(a^4 + b^4) + 2a^2 b^2 \right] - p c_0 a b.
\]

We set \( J'(c_0) = 0 \) to obtain

\[c_0 = c_0^* = \frac{p a^4 b^4}{4\pi^4 \left[ 3(a^4 + b^4) + 2a^2 b^2 \right]}
\]

Now since \( J''(c_0) > 0 \), \( c_0^* \) is indeed a minimum of \( J(c_0) \). The optimizer for \( J(u_0) \) is obtained with

\[u_0 = u_0^* = c_0^* \left[ 1 - \cos \left( \frac{2\pi x}{a} \right) \right] \left[ 1 - \cos \left( \frac{2\pi y}{b} \right) \right].
\]
CONSIDER THE CALCULUS PROBLEM OF MINIMIZING \( f(x_1, \ldots, x_n) \) SUBJECT TO M

CONTRAIN OF THE FORM \( g_i(x_1, \ldots, x_n) = 0 \) FOR \( i = 1, \ldots, m \) WITH \( M < N \). THE IDEA IS TO INTRODUCE

LAGRANGE MULTIPLIER PARAMETERS \( \lambda_1, \ldots, \lambda_m \) AND CONSTRUCT

\[
F(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) + \lambda_1 g_1(x_1, \ldots, x_n) + \cdots + \lambda_m g_m(x_1, \ldots, x_n).
\]

NOW IF \( F \) HAS A MINIMUM AT \( x^0_1, \ldots, x^0_n \) SUBJECT TO \( g_i = 0 \) FOR \( i = 1, \ldots, m \) IT IS NECESSARY THAT

\[
\nabla F|_{x^0} = 0 \quad \Rightarrow \quad \nabla F|_{x^0} = \alpha^T \nabla g|_{x^0}
\]

WITH \( g \in \left( \begin{array}{c} g_1 \\ \vdots \\ g_m \end{array} \right) \), \( \lambda \in \left( \begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_m \end{array} \right) \) AND \( x^0 \in \left( \begin{array}{c} x_1^0 \\ \vdots \\ x_n^0 \end{array} \right) \)

TOGETHER WITH

\[
g_i = 0 \quad \text{AT} \quad x^0.
\]

OVERALL THIS YIELDS \( n + m \) EQUATIONS FOR THE \( n + m \) UNKNOWN \( x^0 \) AND \( \lambda \). FOR A PROOF

SEE ANY CALCULUS TEXT.

WE NOW ILLUSTRATE THIS APPRAOH FOR A CALCULUS OF VARIATIONS PROBLEM

OF MINIMIZING \( J(u) = \int_D f(x, y, u, u_x, u_y) \, dx \) WITH \( u = 0 \) ON \( \partial D \) SUBJECT TO THE

CONSTRAINT THAT \( \int_D g(x, y, u, u_x, u_y) \, dx = M \) WHERE \( M \) IS CONSTANT. WE CAN REDUCE THIS

TO THE LAGRANGE MULTIPLIER PROBLEM AS FOLLOWS:

LET \( u = u^* + \varepsilon_1 w + \varepsilon_2 v \) WHERE \( w = v = 0 \) ON \( \partial D \).

WE WANT TO MINIMIZE

\[
J(\varepsilon_1, \varepsilon_2) = \int_D f(x, y, u^* + \varepsilon_1 w + \varepsilon_2 v, u^* + \varepsilon_1 w + \varepsilon_2 v_x, u^* + \varepsilon_1 w + \varepsilon_2 v_y) \, dx
\]

SUBJECT TO \( G(\varepsilon_1, \varepsilon_2) = M \) WHERE \( G(\varepsilon_1, \varepsilon_2) = \int_D g(x, y, u^* + \varepsilon_1 w + \varepsilon_2 v, u^* + \varepsilon_1 w + \varepsilon_2 v_x, u^* + \varepsilon_1 w + \varepsilon_2 v_y) \, dx \)

THIS BECOMES A LAGRANGE MULTIPLIER PROBLEM. WE INTRODUCE

\[
\hat{\lambda}(\varepsilon_1, \varepsilon_2) = \lambda \left[ G(\varepsilon_1, \varepsilon_2) - M \right],
\]

THE VARIATIONAL PROBLEM IS TO SET \( \hat{\lambda}_{\varepsilon_1} = 0 \) AND \( \hat{\lambda}_{\varepsilon_2} = 0 \) AT \( \varepsilon_1 = 0, \varepsilon_2 = 0 \).

