WAVE EQUATION AND GREEN'S FUNCTIONS

We will consider two different aspects of wave propagation.

(I) Time harmonic solution and scattering theory \(\rightarrow\) Helmholtz equation.

(II) Point source solution due to propagation of signals concentrated in space and time \(\rightarrow\) Analogue of d'Alembert's formula in 2-D and 3-D.

Time-harmonic problems

We begin with the wave equation \(\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u\) where \(u = u(x,t)\) and \(c\) is the speed of the medium (speed of sound or light). We look for time periodic solutions with \(u(x,t) = e^{-i\omega t} u(x)\). This yields Helmholtz's equation

\[ \Delta u + k^2 u = 0 \quad \text{with} \quad k = \frac{\omega}{c} \quad \text{(wavenumber, unit of } 1/\text{length}). \]

The key aspect of this PDF is that we will need to find solutions in complex form that correspond to either inward or outward propagating waves.

One-dimensional case

Consider

\[ \frac{d^2 u}{dx^2} - \frac{1}{c^2} \frac{d^2 u}{dt^2} = \delta(x-x_0) e^{-i\omega t} \]

which corresponds to a localized source at \(x_0\), periodic in time. We put \(u = e^{-i\omega t} u(x)\) and obtain

\[ \frac{d^2 u}{dx^2} + k^2 u = \delta(x-x_0), \quad -\infty < x < \infty. \]

We want to find a solution that corresponds to waves propagating away from the source at \(x = x_0\), i.e., outward propagating waves.

\[ u \sim A e^{i(kx - \omega t)} \quad \text{wave to right} \]

\[ u \sim A e^{i(kx + \omega t)} \quad \text{wave to left} \]

**Fig. 1**

**Remark:** If we simply impose that \(u\) is bounded as \(|x| \to \infty\) for (2), there is no unique solution since \(u = \text{span} \{ \cos(kx), \sin(kx) \}\) are both bounded at infinity.

Moreover, this non-uniqueness is also evident if we take Fourier transforms in (2).

Defining \(\hat{F} = \mathcal{F}(u) = \int_{-\infty}^{\infty} u(x)e^{i\sigma x} dx\) we obtain \((k^2 - \sigma^2)u = e^{i\sigma x_0}\) so that

\[ \hat{u}(\sigma) = \frac{e^{i\sigma x_0}}{k^2 - \sigma^2} \quad \text{and} \quad \hat{u}(\sigma) = \int_{-\infty}^{\infty} e^{i\sigma x_0} e^{-i\sigma x} dx. \]

We observe that the poles at \(\sigma = \pm i k\) are on integration path, we can either go above, or below, each of the pole. Four different possibilities \(\rightarrow\) non-uniqueness.

To eliminate this non-uniqueness we impose a physically-based condition that waves are radiating outward from the source as \(t\) increases.
THE KEY HERE IS THAT WAVES ARE "OUTWARD" WITH TIME FACTOR $e^{-iwt}$ (SEE FIG. 1). FOR $x < x_0$, \( U = \text{span} \left\{ e^{i(kx-x_0)}, e^{-i(kx-x_0)} \right\} \). THE OUTWARD CONDITION IS SATISFIED IF WE SET

\[
U(x) = \begin{cases} 
A e^{i(kx-x_0)}, & x > x_0 \\
A e^{-i(kx-x_0)}, & x < x_0 
\end{cases}
\] (3)

NOW \( U \) IS CONTINUOUS ACROSS \( x = x_0 \) AND BY THE JUMP CONDITION \( U^+(x_0-) - U^-(x_0-) = 1 \) WE GET

\[2Aix = 1 \quad \Rightarrow \quad A = \frac{i}{2k} = e^{-i\pi/2} \]

IN THIS WAY WE OBTAIN

\[U(x) = \frac{i}{2k} e^{i(k|x-x_0| - \pi/2)}\]

AND SO

\[U(x,t) = \frac{i}{2k} e^{i(k|x-x_0| - \omega t - \pi/2)}\] WHICH IS OUTWARD PROPAGATION FROM SOURCE \( x_0 \).

