Green's Functions: Heat Operator

Consider
\[ L = j^2 u_{xx} - u_t = g(x, t) \quad -b < x < c, \ t > 0 \]
\[ u(x, 0) = u_0(x) \]
with boundary conditions on \( x = -b, c. \)

Define the adjoint of \( L \) by \( L^* u \) where
\[ L^* u = j^2 u_{xx} + u_t. \]

We now derive an analogue of Green's second identity.

Let \( u, v \) be any two functions, then
\[ \nabla u \cdot u L^* v = \nabla \left( j^2 u_{xx} - u_t \right) \cdot u \left( j^2 \nabla u_{xx} \right) \cdot u v \]
\[ \nabla L L^* v = \left( j^2 \nabla u_{xx} \cdot u v \right) \cdot u v \]
Thus
\[ \nabla L L^* v = \nabla \left( j^2 \nabla u_{xx} \cdot u v \right) \cdot u v \]

Let \( s \to x, \ t \to t \) and integrate over a space-time rectangle as shown.

Recall - Stokes theorem in the plane
\[ \int_{\partial} \left( Q_s - P_t \right) ds dt = \int_C (P ds + Q dt) \]

Then
\[ \int_{\partial} \left( v L u - u L^* v \right) ds dt = \int_C \left( j^2 (\nabla u_{xx} \cdot u v) - (\nabla u)_{xx} \cdot u v \right) ds dt \]
\[ \int_{\partial} \left( v L u - u L^* v \right) ds dt = \int_C \left( j^2 (\nabla u_{xx} \cdot u v) - (\nabla u)_{xx} \cdot u v \right) ds dt \]

Now integrate explicitly to get
\[ \int_{\partial} \left( v L u - u L^* v \right) ds dt = \int_{-b}^{c} \nabla u \left|_{T=0}^{T=t} \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r
\[ \phi(x,t) \text{ cannot depend on values of } t > t_0 \text{ when } t_0 \text{ is any fixed time.} \]

Thus, we must impose a causality condition that
\[ \phi(s,t) = 0 \text{ for } t > t. \]

From (3) we get
\[ \phi(x,t) = \int_{-b}^{c} \int_{0}^{t} \left[ \psi \left( \frac{t}{s} \right) \right] d\tau + \int_{0}^{t} \left( \psi \left( \frac{t}{s} \right) \right) d\tau + \int_{0}^{t} \left[ \psi \left( \frac{t}{s} \right) \right] d\tau \]
\[ \int_{-b}^{c} \int_{0}^{t} \left( \psi \left( \frac{t}{s} \right) \right) d\tau \]
\[ -k^2 \int_{0}^{t} \left[ \psi \left( \frac{t}{s} \right) \right] d\tau \]
\[ \int_{-b}^{c} \int_{0}^{t} \left( \psi \left( \frac{t}{s} \right) \right) d\tau \]
\[ \int_{0}^{t} \left[ \psi \left( \frac{t}{s} \right) \right] d\tau \]

**Case 1.** Suppose \( b = c = \infty \) and \( \phi(s,0) = \phi_0(s) \). Then we impose \( \psi, \phi \to 0 \) as \( |s| \to \infty \) to obtain from (4)
\[ \phi(x,t) = \int_{-b}^{c} \int_{0}^{t} \psi d\tau + \int_{0}^{t} \psi \phi_0(0) d\tau \]

Here, \( \psi(x,t) \) satisfies
\[ k^2 \psi + \psi = \delta(s-x) \quad \text{in } 0 < t < \infty, \quad -\infty < s < \infty \]
\[ \psi \equiv 0 \text{ for } t > t \]
\[ \psi \to 0 \text{ as } |s| \to \infty. \]