THIS IS EQUIVALENT TO REWRITING THE AUGMENTED FUNCTIONAL AS

\[
\hat{J} = \int_D f(x, y, u, u_x, u_y) \, dx + \lambda \left( \int_D g(x, y, u, u_x, u_y) \, dx - M \right)
\]

AND DETERMINING THE LAGRANGE MULTIPLIER \( \lambda \) AND MINIMIZER \( u^* \) FROM A

NECESSARY CONDITION FOR A MINIMUM:

\[
F_\varepsilon - \frac{\partial}{\partial \varepsilon} F_{u_x} - \frac{\partial}{\partial \varepsilon} F_{u_y} + \lambda \left( g_{u_x} - \frac{\partial}{\partial \varepsilon} g_{u_x} \right) = 0
\]

SUBJECT TO \( \int_D g \, dx = M \),
Another approach to view variational problems with constraints is as follows:

Suppose we want to minimize

$$I(u) = \int \left[ F(x, y, u, u_x, u_y) \right] dx \quad \text{for} \quad u \in H \quad \text{with} \quad H = \left\{ u \mid u \in C^2(\Omega), u = 0 \text{ on } \partial \Omega \right\}$$

subject to

$$M(u) = \int \left[ G(x, y, u, u_x, u_y) \right] dx = \text{constant} \quad \text{(one constraint)}.$$

Claim 1: If $u^x$ solves the variational problem, then $u^x$ is a critical point

for the functional $I(u) + \lambda M(u)$, i.e., $u^x$ satisfies

$$F_u \frac{d}{dx} + F_{u_x} \frac{d}{dy} + F_{u_y} + \lambda \left( G_u \frac{d}{dx} + G_{u_x} \frac{d}{dy} + G_{u_y} \right) = 0$$

subject to $\int \left[ G \right] dx = \text{prescribed constant}$.

Proof: Suppose $u^x(\sigma)$ is a minimizer, thus $M(u) = C$ and $u^x$ minimizes $I(u)$ among all functions $u \in H$ with $M(u) = C$. In particular, let $u_\sigma \in H$ be a one-parameter family of functions in $H$ with $M(u_\sigma) = C$ and $u_0 = u^x$ (i.e., at $\sigma = 0$ it is the minimizer).

Then $I(u_\sigma)$ (being a function of $\sigma$) is minimized at $\sigma = 0$ and so

$$0 = \frac{d}{d\sigma} I(u_\sigma) \bigg|_{\sigma = 0} = \frac{d}{d\sigma} \left[ \int \left[ F(x, u_\sigma, u_{\sigma x}, u_{\sigma y}) \right] dx \right] \bigg|_{\sigma = 0} = \int \left[ F_u \frac{d}{d\sigma} u_{\sigma x} + F_{u_x} \frac{d}{d\sigma} u_{\sigma y} + F_{u_y} \frac{d}{d\sigma} u_{\sigma x} \right] dx.$$

Now label $\zeta = \frac{d}{d\sigma} u_{\sigma x} \bigg|_{\sigma = 0}$. Notice that $\zeta = 0$ on $\partial \Omega$, and so using div. theorem

$$\int \left[ F_u - \frac{d}{dx} F_{u_x} - \frac{d}{dy} F_{u_y} \right] \zeta(x) dx = 0 \quad \text{where} \quad \zeta \text{ indicates evaluation at } \sigma = 0,$$

i.e., at $u^x(x)$.

Now if $\zeta(x)$ could be any smooth function on $\Omega$ (with zero BC on $\partial \Omega$), we would conclude as before that $F_u - \frac{d}{dx} F_{u_x} - \frac{d}{dy} F_{u_y}$ must be zero, and that would be the E-L equation. However, due to the constraint $M(u_\sigma) = C$, $\zeta(x)$ cannot be just any function. Indeed, since $M(u_\sigma) = C$ for $\sigma$, then by differentiating wrt $\sigma$ we have

$$0 = \frac{d}{d\sigma} M(u_\sigma) \bigg|_{\sigma = 0} = \int \left[ G_u \frac{d}{dx} G_{u_x} - \frac{d}{dy} G_{u_y} \right] \zeta(x) dx.$$

Therefore, we have that, with $F(x) = F_u - \frac{d}{dx} F_{u_x} - \frac{d}{dy} F_{u_y}$ and $g(x) = G_u - \frac{d}{dx} G_{u_x} - \frac{d}{dy} G_{u_y}$, we have

$$\int \left[ F(x) \zeta(x) \right] dx = 0 \quad \text{for any} \quad \zeta(x) \text{ with} \quad \int \left[ g(x) \zeta(x) \right] dx = 0.$$

What can we conclude about $F(x)$ from this? We claim that

Claim 2: $F(x) = A g(x)$ for some $A$ real.
Once we establish Claim 2, then Claim 1 is complete. Let $\lambda(x)$ be any smooth function in $\Omega$, and decompose it as

$$\lambda(x) = \lambda g(x) + \hat{\lambda}(x)$$

where $\int_{\Omega} \hat{\lambda} \, g \, dx = 0$.

