Point Source Solution of Wave Equation

We will calculate the Green's function for three problems:

(I-D) (i) \[ \frac{\partial^2 G}{\partial x^2} - \frac{1}{c^2} \frac{\partial G}{\partial t} = \delta(x-x_0) \delta(t) \quad -\infty < x < \infty, \quad t > 0 \]

\[ G = G_0 = 0 \quad \text{at} \quad t = 0 \]

(2-D) (ii) \[ \frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} - \frac{1}{c^2} \frac{\partial G}{\partial t} = \frac{1}{2\pi} \delta(r) \delta(t) \quad 0 < r < \infty, \quad t > 0 \]

\[ G = G_0 = 0 \quad \text{at} \quad t = 0 \]

(3-D) (iii) \[ \frac{\partial^2 G}{\partial r^2} + \frac{2}{r} \frac{\partial G}{\partial r} - \frac{1}{c^2} \frac{\partial G}{\partial t} = \frac{1}{4\pi r^2} \delta(r) \delta(t) \quad 0 < r < \infty, \quad t > 0 \]

\[ G = G_0 = 0 \quad \text{at} \quad t = 0 \]

They represent the 1-D, 2-D and 3-D cases respectively. The form of the Green's function will depend on the dimension.

In (i), (ii) and (iii) we interpret \( \delta(t) \) as \( \delta(t-\varepsilon) \) for \( \varepsilon \to 0^+ \). Thus the source is turned on at \( t = 0^+ \) in an otherwise quiet medium.

To solve (i) take Laplace transform \( \hat{G}(x,s) = \mathcal{L}\{G(x,t)\} \), \( \hat{\delta(t)} = 1 \)

\[ \hat{G}_{xx} - \frac{s^2}{c^2} \hat{G} = \delta(x-x_0) \quad -\infty < x < \infty \]

Imposing decay \( \hat{G}(x,s) \rightarrow 0 \) as \( x \to \infty \) gives

\[ \hat{G}(x,s) = \begin{cases} A e^{-s(x-x_0)} & x > x_0 \quad \text{(Recall} \ s > 0) \\ A e^{s(x-x_0)} & x < x_0 \end{cases} \]

We have imposed continuity at \( x = x_0 \). The jump condition is

\[ \hat{G}_x(x_0) - \hat{G}_x(x_0^{-}) = 1 \quad \Rightarrow \quad -2As/c = 1 \quad \Rightarrow \quad A = -c/(2s) \]

Thus

\[ \hat{G}(x,s) = -c/(2s) e^{-s/c \ |x-x_0|} \]

Now recall \( \mathcal{L}^{-1}[e^{-s/2}] = H(t-\varepsilon) \), \( H \) being Heaviside function.

Thus

\[ G(x,t) = -c/(2e) H\left[t - \frac{1}{c} \ |x-x_0|\right] \]

\[ t = -\frac{1}{c} (x-x_0) \]

\[ G = 0 \quad \text{at} \quad x = 0 \]

\[ x = 0 \]

\[ G = 0 \]

\[ t = \frac{1}{c} (x-x_0) \]

X

\[ x \]
To solve (iii) take Laplace transforms.

\[
\hat{G}_{rr} + \frac{1}{r} \hat{G}_r - \frac{s^2}{c^2} \hat{G} = \delta(r)
\]

Thus, \( \hat{G}(r, s) = \mathcal{A}K_0(\frac{rs}{c}) \), where \( K_0(z) \) is the modified Bessel function satisfying \( z^2K_0'' + zK_0' - z^2K_0 = 0 \).

To find \( \mathcal{A} \), use divergence theorem.

\[
\int_0^{2\pi} \int_0^\infty \Delta \hat{G} \ r \ dr \ dq = \int_0^{2\pi} \int_0^\infty \frac{\partial \hat{G}}{\partial r} \ r \ dr \ dq = 1 + \frac{s^2}{c^2} \int_0^{2\pi} \int_0^\infty \hat{G} \ r \ dr \ dq
\]

\[
\hat{G} \sim -A \log r \quad \text{as} \quad r \to 0
\]

\[
\frac{\partial \hat{G}}{\partial r} \sim -A/r \quad \text{as} \quad r \to 0
\]

Now from a handbook,

\[
d(\frac{1}{\sqrt{t-r^2}}) = K_0(\frac{rs}{c})
\]

If we draw a picture, then

1. Notice that once the signal along \( t = r/c \) has passed, the solution decays like \( \sim (c^2t^2 - r^2)^{-1/2} \) for later times.