**Case 2.** Now suppose that \( \phi \) is given on \( s = -b \) and \( s = c \). Then we take \( \psi \equiv 0 \) on \( s = -b, c \) so that from (4)
\[ \phi(x,t) = \int_{-b}^{c} \int_{0}^{t} \psi d\tau + \int_{0}^{t} \psi \phi_0(0) d\tau - \int_{0}^{t} \psi \phi_0(0) d\tau \]
\[ \int_{-b}^{c} \int_{0}^{t} \psi d\tau \]
\[ \int_{-b}^{c} \int_{0}^{t} \psi d\tau \]
\[ \int_{0}^{t} \psi \phi_0(0) d\tau \]
\[ \int_{0}^{t} \psi \phi_0(0) d\tau \]

where \( \psi(s,t) \) satisfies
\[ k^2 \psi + \psi = \delta(s-x) \quad 0 < t < \infty, \quad -b < s < c \]
\[ \psi \equiv 0 \text{ for } t > t \]
\[ \psi \equiv 0 \text{ on } s = -b, c \]

It is clear that a similar formula for \( \phi \) can be written with many combinations of boundary conditions on \( s = -b, c \).

Before solving for \( \psi \) in (5b) and (6b) we make some remarks about backward heat equations.
Consider \( u_t = D u_{xx} \) with \( D > 0 \) or \( D < 0 \).

Substitute a plane wave \( u(x,t) = e^{i(kx - \omega t)} \) \( \lambda = 2\pi / k \) wavelength in space \( k \) wavenumber.

Then, \( -\omega = -k^2 D \) \( \omega = -ik^2 D \)

\( u(x,t) = e^{-k^2 D t} e^{ikx} \)

Notice (i) as \( t \) increases, a high wavenumber (low wavelength) disturbance grows exponentially if \( D < 0 \) (backward heat equation), it decays exponentially if \( D > 0 \).

(ii) The growth or decay rate becomes larger as \( k \to 1 \) i.e. \( \lambda \rho \)

(iii) Negative diffusion \( D < 0 \) can be realized by letting \( D > 0 \) but by integrating backward in time.

Now consider (55b). Although the diffusion coefficient is negative we are integrating for decreasing values of \( \tau \), \( \Rightarrow \) well posed.

Thus let \( \sigma = \frac{t}{\tau} - 1 \), and \( G(s, \sigma) = -V(s, \frac{t}{\tau} - \sigma) \) where \( G(s, \sigma) \) is the Green's function. It satisfies for case I (recall \( \delta(z) = \delta(-z) \))

\[
(7) \quad G_{\sigma} - k^2 G_{ss} = \delta(1-x, \sigma) \quad \sigma > 0 \quad -\infty < s < \infty \]

\[ G = 0 \] for \( \sigma < 0 \). \( G \to 0 \) as \( |s| \to \infty \).

The way to interpret (7) is the following. Let \( G^E \) solve

\[ G^E_{\sigma} - k^2 G^E_{ss} = \delta(s-x, \sigma - \varepsilon) \quad \varepsilon > 0 \] fixed

\[ G^E = 0 \] for \( \sigma = 0 \)

Then \( G = \lim_{\varepsilon \to 0} G^E \).

Remark (i) we want to solve (7) for \( G(s, \sigma) \) (with this interpretation in mind). Then

\[ V(s, \tau) = -G(s, \frac{t}{\tau} - \sigma) \]

and we can substitute in (55a) to solve the Cauchy problem.
(ii) For Case 2 let \( \sigma = t - \tau \) and \( \nu = -G \) so that

\begin{align*}
G_\sigma - \nu^2 G_{ss} &= \delta (s - x, \sigma) - b s \nu \leq G, \sigma > 0 \\
G (s, \sigma) &= 0 \quad \text{if} \quad G (-b, \sigma) = G (G, \sigma) = 0
\end{align*}

Then \( \nu (s, \tau) = -G (s, t - \tau) \) is to be substituted in (6a).