Thus $\hat{\lambda}$ is $L^1$ to $g$, by multiplying by $g(x)$ and integrating, $\int_{\Omega} \lambda(x) g(x) \, dx = \int_{\Omega} g(x) \lambda(x) \, dx$ or equivalently $u = (\lambda, g)/(g, g)$. Thus,

$$\lambda(x) = \frac{(\lambda, g)}{(g, g)} \, g(x) + \hat{\lambda}(x).$$

Now since $\hat{\lambda}$ is $L^1$ to $g$, we must have (setting $\xi = \lambda$) that

$$\int_{\Omega} F(x) \hat{\lambda}(x) \, dx = 0 \quad \rightarrow \quad \int_{\Omega} F(x) \left[ \lambda(x) - \frac{(\lambda, g)}{(g, g)} \, g(x) \right] \, dx = 0.$$ 

Interchanging integration to get

$$\int_{\Omega} \int_{\Omega} F(x) \lambda(y) \, dy = 0 \quad \forall \lambda(x).$$

We conclude that $F(x) - \frac{(f, g)}{(g, g)} \, g(x) = 0$ in $\Omega$.

In other words, $\exists \xi$ such that the optimizer $u^* \, \text{satisfies:}$

$$\left( \frac{\partial}{\partial x} F_u^* - \frac{\partial}{\partial y} F_y^* \right) + \xi \left( \frac{\partial}{\partial x} u^* - \frac{\partial}{\partial y} u_y^* \right) = 0$$

with $\int_{\Omega} g(x, u^*, u_x^*, u_y^*) \, dx = 0$.

Example: Minimize $\int_{\Omega} (u_x^2 + u_y^2) \, dx \, dy$ subject to $u = 0$ on $\partial \Omega$ with $\int_{\Omega} u^2 \, dx \, dy = 1$.

We let $\hat{u}(u) = \int_{\Omega} (u_x^2 + u_y^2) \, dx \, dy + \lambda \left( \int_{\Omega} u^2 \, dx \, dy - 1 \right) = \int_{\Omega} (u_x^2 + u_y^2) \, dx \, dy + \lambda \int_{\Omega} (u^2 - 1) \, dx \, dy$.

The equation for $\hat{u}(u)$ is

$$2 \Delta u - \frac{\partial}{\partial x} (2 u_x) - \frac{\partial}{\partial y} (2 u_y) = 0 \quad \rightarrow \quad u_{yy} + u_{xx} - \Delta u = 0 \quad \text{in} \quad \Omega.$$ 

Thus $\Delta u - \sigma = 0$ where $\sigma$ is an eigenvalue of $\Delta u + \sigma u = 0$ in $\Omega$; $u = 0$ on $\partial \Omega$,

with corresponding eigenfunction $u = \sigma$, normalized by $\int_{\Omega} \sigma \, dx = 1$.

Then $I(\sigma) = \int_{\Omega} \sigma \phi \sqrt{2 - \psi(x)} \, dx = -\int_{\Omega} \sigma \phi \Delta \phi \, dx + \sigma \int_{\Omega} \phi^2 \, dx = \sigma$.

Thus $I(\phi)$ is minimized when $\sigma = \sigma_1$, the smallest eigenvalue.
Example (Geometry) Consider the Minimum Surface Area Problem of Minimizing

\[ J(u) = \int_{\Omega} \left( 1 + u_x^2 + u_y^2 \right)^{\frac{1}{2}} \, dx \, dy \quad \text{with} \quad u = 0 \quad \text{on} \quad \partial \Omega. \]

Subject to a Fixed Volume Constraint That

\[ \int_{\Omega} u \, dx \, dy = V_0. \]

Find the E-L Equation and Solve It When \( \Omega = \{ (x, y) \mid x^2 + y^2 \leq R^2 \} \).