REMARK THIS OUTWARD PROPAGATING WAVE CONDITION CAN BE IMPOSED VIA A BOUNDARY CONDITION THAT HOLDS AS \( x \to +\infty \) AND AS \( x \to -\infty \). THESE ARE THE SOMMERFELD RADIATION CONDITIONS: WE IMPose

\[U_x - iku = 0 \quad \text{as} \quad x \to +\infty, \quad U_x + iku = 0 \quad \text{as} \quad x \to -\infty.\]

NOTICE IF WE SET \( U = \frac{i}{2k} e^{i(k(x-x_0) - k(x-x_0))} \) WE CONCLUDE THAT \( B = 0 \) FOR \( x \to +\infty \) AND \( A = 0 \) FOR \( x \to -\infty \). THIS IS REFLECTED IN (3).

IN CONCLUSION, FOR HELMHOLTZ'S EQUATION WE MUST CONSIDER COMPLEX-VALUED SOLUTIONS AND AN OUTWARD PROPAGATING WAVE CONDITION (ENFORCED BY A SOMMERFELD RADIATION CONDITION) ELIMINATES NON-UNIQUENESS OF SOLUTIONS.

TWO-DIMENSIONS WE CONSIDER A LOCALIZED TIME HARMONIC SOURCE IN 2-D WHERE

\[
\Delta U - \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} = e^{-iwt} \delta(x-x_0) \quad \text{IN 2-D.}
\]

WE PUT \( U(x,t) = e^{-iwt} U(x) \) AND CONVERT TO A POLAR COORDINATE SYSTEM CENTERED AT \( x_0 \). LABELLING \( \rho = |x-x_0| \) WE GET

\[\Delta U + \frac{\partial^2 U}{\partial \rho^2} = \delta(\rho) \]

AND IF \( U = \bar{U}(\rho) \) THEN

\[\frac{\partial^2 \bar{U}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \bar{U}}{\partial \rho} + \bar{U} \mathcal{L} \bar{U} = \delta(\rho) \] (4)

FOR \( \rho > 0 \), THE TWO SOLUTIONS ARE \( U = \text{span} \left\{ J_0(\rho \gamma), Y_0(\rho \gamma) \right\} \) WHERE \( J_0(\gamma), Y_0(\gamma) \) ARE THE BESSEL FUNCTIONS OF THE FIRST AND SECOND KIND OF ORDER ZERO.
SATISFYING \( y'' + \frac{1}{2} y' + y = 0 \) FOR \( Z > 0 \).

IT IS WELL-KNOWN THAT

\[
\begin{align*}
\psi_0(Z) &\sim -\frac{\log Z}{Z}, \quad \text{and} \quad \psi_0(Z) \sim 1 - Z^2/4, \quad \text{as} \quad Z \to 0^+ \\
\psi_0(Z) &\sim \left( \frac{2}{\pi Z} \right)^{1/2} \cos \left( Z^{1/4} \right), \quad \text{and} \quad \psi_0(Z) \sim \left( \frac{2}{\pi Z} \right)^{1/2} \sin \left( Z^{1/4} \right), \quad \text{as} \quad Z \to \infty.
\end{align*}
\]

NOW IN (4), WE NEED A SINGULAR SOLUTION AS \( \Gamma \to 0 \) WHICH HAS \( U \sim \frac{1}{2} \log \Gamma \) AS \( \Gamma \to 0 \).

THEORETICALLY IT IS NATURAL TO TRY \( U \sim \frac{1}{4} \psi_0(\Gamma) \) AS THE FREE-SPACE

\( G \)-FUNCTION.

HOWEVER, IN WAVE PROPAGATION PROBLEMS, WE MUST IMPOSE PHYSICALLY AN OUTGOING CONDITION AT INFINITY, AND SUCH \( \frac{1}{4} \psi_0(\Gamma) \) IS NOT THE CORRECT CHOICE FOR (4) WHEN SUCH A CONDITION IS NEEDED.