Calculating the Green's function \( G \) explicitly

We now want to solve (7) explicitly. Before doing so we solve the related problem

\begin{align*}
W_\sigma &= D \Delta W \text{ in } \mathbb{R}^\nu \\
\Delta W &= W_1, s_1, \ldots + W_{sN}, \nu_n
\end{align*}

\[ W(s_1, s_2, s_N, 0) = \delta (s_1) \delta (s_2) \ldots \delta (s_N) \]

We then relate the solution to (9) to that of (7), to solve (9) introduce the \( n \)-dimensional Fourier transform

\[ \hat{W} (k_1, \ldots, k_N, 0) = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-i \mathbf{k} \cdot \mathbf{s}} W (s_1, \ldots, s_N) \, ds_1 \ldots ds_N \]

\[ W (s_1, \ldots, s_N, 0) = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i \mathbf{k} \cdot \mathbf{s}} \hat{W} (k_1, \ldots, k_N) \, dk_1 \ldots dk_N \]

Where \( \mathbf{k} \cdot \mathbf{s} = \sum k_i s_i = (k_1, \ldots, k_N) \cdot (s_1, \ldots, s_N) = k_1 s_1 + \ldots + k_N s_N \). Taking Fourier transform gives

\[ \hat{W}_\sigma = -D |k|^2 \hat{W} \quad \Rightarrow \quad \hat{W} = \left( \frac{1}{2\pi} \right)^N e^{-|k|^2/2} \]

\[ \hat{W} (k_1, \ldots, k_N, 0) = 1/(2\pi)^N \]

Now

\[ W (s_1, \ldots, s_N, 0) = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i \mathbf{k} \cdot \mathbf{s} - |k|^2/2} \, dk_1 \ldots dk_N \]

This can be written as the product of \( n \) integrals of the form

\[ \int_{-\infty}^{\infty} e^{i k_1 s_1 - k_1^2/2} \, dk_1 = \left( \frac{\pi}{|D\sigma|} \right)^{1/2} e^{-s_1^2 / 4 |D\sigma|} \]

Thus

\[ W (s_1, \ldots, s_N, 0) = \frac{1}{(2\pi)^{1/2}} \left( \frac{\pi}{|D\sigma|} \right)^{N/2} e^{-(s_1^2 + \ldots + s_N^2) / 4 |D\sigma|} \]

Re-writing

\[ W (s_1, s_2, \ldots, s_N, \tau) = \frac{1}{2^N (\pi |D\sigma|)^{N/2}} e^{-\frac{(s_1^2 + \ldots + s_N^2)}{4 |D\sigma|}} \]
Now if \( N = 1 \), then

\[
W(s, \sigma) = \frac{1}{2 \sqrt{\pi D \sigma}} e^{-\frac{1}{4} \sigma D}
\]

Remark: By translating space it is clear that the solution to

\[
W_\sigma = k^2 W_{ss} \quad \sigma < s < \sigma
\]

(10) \( W(s, 0) = \delta(s - \lambda) \)

\[
W(s, \sigma) = \frac{1}{2 \sqrt{\pi D \sigma}} e^{-\frac{1}{4} \sigma D} \quad \text{for} \quad \sigma > 0.
\]

Now what does the solution to (10) have to do with the solution to (7). What we now show is that

\[
G(s, \sigma) = W(s, \sigma). \quad \text{(with} \quad D = k^2)\]

Theorem: The solution to (7) and (10) are the same.