Solution \( J(u) \) is the surface area of the function \( z = u(x, y) \) in 3-D above \( \Omega \) and the surface is attached to the \( z = 0 \) plane at \( u = 0 \) on \( \partial \Omega \).

\[ z = u(x, y) \quad \text{(side-view)} \]

Then \( V_0 \) is the volume of the "dome" below \( u(x, y) \) and above \( u = 0 \).

Using Lagrange's Multiplier, we write the augmented functional as

\[ \tilde{J}(u) = \int_{\Omega} \left( 1 + u_x^2 + u_y^2 \right)^{\frac{1}{2}} \, dx + \lambda \left( \int_{\Omega} u \, dx - V_0 \right) = \int_{\Omega} \sqrt{1 + |u_x|^2} \, dx + \lambda \left( u - \frac{V_0}{|\Omega|} \right) \, dx. \]

Now

\[ F = \sqrt{1 + |u_x|^2} + \lambda \left( u - \frac{V_0}{|\Omega|} \right) \]

and so

\[ F_{u_x} = \frac{u_x}{\sqrt{1 + |u_x|^2}}, \quad F_{u_y} = \frac{u_y}{\sqrt{1 + |u_x|^2}}. \]

This yields that

\[ F_{u_x} \frac{\partial}{\partial x} - F_{u_y} \frac{\partial}{\partial y} = 0 \quad \text{so that} \]

\[ \nabla \cdot (Tu) = \lambda \]

\[ \nabla \cdot (Tu) = \frac{1}{\sqrt{1 + |u_x|^2}}. \]

The Lagrange multiplier parameter \( \lambda \) is to be found to satisfy the Volume Constraint.

Now from Geometry \( 2H = -\nabla \cdot (Tu) = \nabla \cdot \lambda \) where \( \nabla = \frac{(\nabla u, -1)}{\sqrt{1 + |u_x|^2}} \)

and \( \nabla \) is the 3-D gradient \( \nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \) and \( \nabla \) is unit normal to \( z = u(x, y) \) in 3-D. Here \( H \) is mean curvature of surface.

As a result, if we want \( \nabla \cdot (Tu) = \) constant, then we should take \( z = u(x, y) \) to correspond to part of a sphere in 3-D for which we know \( H = \) constant. In particular, for a sphere of radius \( C_0 \), i.e. \( x^2 + y^2 + z^2 = C_0^2 \) then \( H = 1/C_0 \).
This suggests that if \( \Omega \) is the domain \( x^1 + y^1 = \gamma_0^2 \), we should look for \( u \) in the form
\[
(u + A)^2 + x^1 + y^1 = c_0^2.
\]
When \( x^1 + y^1 \rightarrow 0 \rightarrow \gamma = 0 \rightarrow A = \sqrt{c_0^2 - \gamma_0^2}. \) Thus
\[
\nabla \cdot (Tu) = \gamma \left( \frac{c_0^2 - \gamma_0^2}{\gamma_0^2} \right)^{1/2} - \left( \frac{c_0^2 - \gamma_0^2}{\gamma_0^2} \right)^{1/2}.
\]
For this \( \nabla \cdot (Tu) \) should be constant and \( u = 0 \) on \( \partial \Omega \) where \( x^1 + y^1 = \gamma_0^2 \).

We calculate
\[
\nabla = \gamma \left( \frac{c_0^2 - \gamma_0^2}{\gamma_0^2} \right)^{1/2},
\]
so
\[
1 + u_x^1 + u_y^1 = 1 + \frac{x^1 + y^1}{c_0^2 - \gamma_0^2} = \frac{c_0^2}{c_0^2 - x^1 - y^1}, \quad \Rightarrow \quad 1 + |\nabla|^2 = \frac{c_0^2}{\sqrt{c_0^2 - x^1 - y^1}}.
\]
Thus,
\[
\frac{u_x}{\sqrt{1 + |\nabla|^2}} = \frac{-x}{c_0}, \quad \frac{u_y}{\sqrt{1 + |\nabla|^2}} = \frac{-y}{c_0}
\]
so that
\[
Tu = \frac{1}{c_0} (x, y).
\]

Hence \( \nabla \cdot (Tu) = -2/c_0 \). This implies from (1) that \( \gamma_0 = -2/c_0 \).