THIS MOTIVATES THE DEFINITION OF THE HANKEL FUNCTIONS \( H_0^{(1)}(Z) \) AND \( H_0^{(2)}(Z) \) OF THE FIRST AND SECOND KIND OF ORDER ZERO. THEY ARE DEFINED BY

\[
H_0^{(1)}(Z) = J_0(Z) + i Y_0(Z), \quad H_0^{(2)}(Z) = J_0(Z) - i Y_0(Z).
\]

THEY ARE TWO LINEARLY INDEPENDENT SOLUTIONS TO \( y'' + \frac{1}{Z} y' + y = 0 \).

NOW FROM (5), WE IDENTIFY THAT

\[
H_0^{(1)}(Z) \sim \frac{2i}{\pi} \log Z, \quad \text{as} \quad Z \to 0^+
\]

AND

\[
H_0^{(2)}(Z) \sim \left( \frac{2}{\pi Z} \right)^{1/2} e^{i(Z^{1/4})}, \quad \text{as} \quad Z \to \infty, \quad H_0^{(2)}(Z) \sim \left( \frac{2}{\pi Z} \right)^{1/2} e^{-i(Z^{1/4})}, \quad \text{as} \quad Z \to \infty.
\]

AS SUCH, THE SOLUTION OF (4) THAT SATISFIES AN OUTGOING CONDITION AS \( \Gamma \to \infty \), AND HAS A \( \frac{1}{2} \log \Gamma \) SINGULARITY AT \( \Gamma = 0 \) IS

\[
U = -\frac{i}{4} H_0^{(1)}(\Gamma \Gamma \Gamma \Gamma) \quad \text{which yields} \quad \tilde{U}(x, t) = \frac{i}{4} e^{-i \omega t} H_0(z(t - x - t)) \quad (7)
\]

NOW AS \( \Gamma \to \infty \) WE HAVE

\[
U \sim \frac{i}{4} \left( \frac{2}{\pi \Gamma \Gamma \Gamma \Gamma} \right)^{1/2} e^{i(\Gamma \Gamma \Gamma \Gamma - \omega t - \Gamma \Gamma \Gamma \Gamma)} \quad \text{as} \quad \Gamma \to \infty, \quad \text{where} \quad \Gamma = |x - x_0|.
\]

THIS IS AN OUTGOING WAVE AS \( \Gamma \to \infty \).

THE AMPLITUDE DECAYS LIKE \( C / \Gamma^{1/2} \) AS \( \Gamma \to \infty \) DUE TO A GEOMETRICAL SPREADING OF WAVES.
To obtain an outgoing wave we can append to (4) an outgoing wave condition
\[ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u = \frac{\delta(r)}{2\pi r} \quad \text{on} \quad r > 0 \]
with \[ \frac{\partial}{\partial r}(u_r - iu) \rightarrow 0 \quad \text{as} \quad r \rightarrow 0^+ \] (Sommerfeld radiation condition).
The function \( u = \frac{i}{4} \mathbb{H} \Omega(r) \) which has \( u \sim \frac{i}{4} \left( \frac{2}{\pi \text{im} r} \right)^{1/2} e^{i(mr - \text{im} t)} \) as \( r \rightarrow \infty \) satisfies this radiation condition.

Three dimensions in 3-D we have
\[ \Delta u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = e^{-i\omega t} \delta(x - x_0). \]
We want an outward propagating wave. We put \( u = e^{-i\omega t} \Omega(r) \) with \( r = |x - x_0| \) to get
\[ u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} u = \frac{\delta(r)}{4\pi r^2} \quad \text{in} \quad r > 0. \tag{8} \]
Now for \( r > 0 \) there are two exact solutions \( u = \text{Span} \left\{ \frac{e^{i\text{im} r}}{r}, \frac{e^{-i\text{im} r}}{r} \right\} \)
and we need \( u \sim -\frac{1}{4\pi r} \) as \( r \rightarrow 0^+ \). The outward propagating solution is
\[ u = -\frac{1}{4\pi r} e^{i\text{im} r}. \tag{9} \]