   \( \Rightarrow \) signal is not crisp and there is a "tail" that remains behind \( \Rightarrow \) poor for sound communication.

Now solve (iii). Take Laplace transforms.

\[
\hat{G}_{rr} + \frac{2}{r} \hat{G}_r - \frac{s^2}{c^2} \hat{G} = \frac{\delta(r)}{4\pi r^2}
\]

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \hat{G}_r \right) - \frac{s^2}{c^2} \hat{G} = \frac{\delta(r)}{4\pi r^2}
\]

To solve the homogeneous problem let \( \hat{G} = F(r)/r \). Then

\[
\frac{1}{r^2} \left( r^2 \hat{G}_r \right)_r - \frac{s^2}{c^2} \hat{G} = 0 \quad \text{reduces to} \quad \frac{F''(r)}{r} - \frac{s^2}{c^2} F(r) = 0.
\]

Imposing decay as \( r \to \infty \) gives

\[
F(r) = Ae^{-s/r/c}
\]

Thus, \( \hat{G}(r, s) = A/r e^{-s/r/c} \).
THE CONSTANT $A$ CAN BE FOUND USING THE DIVERGENCE THEOREM TO GET

$$A = -\frac{1}{4\pi}$$

THUS

$$G(r, t) = -\frac{1}{4\pi r} e^{-\frac{r}{c}}$$

INVERTING

$$G(r, t) = -\frac{1}{4\pi r} \delta(t - \frac{r}{c})$$

NOTICE THAT $G = 0$ BEHIND THE LINE $t = \frac{r}{c}$ \implies \text{SIGNAL IS CRISP!}

HELMHOLTZ EQUATION

START WITH THE WAVE EQUATION

$$\nabla^2 u + \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0$$

ASSUME THAT THERE IS A TIME PERIODIC DISTURBANCE OF THE FORM

$$u = e^{i\omega t} u(x)$$

THEN SUBSTITUTING GIVES HELMHOLTZ EQUATION

$$\nabla^2 u + k^2 u = 0 \quad k = \omega/c \quad \text{WAVENUMBER}.$$ 

NOW (\ref{eq:helmholtz}) GOVERNS WAVE PROPAGATION WHEREAS $\nabla^2 u - k^2 u = 0$ GOVERNS EXPONENTIAL DECAY (NO PROPAGATION).

NOW CALCULATE THE FREE-SPACE GREEN'S FUNCTION FOR

$$\nabla^2 u + k^2 u = \delta(x-x', y-y') \quad \text{(2-DIMENSIONAL)}$$

$$\nabla^2 u + \frac{1}{r} \nabla u + k^2 u = \frac{\delta(r)}{2\pi r}$$

THE SOLUTION TO THE HOMOGENEOUS PROBLEM IS

$$u = A J_0(kr) + B Y_0(kr)$$

$J_0(z)$, $Y_0(z)$ SATISFY

$$z^2 y'' + z y' + z^2 y = 0$$

$J_0(z)$ REGULAR AS $z \to 0$

$Y_0(z) \sim z^2/\pi \log z$ AS $z \to 0$. 

$$G = 0 \quad t = \frac{r}{c}$$
The constant $A$ can be found using the divergence theorem to get

$$A = -\frac{1}{4\pi}$$

Thus

$$G(r, \rho) = -\frac{1}{4\pi} e^{-\rho/c}$$

Inverting

$$G(r, t) = -\frac{1}{4\pi} \delta(t-\rho/c)$$

NOTE: $G = 0$ behind signal

$\Rightarrow$ signal is crisp.

**Helmholtz Equation**

Start with the wave equation

$$(1) \quad \nabla^2 u + \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0 \quad u = u(x,t)$$

Assume there is a periodic disturbance of the form

$$u = e^{i\omega t} \phi(x)$$

Substituting gives Helmholtz's equation

$$(2) \quad \nabla^2 u + k^2 u = 0 \quad k = \frac{\omega}{c} \text{ wave number}$$

(2) governs wave propagation whereas $\nabla^2 u - k^2 u = 0$ has exponential decaying and growing solutions. We look at this problem in (1-D) and (2-D).