Now we find \( c_0 \) by imposing volume constraint. We integrate in polar coordinates to get
\[
\frac{1}{2} \pi \int_0^{\gamma_0} \left[ \gamma_0^2 - \left( \frac{c_0^2 - \gamma_0^2}{c_0^2} \right)^{1/2} \right] \gamma \, d\gamma = V_0 \quad \Rightarrow \quad 2\pi \int_0^{\gamma_0} \left[ \gamma_0^2 - \left( \frac{c_0^2 - \gamma_0^2}{2c_0^2} \right)^{1/2} \right] \gamma \, d\gamma = V_0.
\]

We integrate to obtain
\[
-\frac{1}{3} \left( c_0^2 - \gamma_0^2 \right)^{3/2} \left[ \gamma_0^2 - \left( \gamma_0^2 - \frac{c_0^2}{2} \right)^{1/2} \right] \gamma_0^2 = \frac{V_0}{2\pi} \quad \Rightarrow \quad \frac{1}{3} c_0^3 \left[ \gamma_0^2 - \left( \gamma_0^2 - \frac{c_0^2}{2} \right)^{1/2} \right] = \gamma_0^2.
\]

Rewriting this we obtain
\[
c_0^3 - \left( c_0^2 - \gamma_0^2 \right)^{1/2} \left[ \gamma_0^2 + \gamma_0^2/2 \right] = \frac{3V_0}{2\pi}.
\]

Hence we have a nonlinear algebraic equation for \( c_0 \) in terms of \( V \).
\[
\frac{3V_0}{2\pi} = c_0^3 - \left( c_0^2 - \gamma_0^2 \right)^{1/2} \left[ c_0^2 + \gamma_0^2/2 \right].
\]

Now introduce a new parameter \( \gamma \) by \( c_0^2 = \gamma \gamma_0^2 \rightarrow c_0 = \sqrt{\gamma} \gamma_0 \).

Then we obtain that
\[
\frac{3V_0}{2\pi} = \mathcal{F} (\gamma) \equiv \gamma^{3/2} - (\gamma - 1)^{1/2} (\gamma + 1/2) \quad (3)
\]

Now if (3) has a solution \( \gamma \) for a given \( V \), it follows that the solution to (1) is given by (2) where \( c_0 = \sqrt{\gamma} \gamma_0 \).
Now for \( \gamma > 1 \) we can show that \( F'(x) < 0 \) and \( F(1) = 1 \).

Looking for intersection with \( 3V_0/2\pi r_0^3 \) we have

\[
\frac{3V_0}{2\pi r_0^3} > 1 \quad \text{and} \quad \frac{3V_0}{2\pi r_0^3} < 1
\]

We conclude from (3) that if \( V_0 < V_{Ca} = \frac{2\pi r_0^3}{3} \left( \frac{1}{2} \right) \), a sphere of radius \( r_0 \) then \( \exists \) a unique value of \( \gamma \) such that (3) has a solution \( \gamma^* \). In this case \( \exists \) a unique \( r_0 = r_0^* \) and we conclude that \( u = \left[ r_0^2 - r_0^4 \right]^{1/2} \cdot (r_0^2 - r_0^4)^{1/2} \).

As such we can have a picture as shown:

\[
u(x, y) \quad \text{and not} \quad u(x, y)
\]

This yields a multi-valued

For other applications a sphere may not be only shape that gives \( \nabla \cdot (TM) = 0 \) constant. In general \( \nabla \cdot (TM) = 2H = \frac{1}{2} \left( \frac{1}{r_1} + \frac{1}{r_2} \right) \) is mean curvature of the surface at point \( x \), with \( r_1, r_2 \) being the principal radii of curvature at \( (x, y) \).

Remark consider a liquid bridge symmetric in z-direction with a circular cross-section as shown.

It can be shown that the catenoid (see 1-D notes)

\[
cosh \left( \frac{u}{c_0} + c_1 \right) = \frac{1}{c_0} \left( x^2 + y^2 \right)^{1/2}
\]

satisfies \( \nabla \cdot (TM) = \text{constant} \).

To satisfy the b.c.:

\[
cosh \left( \frac{u}{c_0} + c_1 \right) = A/c_0 \quad \text{and} \quad cosh \left( -\frac{u}{c_0} + c_1 \right) = A/c_0 .
\]

We must take \( c_1 = 0 \) in (x) so that \( \cosh \left( \frac{u}{c_0} \right) = A/c_0 \).

This is a nonlinear algebraic equation for \( c_0 \) in terms of \( A \) and \( a \).

Let \( z = u/c_0 \) so that \( A/z = \cosh z \) determines \( z \) in terms of \( A/a \).

\[
\text{either } 0, 1, \text{ or } 2 \text{ solutions} \]

if \( A/a \) large enough. Let \( r = (x^2 + y^2)^{1/2} \).