For (10) \( W = \frac{1}{2 \sqrt{\pi D \sigma}} \frac{H'(\sigma)}{2 \sqrt{\pi D \sigma}} e^{-\frac{1}{4} \sigma D} \quad \text{with} \quad D = k^2 \quad H \text{is Heavyside function.}

Then \( W_\sigma = k^2 W_{ss} = \frac{H'(\sigma)}{2 \sqrt{\pi D \sigma}} e^{-\frac{1}{2} \sigma D} \frac{d}{d\sigma} \frac{H(\sigma)}{\sqrt{\pi D \sigma}} [\frac{1}{2 \sqrt{\pi D \sigma}} e^{-\frac{1}{4} \sigma D}]

Thus \( W_\sigma = k^2 W_{ss} = \delta(\sigma) \frac{1}{2 \sqrt{\pi D \sigma}} e^{-\frac{1}{4} \sigma D} \)

Now as \( \sigma \to 0 \) we have

\[
\frac{1}{2 \sqrt{\pi D \sigma}} e^{-\frac{1}{4} \sigma D} \to \delta(s - \lambda)
\]

Thus \( W_\sigma = k^2 W_{ss} = 0 \quad \delta(\sigma) \delta(s - \lambda) \quad \text{which is} \quad (7).

Thus for (7),

\[
G(s, \sigma) = \frac{1}{2 \sqrt{\pi D \sigma}} e^{-\frac{1}{4} \sigma D} \quad D = k^2
\]

And in (5b) we get

\[
W(s, \tau) = -\frac{1}{2 \sqrt{\pi D (t-\tau)}} e^{-\frac{1}{4} \sigma D (t-\tau) k^2}
\]
Therefore the solution to

\[ K^2 u(x, t) - u_t = g(x, t) \text{ in } -\infty < x < \infty, \ t > 0 \]

\[ u(x, 0) = u_0(x) \text{ if } (5a), \]

\[ (I) \quad u(x, t) = \int_{-\infty}^{t} \int_{x}^{t} -\frac{9}{4(\tau-s)} e^{-\frac{(s-x)^2}{4(t-\tau)} \frac{d\tau d\tau}{2 \sqrt{\pi d(t-\tau)}}} + \int_{-\infty}^{t} \frac{u_0(s)}{2 \sqrt{\pi d(t-\tau)}} e^{-\frac{(s-x)^2}{4t}} ds \]

Example find the solution to

\[ K^2 u(x, t) = g(x, t) \text{ in } 0 < x < a, \ t > 0 \]

\[ u(x, 0) = u_0(x) \text{ and } u \text{ given on } x = 0 \text{ and } x = a. \]

From (6a, b) we have the integral representation

\[ u(x, t) = \int_{0}^{a} \int_{x}^{t} g(s, t-s; x) g(s, t-s; x) ds dt + \int_{0}^{a} g(s, t-s; x) du_0(s) \ ds \]

\[ - \int_{x}^{t} \partial_{1} G(x, t-s; x) ds + \int_{0}^{a} \partial_{1} G(x, t-s; x) \ ds \]

Where \( G(\sigma, \tau; x) \) solves

\[ G_{\sigma} - G_{\tau} = \delta(s-x, \sigma) \quad 0 < x < a, \ \sigma > 0 \]

\[ G = 0 \text{ on } \sigma = 0 \text{ and on } s = 0 \text{ and } s = a. \]

We expand \( G(\sigma, \tau; x) = \sum_{n=1}^{\infty} T_n(\sigma) \sin\left(\frac{n\pi x}{a}\right) \rightarrow T_n(\sigma) = \frac{2}{a} \int_{0}^{a} G(s, \sigma) \sin\left(\frac{n\pi s}{a}\right) ds \]

Thus \[ \frac{2}{a} \int_{0}^{a} \left[ G_{\sigma} - G_{\tau} \right] \sin\left(\frac{n\pi x}{a}\right) ds = \frac{2}{a} \delta(\sigma) \sin\left(\frac{n\pi x}{a}\right) \]

\[ \Rightarrow \quad T_n(\sigma) + \frac{n^2\pi^2}{a^2} T_n(\sigma) = \frac{2}{a} \delta(\sigma) \sin\left(\frac{n\pi x}{a}\right) \]

\[ T_n(0) = 0 \]

Now take Laplace transforms in \( \sigma \), keeping in mind that \( \delta(\sigma) \rightarrow \delta(\sigma-e) \ e^\sigma. \)