**One-Dimension**

Suppose in (1-D) we had a localized source, periodic in time

$$\nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \delta(x-x_0) e^{-i\omega t}$$

Then if $u = e^{i\omega t} \phi(x)$ we get

$$(3) \quad \phi'' + k^2 \phi = \delta(x-x_0)$$

The problem is ill-posed in that there is no unique solution. This can be seen in two ways:

(i) Solution of $\phi'' + k^2 \phi = 0 \rightarrow$ sing and cosine, both of which are bounded at infinity.

(ii) Taking the Fourier Transform we get poles on the axis of integration.
To eliminate the non-uniqueness problem we impose a physically based condition that the waves are radiating outward from the source as $t$ increases.

With a time factor $e^{-i\omega t}$, this means that

$$W(x) = e^{i(k|x-x_0|-\omega t)} \quad \text{for } |x| \to \infty$$

SOMMERFELD RADIATION CONDITION

and not

$$W(x) = e^{-i(k|x-x_0|+\omega t)} \quad \text{for } |x| \to \infty.$$ 

Now solve (3) subject to (4). It is clear that we want to introduce complex-valued solutions.

$$W(x) = \begin{cases} 
A e^{ik(x-x_0)} + B e^{-ik(x-x_0)} & x > x_0 \\
C e^{ik(x-x_0)} + D e^{-ik(x-x_0)} & x < x_0 
\end{cases}$$

For outgoingness $A$, $x \to +\infty$ we need $B = 0$.

For outgoingness $A$, $x \to -\infty$ we need $C = 0$.

Continuity of $W(x)$ at $x = x_0 \to A = D$.

$$W(x) = \begin{cases} 
A e^{ik(x-x_0)} & x > x_0 \\
A e^{-ik(x-x_0)} & x < x_0 
\end{cases}$$

Now

$$W'_+(x_0) - W'_-(x_0) = 1 \quad \rightarrow \quad 2A i\kappa = 1 \quad A = -\frac{i}{2\kappa}$$

$$W(x) = \frac{i}{2\kappa} e^{ik|x-x_0|}$$

or

$$W(x) = e^{i(k|x-x_0| - \frac{\pi}{2})}$$

Now how can we impose the radiation condition via a boundary condition as $|x| \to \infty$ rather than by inspection.

From (5)

$$W_x = i\kappa W \quad \text{for } x > x_0$$

$$W_x = -i\kappa W \quad \text{for } x < x_0$$

Thus take

$$W_x - i\kappa W = 0 \quad \text{as } x \to +\infty$$

$$W_x + i\kappa W = 0 \quad \text{as } x \to -\infty$$

These radiation boundary conditions guarantee outgoing waves and the solution to (3) with (6) is then unique.
TWO-DIMENSIONS

\[ \Delta U - \frac{\partial^2 U}{\partial z^2} = e^{i\omega t} \delta(x-\xi, y-\eta) \]

Now let \( U(x, t) = e^{i\omega t} \tilde{U}(x) \) and convert to a polar coordinate system centered at \( x = \xi, y = \eta \).

\[ \rho = \sqrt{(x-\xi)^2 + (y-\eta)^2} \]

\[ \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial U}{\partial \rho} + \lambda^2 U = \frac{\delta(\rho)}{2\pi \rho} \]

There are two solutions to this problem (inhomogeneous):

\[ U = A J_0(\rho r) + B Y_0(\rho r) \]

Remark:

\[ J_0(\rho r) \sim \left( \frac{2}{\rho \pi r} \right)^{1/2} \cos \left( \frac{\rho r - \pi}{2} \right) \quad \text{as} \quad \rho \to 0 \]

\[ Y_0(\rho r) \sim \left( \frac{2}{\rho \pi r} \right)^{1/2} \sin \left( \frac{\rho r - \pi}{2} \right) \quad \text{as} \quad \rho \to 0 \]

\[ Y_0(z) \sim \frac{2}{\pi z} \log z \quad \text{as} \quad z \to 0 \]

\[ J_0(z) \sim 1 \quad \text{as} \quad z \to 0. \]

If we impose the singularity condition we get from the divergence theorem that

\[ U = A J_0(\rho r) + B Y_0(\rho r) \]

A is arbitrary (non-uniqueness).