Finally \( \cosh \left( \frac{u}{c_0} \right) = \frac{1}{c_0} \left( x^2 + y^2 \right)^{1/2} \rightarrow r = c_0 \cosh \left( \frac{u}{c_0} \right) \). The volume constraint is

\[
\overline{V}_0 = \pi \int_{-a}^{a} \left( r(u) \right)^2 \, du \rightarrow \overline{V}_0 = 2\pi c_0^2 \int_{-a}^{a} \cosh^2 \left( \frac{u}{c_0} \right) \, du
\]

Question: given \( \overline{V}_0 > 0 \) does \( \exists \) a value of \( c_0 \) and \( A \) such that both \( \cosh \left( \frac{u}{c_0} \right) = A/c_0 \) and the constraint \( \overline{V}_0 = 2\pi c_0^2 \int_{-a}^{a} \cosh^2 \left( \frac{u}{c_0} \right) \, du \) can be satisfied? → nonlinear algebraic system for \( A \) and \( c_0 \) in terms of \( a, \overline{V}_0 \).
Applications (Physics)

In the continuum theory of phase transitions we encounter variational problems of the form in 2-D:

$$\min \left[ \int_\Omega \left( \frac{\epsilon^2}{2} |\nabla u|^2 + V(u) \right) dx \right] \quad \text{subject to the mass constraint} \quad \int_\Omega u \, dx = M$$

where \( M \) is constant. The potential \( V(u) \) has a double-well structure of the form

$$V(u) = \begin{cases} \frac{1}{4} (1-u^2)^2 & \text{for } |u| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and is small and positive. No BC are given → natural BC hold.

To derive the Euler-Lagrange equation we begin with the augmented functional

$$I(u) = \int_\Omega F(x, y, u, \nabla u, u_x, u_y) \, dx + \int_\Omega \left[ G(x, y, u) - M \right] \, dx$$

where

$$F = \frac{\epsilon^2}{2} |\nabla u|^2 + V(u), \quad G = u, \quad \text{with } |\nabla u|^2 = u_x^2 + u_y^2.$$  

The E.L. equation is

$$F_u - \frac{\partial F}{\partial u} u_x - \frac{\partial F}{\partial y} u_y + \lambda (G_u) = 0, \quad \text{(A1)}$$

And the natural boundary condition are \( (F_u, F_y), \ \hat{n} = 0 \) on \( \partial \Omega \).

We calculate

$$F_u = V'(u), \quad F_{u_x} = \epsilon^2 u_x, \quad F_{u_y} = \epsilon^2 u_y, \quad G_u = 1.$$  

Thus we obtain that \((A1)\) becomes

$$V'(u) - \epsilon^2 \Delta u + \lambda = 0, \quad \forall u : \hat{n} \cdot \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega.$$  

Thus the minimizer would satisfy

$$\begin{cases} \epsilon^2 \Delta u - V'(u) = \lambda & \text{in } \Omega \\ \hat{n} \cdot u = 0 & \text{on } \partial \Omega \\ \int_\Omega u \, dx = M \end{cases}, \quad \text{(A2)}$$

where the mass constraint is the effective equation for \( \lambda \).

This PDE (A2) is called the constrained Allen-Cahn equation.
The corresponding time dependent problem is for \( \textbf{u}(x, y, t) \)

\[
\begin{align*}
\frac{\partial \textbf{u}}{\partial t} &= \textbf{e}^2 \Delta \textbf{u} - \nabla \cdot (\textbf{u} \nabla \textbf{u}) - \lambda \\
\text{on } \Omega \text{ and } \int_{\Omega} \text{u} \, d\Omega &= M,
\end{align*}
\quad (A3)
\]

with some initial condition \( \textbf{u}(x, 0) \).

We define the time-dependent energy \( E[\textbf{u}(x, t)] \) by

\[
E(\textbf{u}) = \int_{\Omega} \left( \frac{\varepsilon^2}{2} \nabla \textbf{u}^2 + \textbf{u}(\textbf{u}) \right) \, dx.
\]

It is clear that \( E > 0 \) always. We now claim the following

Claim

\[
\frac{d}{dt} E(\textbf{u}) = -\int_{\Omega} (\textbf{u}_t)^2 \, dx < 0 \text{ if } \textbf{u} \text{ satisfies } (A3).
\]

This indicates that \( \textbf{u} \) evolves in \((A3)\) so as to minimize the energy \( E \).