To ensure an outward propagating wave we can impose the Sommerfeld radiation condition
\[ \lim_{r \rightarrow 0^+} (u_r - iu) = 0. \tag{10} \]
The function (9) satisfies this. Then we have
\[ u = \frac{1}{4\pi |x - x_0|} e^{i(k|x - x_0| - \omega t)} \quad \text{as an outward propagating wave}. \tag{11} \]

Now with these free-space Green's function we can solve various scattering problems in 2-D and in 3-D.

Example 1 Suppose that in 2-D, \( u(x, t) \) satisfies
\[ \Delta u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = e^{-i\omega t} \delta(x - x_0) \quad \text{in} \quad y > 0 \quad \text{with} \quad x = (x, y) \]
\[ u = 0 \quad \text{on} \quad y = 0, -\omega < x < \omega \quad \text{and} \quad x_0 = (x_0, y_0) \quad \text{with} \quad y_0 > 0. \]

We want to find the solution using method of images.
We then put \( u(x, t) = e^{-i\omega t} u(x) \) so that
\[
\Delta u + u^2 u = \delta(x - x_0) \text{ in } y > 0
\]
\( u = 0 \text{ on } y = 0, -\omega \leq x \leq 0 \)

Now the free space G-function for outward propagating waves is
\[
\psi_0 = \frac{-i}{4} H_0^{(1)}(2\pi|x - x_0|).
\]

To find the solution we put an image charge at \( x_t = (x_0, -y_0) \). This yields that
\[
\psi = \frac{-i}{4} H_0^{(1)}(2\pi|x - x_0|) + \frac{i}{4} H_0^{(1)}(2\pi|x - x_t|)
\]

\( \rightarrow \text{incident wave} \quad \rightarrow \text{reflected wave} \)

We can interpret \( \psi \) as a wave propagating outward from the source point \( x = x_0 \). It then generates a reflected wave that bounces off the barrier \( y = 0 \).

We now discuss briefly the idea of wave scattering in 2-D. We consider wave scattering by an infinite cylinder either due to a plane wave or an oscillating point source.

**Plane-wave scattering**

We consider a plane wave solution
\[
\psi_{\text{inc}} = e^{-i\mathbf{k} \cdot \mathbf{r}} \quad \text{with } \mathbf{k} = \omega / c
\]

that propagates to the right as \( t \uparrow \) and is incident on an infinite cylinder \( x^2 + y^2 < a^2 \). We have that \( \psi(x, t) \) satisfies
\[
(12) \begin{cases}
\Delta \psi = c^2 \psi \quad \text{outside } \Omega \\
\psi = 0 \quad \text{on } \partial \Omega
\end{cases}
\]

where \( \Omega = \{ (x^2 + y^2) < a^2 \} \). Now \( \psi_{\text{inc}} = e^{i(kx - \omega t)} \) is constant on \( x^2 + y^2 = \text{constant} \). We decompose the total field \( \psi \) as the incident field \( \psi_{\text{Inc}} \) and the scattered field \( \psi_{\text{sca}} \) by
\[
\psi = \psi_{\text{inc}} + \psi_{\text{sca}}
\]

Then putting this into the PDE and writing \( \psi_{\text{sca}}(x, t) = u_s(x) e^{-i\omega t} \)

we obtain that \( u_s(x) \) satisfies
\[
(14) \begin{cases}
\Delta u_s + u_s^2 u_s = 0 \quad \text{outside } \Omega \\
u_s = -e^{i\mathbf{k} \cdot \mathbf{r}} \quad \text{on } \partial \Omega
\end{cases}
\]
with $U_5$ satisfying the outward propagating wave condition

$$\lim_{r \to \infty} \left( \frac{d}{dr} U_5 + i \omega U_5 \right) = 0.$$ 

Now although for the cylinder (14) can be solved using an eigenfunction expansion method, we now formulate a solution using a Green's function.