The function \( U \) is bounded for all \( \rho \).

To eliminate the non-uniqueness, we impose an outgoing radiation condition with respect to time factor \( e^{-i\omega t} \). It is convenient to introduce complex valued solutions as follows:

Define

\[ H_0^{(1)}(z) = J_0(z) + i Y_0(z) \]

\[ H_0^{(2)}(z) = J_0(z) - i Y_0(z) \]

Then

\[ H_0^{(1)}(z) \sim \left( \frac{2}{\pi z} \right)^{1/2} e^{i(z-\pi/4)} \quad \text{as} \quad z \to +0 \]

\[ H_0^{(2)}(z) \sim \left( \frac{2}{\pi z} \right)^{1/2} e^{-i(z-\pi/4)} \quad \text{as} \quad z \to 0 \]

\[ H_0^{(3)}(z) \sim \frac{2i}{\pi} \log(z) \quad \text{as} \quad z \to 0 \]
Thus to ensure that \( u \) is outgoing we have

\[
u = A H_0^{(1)}(kr) \quad \text{with} \quad u \to 2i A \log r \quad \text{as} \quad r \to 0
\]

We want \( u \to \frac{1}{2\pi} \log r \quad \text{as} \quad r \to 0 \) thus \( A = -i/4 \).

Thus \( u = -\frac{i}{4} H_0^{(1)}(kr) \) is outgoing free space Green's function.

Since \( u \approx -\frac{i}{4} \left( \frac{2}{\pi kr} \right)^{1/2} e^{i(kr-\pi/4)} \quad \text{as} \quad r \to \infty \),

we can impose an outgoing condition by demanding that \( \partial_r u - ik u \to 0 \quad \text{as} \quad r \to \infty \).

**Summary:**

The problem \( \nabla^2 u + \frac{1}{r} \frac{\partial u}{\partial r} + k^2 u = \delta(r) \)

with \( \nabla u - ik u \to 0 \quad \text{as} \quad r \to \infty \) \( \rightarrow \) Sommerfeld radiation condition.

has a unique solution \( u = -\frac{i}{4} H_0^{(1)}(kr) \) which is outgoing \( \text{as} \quad r \to \infty \).

Then

\[
U(r, t) = -\frac{i}{4} H_0^{(1)}(kr)e^{-i\omega t} \approx -\frac{i}{4} \left( \frac{2}{\pi kr} \right)^{1/2} e^{i(kr-\omega t-\pi/4)} \quad \text{as} \quad r \to \infty.
\]

**Remark:**

If we impose \( \nabla u + ik u \to 0 \quad \text{as} \quad r \to \infty \) we would get a unique ingoing solution \( u = -\frac{i}{4} H_0^{(2)}(kr) \).

**Three-Dimensional** in 3-D we have

\[
\nabla^2 u + \frac{2}{r} \frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial r^2} = e^{-i\omega t} \delta(r)/4\pi r^2
\]

Now let \( U(r, t) = u(r)e^{-i\omega t} \). Then

\[
(\Phi) \quad LU = u(r) + \frac{2}{r} \frac{\partial u}{\partial r} + k^2 u = \delta(r)/4\pi r^2 \quad \text{oc} \quad r > 0
\]

There are two solutions to \( LU = 0 \) \( u = e^{ikr}/r, e^{-ikr}/r \).

The solution which is outgoing at infinity is

\[
u = A e^{ikr}/r \quad \text{since} \quad u \to 0 \quad \text{as} \quad r \to \infty \quad \text{we need} \quad A = -1/4\pi.
\]

The outgoing solution is \( u = -\frac{1}{4\pi} e^{ikr} \). By putting in an radiation boundary condition as in 2-D we have \( \nabla u - ik u \to 0 \quad \text{as} \quad r \to \infty \)

\[
\frac{1}{r} \frac{\partial u}{\partial r} \quad \text{which guarantees a unique solution to (9),}
\]
Example: Find the solution to
\[ u_{xx} + u_{yy} + k^2 u = f(x, y) \] in \( 0 < x < a, \ -\sigma < y < \sigma \)

with \( u_x = g(y) \) on \( x = 0 \) and \( u \) outgoing as \( (x^2 + y^2)^{1/2} \to 0 \).