Proof

\[
\frac{d}{dt} E = \int_{\Omega} \left( \varepsilon^2 \Delta \textbf{u} \cdot \nabla \textbf{u}_t + \nabla \cdot (\textbf{u}_t \nabla \textbf{u}) \right) \, dx
\]

Recall that \( \nabla \cdot (\nabla \textbf{v}) = \nabla \cdot \nabla \textbf{v} + \Delta \textbf{v} \) for any \( \textbf{v}, \textbf{w} \). Let \( \textbf{w} = \textbf{u}_t \), \( \textbf{v} = \textbf{u}_t \)

so that \( \nabla \cdot \nabla \textbf{u}_t = \nabla \cdot \textbf{u}_t \nabla \textbf{u} \nabla \textbf{u} \).

We substitute to get

\[
\frac{dE}{dt} = \int_{\Omega} \left[ -\varepsilon^2 \textbf{u}_t \Delta \textbf{u} + \varepsilon^2 \nabla \cdot (\textbf{u}_t \nabla \textbf{u}) + \nabla \cdot (\textbf{u}_t \nabla \textbf{u}) \right] \, dx
\]

\[
= \int_{\Omega} \left[ \nabla \cdot (\textbf{u}_t \nabla \textbf{u}) - \varepsilon^2 \Delta \textbf{u} \right] \, dx + \varepsilon^2 \int_{\Gamma} \textbf{u}_t \nabla \textbf{u} \cdot \textbf{n} \, ds
\]

\[
= \int_{\Omega} \left[ -\Delta \textbf{u} - \lambda \right] \, dx + 0
\]

\[
= -\lambda \int_{\Omega} (\textbf{u}_t^2) \, dx - \int_{\Omega} \textbf{u}_t^2 \, dx + \int_{\Omega} \textbf{u}_t^2 \, dx
\]

Since \( M \) is constant, we get

\[
\frac{dE}{dt} = -\int_{\Omega} \textbf{u}_t^2 \, dx < 0.
\]

The next application involves formation of microstructure in material science (viscoelasticity). Consider the functional in 1-D for \( u(x) \)

\[
I(u) = \int_0^1 \left( \frac{1}{2} \left( u_x^2 - 1 \right)^2 + \frac{d^2 u^2}{x^2} \right) \, dx
\]

with \( u(0) = 0 \), \( u(1) = 0 \) and \( \alpha > 0 \).

The "minimizer" of \( I(u) \) wants to decrease \( u \) (due to \( u_x^2 \) term)

but simultaneously have \( u_x^2 \geq 1 \) \( \Rightarrow u_x \leq \pm 1 \), which is impossible.
THE ENERGY $I_1(V)$ DECREASES THE MORE SAW-TOOTHED THE FUNCTION BECOMES (WITH SMALLER $I$)

\[ I_1 = \begin{cases} 0 & \text{if } 0 < x < \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} < x < \pi \end{cases} \quad \text{slope =} \quad \frac{2}{\pi} \]

\[ I_2 \quad \text{(smaller energy)} \]

IT IS CLEAR THAT THE "MINIMIZER" IS NOT ACHIEVED IN THE SPACE OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS. TO ILLUSTRATE THIS CONSIDER \[ V(x) = \begin{cases} x & \text{if } 0 < x < \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} < x < \pi \end{cases} \]

DEFINE A SAW-TOOTH PROFILE $\tilde{U}^k(x)$ BY

\[ \tilde{U}^k(x) = \frac{1}{k} V(x) \quad \text{with } k \text{ an integer AND LET } \tilde{V}(x+\pi) = \tilde{V}(x). \]

AS $k \uparrow$ THE AMPLITUDE DECREASES AND $\tilde{U}^k(x)$ ON $0 < x < \pi$ BECOMES MORE JAGGED.

WE NOW CALCULATE THE ENERGY ALONG $\tilde{U}^k(x)$.

NOW \[ \tilde{U}'_x = \frac{\tilde{V}'(x)}{k} \] SO THAT \[ (\tilde{U}_x^k)'^2 - 1 = 0. \]

WE OBTAIN

\[ I(\tilde{U}^k(x)) = \int_0^\pi \frac{d}{dx} (\tilde{U}^k(x))^2 \, dx = \frac{d}{2} \int_0^\pi \left( \tilde{V}(x) \right)^2 \, dx = \frac{d}{2 \pi^2} \int_0^\pi (\tilde{V}(x))^2 \, dx. \]