Thus \[ T_n(s + \frac{n^2\pi^2}{a^2}) = \frac{2}{a} e^{-se} \sin\left(\frac{n\pi x}{a}\right) \]

\[ \tilde{T}_n(s) = \frac{2}{a} e^{-se} \sin\left(\frac{n\pi x}{a}\right) \]

Inverting gives \[ T_n(\sigma) = \frac{2}{a} H(\sigma-e) \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n^2\pi^2}{a^2} s} \]
Letting $\varepsilon \to 0$ given $T_n(\sigma) = \frac{2}{a} \frac{H(\sigma)}{\sin(\frac{n \pi x}{a})} e^{-n^2 \pi^2 \sigma/a^2}$

so that

$$G(s, \sigma; x) = \sum_{n=1}^{\infty} \frac{2}{a} \frac{H(\sigma)}{\sin(\frac{n \pi x}{a})} \sin(\frac{n \pi s}{a}) e^{-n^2 \pi^2 \sigma/a^2}$$

Finally, in (13) we substitute

$$G(s, \tau - \tau; x) = \sum_{n=1}^{\infty} \frac{2}{a} \frac{H(\tau - \tau)}{\sin(\frac{n \pi x}{a})} \sin(\frac{n \pi s}{a}) e^{-n^2 \pi^2 \tau/a^2}$$

Example (Method of Images)

Find an integral representation for the solution to

$u_{xx} - u_x = g(x, t) \quad 0 < x < \sigma, \quad t > 0$

$u(0, t) = 0 \quad u(x, 0) = u_0(x) \quad u \to 0 \quad \text{as} \quad x \to \sigma$

From (14) page 112 we can derive

$$-u(x, t) = \int_0^x \int_0^t G(s, \tau - \tau; x) g(s, \tau) ds \, d\tau = -\int_0^t G(0, \tau; t \tau; x) u_0(s) ds$$

or

$$u(x, t) = \int_0^t \int_0^x G(s, \tau - \tau; x) g(s, \tau) ds \, d\tau + \int_0^t G(0, \tau; t \tau; x) u_0(s) ds$$

where $G(s, \sigma; x)$ solves

$$G_{\sigma} - G_{ss} = 0 \quad 0 < s < \sigma \quad \sigma > 0$$

$G = 0$ on $s = 0 \quad G(s, 0; x) = \delta(1 - x)$

By the method of images we put an image charge of opposite sign at $s = -x$ so that we want the solution to

$$G_{\sigma} - G_{ss} = 0 \quad -\sigma < s < \sigma \quad \sigma > 0$$

$$G(s, 0; x) = \delta(1-x) - \delta(1+x)$$

Thus will automatically satisfy $G(0, \sigma; x) = 0$

Thus by superposition we get

$$G(s, \sigma; x) = \frac{1}{2 \sqrt{\pi} \sigma} \left[ e^{-(s-x)^2/4 \sigma} - e^{-(s+x)^2/4 \sigma} \right]$$

or

$$G(s, \tau - \tau; x) = \frac{1}{2 \sqrt{\pi} \tau} \left[ e^{-(s-x)^2/4(t-\tau)} - e^{-(s+x)^2/4(t-\tau)} \right]$$

which is to be substituted in the equation above.
WE CONSIDER \( \Delta u - u_t = g(x,t) \) FOR \( x \in \Omega \)

\( u \) GIVEN ON \( \partial \Omega \)

\( u(x,0) = u_0(x) \)

HERE \( \Omega \) IS A DOMAIN IN \( \mathbb{R}^n \). WE DEFINE \( L u \) AND \( L^\times u \) BY

\[
L u = \Delta u - u_t \quad \text{AND} \quad L^\times u = \Delta u + u_t.
\]

WE LET \( C_T = \Omega \times [0,T] \) BE THE SPACE-TIME CYLINDER AS SHOWN.