We let $V(x', x)$ satisfy

$$\begin{cases} 
L V = \Delta' V + U^2 V = \delta(x' - x) \text{ outside } \varnothing \setminus \bar{\varnothing}, \\
V = 0 \text{ on } d\varnothing, \\
V \text{ outgoing as } |x'1| \to \infty, \text{ i.e. } \lim_{r' \to \infty} (r')^{1/2}(\frac{d}{dr'} V - i \omega V) = 0. 
\end{cases}$$

We then integrate over shaded region using Green's identity, and let $R \to \infty$.

$$\int_{R^2 \setminus \varnothing} (V L U_5 - U_5 LV) \, dx' \to \lim_{R \to \infty} \int_{\varnothing} \left[ V \left( \frac{d_0 U_5 - i \omega U_5}{d_0 V - i \omega V} \right) + \int \frac{(V U_5 - U_5 V)}{d_0 d_0} \, ds \right].$$

Now since $ds = R \, d\varrho$ and $V = O(R^{-1/2})$ while $d_0 (U_5 - i \omega U_5) \to 0$ as $R \to \infty$ the integral over big circle vanishes and we use $L V = \delta(x' - x)$ to get

$$- U_5(x) = \int_{\varnothing} U_5 \frac{d_0 V}{d_0} \, ds \quad \Rightarrow \quad U_5(x) = \int_{\varnothing} U_5 \frac{d_0 V}{d_0} \, ds.$$

Now for the cylinder, $d_0 V = - \frac{d}{dr} V$ and $U_5 = - e^{i \omega x}$ on $d\varnothing$.

Thus,

$$U_5(x) = \left| e^{i \omega x} \frac{d}{dr} V \right|_{r' = a} \, ds = a^{1/2} e^{i \omega a \cos \vartheta} \frac{d}{dr} V \Big|_{r' = a} \, d\vartheta.$$

This determines $U_5(x)$ in terms of solution to (15) and this procedure can be worked for arbitrary shape $d\varnothing$.

Scattering due to a point source

We now consider scattering of waves by an infinite cylinder due to an oscillating point source. Then,

$$\Delta U - \frac{1}{c^2} \frac{1}{t^2} U = e^{-i \mu t} \delta(x - x_0) \text{ in } a < |x| < \infty,$$

$U = 0$ on $|x| = a$,

$U$ outgoing as $|x| \to \infty$; $U, \frac{d}{dx} U \text{ periodic}.$
We now write \( U(x,t) = e^{-i\omega t} U(x) \) so that

\[
\begin{align*}
\Delta U + \mu^2 U &= \delta(x - x_0) \quad \text{in} \quad \alpha \leq |x| \leq \omega, \quad 0 \leq \omega < 2\pi \\
U &= 0 \quad \text{on} \quad r = 1 \quad |x| = 1 \\
U \text{ outgoing} A_1 &\to \infty \text{ with} e^{-i\omega t} \\
U, U_t \text{ are} 2\pi \text{ periodic.}
\end{align*}
\]

We put the source point at \( x_0 = p (\cos \theta, \sin \theta) \) with \( p > r \).

The incoming wave is the free-space Green function

\[
U_{\text{inc}} = -\frac{i}{4} H_0^{(1)}(\mu r) \quad (r, \theta)
\]

with \( R^2 = |x - x_0|^2 = r^2 + p^2 - 2rp \cos (\phi - \phi_0) \).