NOW BY PERIODICITY $\tilde{V}(x + \pi) = \tilde{V}(x)$ WE HAVE

\[ I(\tilde{U}^k(x)) = \frac{d}{2 \pi^2} \int_0^\pi (\tilde{V}(x))^2 \, dx \]

\[ = \frac{d}{2 \pi^2} \left[ \int_0^{\pi/2} t^2 \, dt + \int_{\pi/2}^\pi (\pi - t)^2 \, dt \right] \]

\[ = \frac{d}{2 \pi^2} \left[ \frac{\pi^3}{3} + \frac{\pi^3}{2} \right] = \frac{d}{2 \pi^2} \left[ 3 - 2 \pi \right]. \]

WE CONCLUDE THAT

\[ I(\tilde{U}^k(x)) = \frac{d}{24 \pi^2} \rightarrow 0 \quad \text{as } \quad k \rightarrow \infty \quad \text{so that the energy decreases as the function become increasingly sawtoothed but with a smaller amplitude. THE "MINIMIZER" IS NOT A DIFFERENTIABLE FUNCTION.} \]

NOW IF WE CONSIDER THE TIME-DEPENDENT PROBLEM FOR $U(x,t)$ GIVEN BY

\[ U_t = \left( U_{xx} - U_x + B U_xt \right)_x - \alpha U_x, \quad B > 0, \quad \alpha > 0 \]

WITH \[ U(0,t) = U(\pi,t) = 0, \quad U(x,0) = f(x), \quad U_t(x,0) = 0 \] \[ (A4) \]

$B > 0$ IS A VISCO-ELASTIC DAMPING PARAMETER.
Then for steady states we have that \( u \) satisfies

\[
(\frac{u^3}{x} - u_x)_x = \alpha \ u \quad \text{on} \quad 0 < x < 1 \quad \text{with} \quad u(0) = u(1) = 0.
\]

Thus the EL equation for \( I(u) = \int_0^\infty \left( \frac{1}{4} (u_x^2 - 1)^2 + \frac{\alpha}{2} u^2 \right) \ dx \).

Now if we define an energy \( E(u) = \int_0^\infty \left( \frac{1}{2} u_t^2 + \frac{\alpha}{4} (u_x^2 - 1)^2 + \frac{\alpha}{2} u^2 \right) \ dx \)

then we claim that

**Claim:** \( \frac{dE}{dt} = -\beta \int_0^\infty (u_x t)^2 \ dx < 0 \). This suggests that starting with smooth initial data \( u(x,0) = f(x) \) we have that \( E \to 0 \) as \( t \to \infty \), which suggests that \( u \) evolves to an increasingly jagged sawtooth function as \( t \uparrow \).

Proof we calculate:

\[
\frac{dE}{dt} = \int_0^\infty \left( u_t u_{tt} + (u_x^2 - 1) u_x u_{xt} + \alpha u u_t \right) \ dx.
\]

Now since \( u_{tt} = (\frac{u^3}{x} - u_x + 8 u_x t)_x - \alpha u \) we substitute for \( u_{tt} \) to get

\[
\frac{dE}{dt} = \int_0^\infty \left( (u_x^3 - u_x + 8 u_x t)_x u_t + (u_x^2 - 1) u_x u_{xt} \right.
\]

\[
- \int_0^\infty \left( (u_x^3 - u_x) u_t^2 + (u_x^2 - u_x) u_x u_{xt} \right) \ dx.
\]

Now

\[
\frac{d}{dx} \left[ \frac{1}{2} (u_x^3 - u_x) u_t \right] = (u_x^3 - u_x) u_t + (u_x^3 - u_x) u_{xt}
\]

And

\[
\frac{d}{dx} \left[ u_x u_t \right] = u_{xxt} u_t + (u_x t)^2.
\]

Thus

\[
\frac{dE}{dt} = \int_0^\infty \frac{d}{dx} \left[ (u_x^3 - u_x) u_t \right] \ dx + \int_0^\infty \left( \frac{\beta}{2} \frac{d}{dx} (u_x t u_t) - \beta (u_x t)^2 \right) \ dx.
\]

which becomes

\[
\frac{dE}{dt} = (u_x^3 - u_x) u_t \int_0^\infty + \beta u_x t u_t \bigg|_x^\infty - \beta \int_0^\infty (u_x t)^2 \ dx.
\]

Since \( u(0,t) = 0 \) and \( u(\infty, t) = 0 \) \Rightarrow \( u_t(0,t) = u_t(\infty, t) = 0 \) \forall t \).

Thus yields that

\[
\frac{dE}{dt} = -\beta \int_0^\infty (u_x t)^2 \ dt < 0.
\]