Apply Green's identity to the region shown below.

\[
\int \left[ v(\Delta u + k^2 u) - u(\Delta v + k^2 v) \right] ds \, da = \int_\Gamma \left[ v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right] ds + \int_R \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds
\]

Thus,
\[
\int_R \left[ v [\Delta u + k^2 u] - u [\Delta v + k^2 v] \right] ds \, da = \int_{\Gamma} \left[ v \left( u_x + i k u_y \right) - u \left( v_x + i k v_y \right) \right] \left| ds \right| R \, dq
\]

Now \( \Delta u + k^2 u = f(x, y), \ \Delta v + k^2 v = \delta(x, y) \). Let \( R \to \infty \) and imposing outgoing conditions for \( u \) and \( v \), also take \( V = 0 \) on \( \sigma = 0 \). Thus, letting \( R \to \infty \)

\[
\int_{\Gamma} v f(x, y) ds \, da - u(x, y) = \int_{\Gamma} v \bigg|_{\sigma = 0} g(a) \, da
\]

\( u(x, y) = \int_{\sigma = 0} v f(x, y) ds \, da - \int_{\sigma = 0} v \bigg|_{\sigma = 0} g(a) \, da \)

Where \( V \) satisfies,

\[ V_{ss} + V_{yy} + k^2 V = \delta(s-x, y) \quad 0 < s < a, \ -\sigma < y < \sigma \]

\[ V = 0 \text{ on } s = 0 \quad V \text{ outgoing at } s = \sigma \]

By the method of images and p.37 we have

\[ V = -\frac{i}{4} H_0^{(1)} \left[ K((x-s)^2 + (y-y)^2)^{1/2} \right] - \frac{i}{4} H_0^{(1)} \left[ K((x-s)^2 + (y-y)^2)^{1/2} \right] \]

Notice that on \( s = 0 \); \[ V = -\frac{i}{2} H_0^{(1)} \left[ K(x^2 + (y-y)^2)^{1/2} \right] \]

This is to be substituted in \( (y) \).

Remark: (i) It is clear that we can formulate similar problem in 3-D

where the method of images is needed.

(ii) Sometimes separation of variables is more convenient.
SCATTERING PROBLEMS - WAVE PROPAGATION

Consider the wave equation

\[ \nabla^2 u + k^2 u = f \quad \text{outside } D \]

(1) \[ u = 0 \quad \text{on } \partial D \]

Assume that far from the body there is an incident plane wave of the form \( u_{\text{inc}} = e^{-i(kx - \omega t)} \) where \( k = \omega/c \).

Now we decompose the total field \( u \)

\[ u = u_s + u_{\text{inc}} \]

where \( u_s \) is the scattered field and is outgoing at infinity. Thus we write \( u_s(x, t) = u_s(x) e^{-i\omega t} \).

Then, upon substituting in (1) we get

\[ \Delta u_s + k^2 u_s = 0 \quad \text{outside } D \]

(2) \[ u_s = -e^{-i(kx)} \quad \text{on } \partial D \]

\[ \partial_n u_s - iku_s \rightarrow 0 \quad \text{as } r \rightarrow \infty \]

Outgoing condition

To solve (2) we impose Green's identity to the domain shown below to get

\[ \int_D \left( \nabla u_s \cdot \nabla v - u_s \nabla^2 v \right) \, ds = \int_{\partial D} \left( v (u_{sR} - iku_s) - u_s (\nabla R \cdot i\omega) \right) \, ds + \int_D \left( \nabla \cdot (\delta u_s - u_s \nabla v) \right) \, ds \]

Now let \( R \rightarrow \infty \) and impose outgoing condition which eliminates \( \int_{\partial D} \delta \, ds \)

term. Then take

\[ v = 0 \quad \text{on } \partial D \]

\[ \nabla R - i\omega v \rightarrow 0 \quad \text{as } r \rightarrow \infty \]

This gives

(\text{Note change in sign})

\[ u(x) = \int_{\partial D} u_s \frac{1}{r} \, ds \]

Where \( \frac{1}{r} \) now denotes the outward normal derivative as shown.

\[ u_s = \int_{\partial D} e^{-i(kx)} \, ds \]