Next we will represent \( U_{\text{inc}} \) in terms of a Fourier series:

\[
U_{\text{inc}} = \sum_{n=0}^{\infty} C_n(\Gamma) e^{in\phi} \quad \text{with} \quad C_n(\Gamma) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-in\phi} U(\Gamma, \phi) d\phi.
\]

We now use the PDE \( \partial r U_{\text{inc}} + \frac{1}{r} \partial r U_{\text{inc}} + \frac{1}{\Gamma^2} U_{\text{inc}} + \mu^2 U \) to obtain

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \left( \partial r U_{\text{inc}} + \frac{1}{r} \partial r U_{\text{inc}} + \frac{1}{\Gamma^2} U_{\text{inc}} + \mu^2 U \right) e^{-in\phi} d\phi = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\delta(\Gamma - p) \delta(\phi - \phi_0)}{r} e^{-in\phi} d\phi = \frac{\delta(\Gamma - p)}{2\pi r} e^{-in\phi_0}
\]

Now the left side gives

\[
C_n^{\mu} + \frac{1}{\Gamma} C_n^{\mu} + \left( \frac{\mu^2 - n^2}{\Gamma^2} \right) C_n = \frac{\delta(\Gamma - p) e^{-in\phi_0}}{2\pi r} \quad \text{on} \quad 0 \leq r < p
\]

with \( C_n \) bounded as \( r \to 0 \) and \( C_n \) outgoing as \( r \to \infty \).

The solution is

\[
C_n(\Gamma) = \begin{cases} AJ_n(\mu r), & 0 \leq r < p \\ BH_n^{(1)}(\mu r), & p < r < \infty. \end{cases}
\]

Imposing continuity gives

\[
C_n(\Gamma) = \begin{cases} D J_0(\mu r) H_n^{(1)}(\mu r), & 0 \leq r < p \\ D H_0^{(1)}(\mu r) J_n(\mu r), & p < r < \infty. \end{cases}
\]

where \( D \) is found from the jump condition

\[
C_n^{\mu}(r) - C_n^{\mu}(p) = \frac{e^{-i\phi_0}}{2\pi r}.
\]

We calculate:

\[
K D \left[ H_n^{(1)}(\mu r) J_n(\mu r) - J_n^{\prime}(\mu r) H_n^{(1)}(\mu r) \right] = \frac{e^{-i\phi_0}}{2\pi r}.
\]
Using the Wronskian Relation:
\[ H_0^{(1)}(iz)J_n(iz) - J_0^{(1)}(iz)H_n^{(1)}(iz) = \frac{2i}{\pi z} \]

We obtain:
\[ W D \left( \frac{2i}{\pi z} \right) = \frac{e^{-i\eta \Psi}}{2\pi \rho} \rightarrow D = -\frac{i}{4} e^{-i\eta \Psi} \]

Therefore,
\[ C_n(r) = -\frac{i}{4} J_n(i\Gamma)H_n^{(1)}(i\Gamma) e^{-i\eta \Psi} \]

with \( \Gamma = \min(\Gamma, \rho), \Gamma = \max(\Gamma, \rho) \).

Thus, we conclude that
\[ (17) \quad U_{inc} = \frac{i}{4} H_0^{(1)}(i\Gamma) = -\frac{i}{4} \sum_{m=-\infty}^{\infty} J_n(i\Gamma)e^{in\Phi} \]

Next, we write \( U = U_{inc} + U_s \) where \( U_s \) is the scattered wave.

We obtain upon substituting into (16) that \( U_s \) satisfies
\[ (18) \begin{cases} \Delta U_s + \eta^2 U_s = 0 \quad \text{in} \quad \Gamma = \alpha \\ U_s = -U_{inc} = -\frac{i}{4} \sum_{m=-\infty}^{\infty} J_n(i\Delta)H_n^{(1)}(i\Delta) e^{in\Phi} \quad \text{on} \quad \Gamma = \alpha \\ U_s \text{ outgoing as } \Gamma \to \infty \end{cases} \]

Here on \( \Gamma = \alpha \), we used \( \Gamma = \alpha \) and \( \Gamma = \rho \).