WE LET \( s, \tau \) BE DUMMY VARIABLES, FOR \( x, t \) AND WE INTEGRATE OVER \( C_T \) AS

\[
\int_{C_T} \nabla (\Delta u - u_t) \, ds \, d\tau = \int_{\Omega} \left( \int_0^T \nabla u \, ds \right) \, d\tau - \int_{\partial \Omega} \left( \int_0^T \nabla u \, ds \right) \, d\tau.
\]

NOW USE GREEN'S IDENTITY ON FIRST TERM AND \( \nabla u_\tau = (u \nabla)_\tau - \nabla u \).

WE GET

\[
\int_{C_T} \nabla L u \, ds \, d\tau = \int_{\Omega} \left( \int_0^T \nabla u \, ds \right) \, d\tau - \int_{\partial \Omega} \nabla u_j \left|_{\tau = 0} \right. \, ds + \int_{\partial \Omega} \left( \int_0^T \nabla u \, ds \right) \, d\tau.
\]

THIS YIELDS

\[
\int_{C_T} \nabla L u \, ds \, d\tau = \int_{C_T} \nabla L^\times u \, ds \, d\tau + D \left( \int_0^T \nabla u \, ds \right) \, d\tau - \int_{\partial \Omega} \left( \int_0^T \nabla u \, ds \right) \, d\tau.
\]

NOW WE CHOOSE \( \nabla \) TO SATISFY

\[
L^\times \nabla = \delta(x-x,x) \quad \text{WITH} \quad 0 < t < T \quad \text{AND} \quad x \in \Omega.
\]

\[
(2) \quad \nabla \equiv 0 \quad \text{FOR} \quad \tau > t \quad \text{CAUSALITY CONDITION}
\]

\[
\nabla = 0 \quad \text{ON} \quad \partial \Omega.
\]

THEN (1) BECOMES

\[
\int_{C_T} \nabla g(s,T) \, ds \, d\tau = \int_{\partial \Omega} \left( \int_0^T \nabla u \, ds \right) \, d\tau + \int_{\partial \Omega} \left( \int_0^T \nabla u_\tau \, ds \right) \, d\tau.
\]
This yields

\[
(3) \quad \|u(x, t)\|^2 = \int_0^t \int_{\Omega} V g(s, T) d\xi d\tau + D \int_0^t \left( \int_{\partial\Omega} \nu_0 V \, dS \right) d\tau - \int_{\Omega} V |u_0(s) d\xi.
\]

As the representation of the solution.

Now the problem for \( V \) is

\[
(4) \begin{cases}
V_\tau = -D \Delta V + \delta(s-X, \tau-t) & \text{in } \Omega \times \{ \tau \leq t \} \\
V = 0 & \text{for } \tau > t \\
V = 0 & \text{on } \partial\Omega.
\end{cases}
\]

Now define \( \sigma = -s + t \) and \( G(s, \sigma) = -V(s, -\sigma + t) \)

We obtain that

\[
(5) \begin{cases}
G_\sigma = D \Delta G + \delta(s-X, \sigma) & \text{in } \sigma > 0, s \in \Omega \\
G = 0 & \text{on } \partial\Omega \\
G = 0 & \text{for } \sigma < 0.
\end{cases}
\]

Then
\[
V(s, \tau) = -G(s, \tau - \tau) \text{ is to be used in (3)}.
\]

We showed earlier that this problem is equivalent to the following:

\[
(6) \begin{cases}
G_\sigma = D \Delta G & \text{in } \sigma > 0, s \in \Omega \\
G = 0 & \text{on } \partial\Omega \\
G(s, 0) = \delta(s-X(s))
\end{cases}
\]

Now if \( \Omega \) is the half-space \( s_2 \geq 0, -\infty < s_2 < \infty \)

then we can use the method of images. We then let \( G \)

satisfy the following problem on \( \mathbb{R}^2 \) with image "charge" at \( X_1 = (-s, y) \).