We will now solve for \( U_s \) by separation of variables. We put
\[ U_s = \sum_{m=-\infty}^{\infty} d_n(i\Delta) e^{in\Phi} \]

where we obtain that \( d_n(i\Delta) \) satisfies
\[ d_n'' + \frac{\Gamma}{\Gamma} d_n' + (\eta^2 - \frac{\rho^2}{\Gamma^2}) d_n = 0 \quad \text{in} \quad \Gamma = \alpha \]

with \( d_n(\alpha) = \frac{i}{4} J_n(i\Delta)H_n^{(1)}(i\Delta), \quad d_n \text{ outgoing as } \Gamma \to \infty. \)

We calculate
\[ d_n(i\Delta) = \frac{i}{4} J_n(i\Delta)H_n^{(1)}(i\Delta) \frac{H_\Delta}{H_0(i\Delta)} \]

We conclude that
\[ U_s = \frac{i}{4} \sum_{m=-\infty}^{\infty} J_n(i\Delta)H_n^{(1)}(i\Delta) \frac{H_n(i\Delta)}{H_0(i\Delta)} e^{in\Phi} \]

and then \( U = U_{inc} + U_s \) where \( U_{inc} = \frac{i}{4} H_0^{(1)}(i\Delta) \) (incident wave).
APPENDIX A (OTHER IMPORTANT SPECIAL FUNCTIONS)

Suppose that in 2-D we look for general solutions to Helmholtz's equation

$$\Delta u + \eta^2 u = 0 \rightarrow u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + \eta^2 u = 0.$$  

Now put either $u = \phi(r) \cos(\eta \theta)$ or $u = \phi(r) e^{i\eta \theta}$, in either case we end up with

$$r^2 \phi'' + r \phi' + \left[ \eta^2 r^2 - \eta^2 \right] \phi = 0 \quad \text{for } \eta > 0. \quad (*)$$

Now if we let $z = kr$ we obtain for $y(z) = \phi \left[ z/r \right]$ that

$$z^2 y'' + z y' + \left[ z^2 - \eta^2 \right] y = 0.$$  

This is the Bessel equation of order $\eta$. We will consider only the case $\eta = 0, 2, \ldots$ so that $u$ is $2\pi$ periodic in $\theta$. (previously we considered $\eta = 0$ for which $y = \sin \theta$, $j_0(z)$, $y_0(z)$).

For $\eta = 1, 2, \ldots$ the two linearly independent solutions are $j_\eta(z)$, $y_\eta(z)$ so that

$$y = C_1 j_\eta(z) + C_2 y_\eta(z).$$

One solution ($y_0(z)$) is unbounded as $z \to 0$, while the other is bounded as $z \to 0$. We have

$$j_\eta(z) \sim \frac{z^n}{2^n n!} \quad \text{as } z \to 0, \quad y_\eta(z) \sim -\frac{1}{n!} (\eta - 1)! \left( \frac{2}{z} \right)^n \quad \text{as } z \to 0.$$  

The plots for a given fixed $\eta$ are

\begin{align*}
\text{Graph of } j_\eta(z) & \quad \text{Graph of } y_\eta(z)
\end{align*}

Moreover, one can show that as $z \to +\infty$ we have

$$j_\eta(z) \sim \sqrt{\frac{2}{\pi z}} \cos \left( z - \frac{\eta \pi}{2} - \frac{\pi}{4} \right), \quad y_\eta(z) \sim \sqrt{\frac{2}{\pi z}} \sin \left( z - \frac{\eta \pi}{2} - \frac{\pi}{4} \right). \quad (1)$$

$j_\eta$, $y_\eta$ are called the Bessel function of the first and second kind of order $\eta$. Therefore in $(*)$ we can write $\phi = C_1 j_\eta(kr) + C_2 y_\eta(kr).$
Now in wave propagation problems, it is convenient to define the Hankel functions of the first and second kind of order \( \eta \) via
\[
\begin{align*}
H_n^{(1)}(z) &= J_\eta(z) + iY_\eta(z), \quad \text{(first kind)} \\
H_n^{(2)}(z) &= J_\eta(z) - iY_\eta(z), \quad \text{(second kind)}.
\end{align*}
\]

The key point is that using the behavior of \( J_\eta(z) \) and \( Y_\eta(z) \) in (1) \( z \to \pm \infty \):
\[
\begin{align*}
H_n^{(1)}(z) &\sim \sqrt{\frac{2}{\pi z}} e^{i(z - \eta \pi/2)} \quad \text{as } z \to +\infty \\
H_n^{(2)}(z) &\sim \sqrt{\frac{2}{\pi z}} e^{-i(z - \eta \pi/2)} \quad \text{as } z \to +\infty.
\end{align*}
\]

Thus for (4) we can write
\[
\Phi = c\, H_n^{(1)}(\kappa r) + c_2\, H_n^{(2)}(\kappa r)
\]
and observe that with a time factor \( e^{-i\omega t} \), we identify that \( H_n^{(1)}(\kappa r) \) is an outward propagating wave, i.e.
\[
U = c\, H_n^{(1)}(\kappa r) e^{i\eta \gamma} \sim c\, \sqrt{\frac{2}{\pi \kappa r}} e^{i(\kappa r - \eta \pi/2 - \eta \gamma)} \quad \text{as } r \to \infty
\]
so that
\[
\tilde{U} = e^{-i\omega t} c\, H_n^{(1)}(\kappa r) e^{i\eta \gamma} \sim c\, \sqrt{\frac{2}{\pi \kappa r}} e^{i(\kappa r - \omega t + \eta \gamma - \eta \pi/2 - \eta \gamma)} \quad \text{as } r \to \infty
\]

Next consider the reduced wave equation defined by the solution to
\[
\Delta U - \eta^2 U = 0 \quad \Rightarrow \quad \Delta \eta + \frac{1}{\eta} \frac{d}{d\eta} + \frac{1}{\eta^2} \eta \frac{d^2}{d\eta^2} - \eta^2 U = 0.
\]

Now put either \( U = \Phi(\eta) \cos(\eta \gamma) \) or \( U = \Phi(\eta) e^{i\eta \gamma} \) to end up with
\[
r^2 \Phi'' + r \Phi' - \left[ \eta^2 + \eta^2 + \eta^2 \right] \Phi = 0 \quad \text{for } \eta > 0.
\]

If we let \( \eta = \kappa r \) and \( y(z) = \overline{\Phi(z/\kappa)} \) satisfies
\[
z^2 y'' + zy' - \left[ z^2 + \eta^2 \right] y = 0.
\]

The two linearly independent solutions are \( J_\eta(z) \) (bounded as \( z \to 0 \)) and \( Y_\eta(z) \) (unbounded as \( z \to 0 \)). They are the modified Bessel functions of first and second kind of order \( \eta \). We will consider \( \eta = 0, 1, 2, \ldots \).
The asymptotic are

\[ I_0(z) \sim 1 + z^{3/4} + \cdots \quad \text{as} \quad z \to 0, \quad I_n(z) \sim -\log z \quad \text{as} \quad z \to 0. \]

\[ I_n(z) \sim \frac{z^n}{2^n n!} \quad \text{as} \quad z \to 0, \quad K_n(z) \sim \frac{1}{2} (\pi-1)^{1/2} \left( \frac{z}{\pi} \right)^{n+1/2} \quad \text{as} \quad z \to 0. \]

For a fixed \( n \), the plots are

- \( I_0(z) \) and \( I_n(z) \) are unbounded while \( K_0(z) \) is bounded as \( z \to \infty \) and decay to zero. We have as \( z \to +\infty \) that

\[ I_n(z) \sim \frac{z^n}{(2^n n!)^{1/2}} + \cdots \quad K_n(z) \sim \left( \frac{\pi}{2z} \right)^{1/2} e^{-z} + \cdots. \]

Therefore

\[ \Phi(\gamma) = c_1 I_0(\gamma R) + c_2 K_0(\gamma R). \]

If we need \( \Phi \) bounded as \( \gamma \to 0 \) we set \( c_2 = 0 \).

If we need \( \Phi \) bounded as \( \gamma \to \infty \) we set \( c_1 = 0 \).

These special functions \( K_n, I_n \) do not arise in wave propagation problems. They typically arise when there is a decay mechanism.