GREEN'S FUNCTIONS: HEAT OPERATOR

CONSIDER $\quad Lu = K^2 u_{xx} - u_t = g(x,t) \qquad -b < x < c\,,\ t > 0$

$$u(x,0) = u_0(x) \quad \text{WITH BOUNDARY CONDITIONS ON } x = -b, c.$$

DEFINE THE ADJOINT OF L BY $L^* u$ WHERE

$$L^* u \equiv K^2 u_{xx} + u_t .$$

WE NOW DERIVE AN ANALOGUE OF GREEN'S SECOND IDENTITY.

LET u, v BE ANY TWO FUNCTIONS. THEN

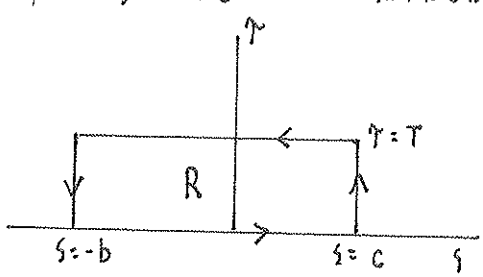
$$v Lu - u L^* v = v(K^2 u_{xx} - u_t) - u(K^2 v_{xx} + v_t)$$

$$(v Lu - u L^* v) = K^2 (v u_{xx} - u v_{xx}) - (vu)_t$$

NOW $(v u_{xx} - u v_{xx}) = (v u_x - u v_x)_x$. THUS

$$(1) \qquad (v Lu - u L^* v) = K^2 (v u_x - u v_x)_x - (vu)_t$$

LET $\xi \to x$, $\tau \to t$ AND INTEGRATE OVER A SPACE-TIME RECTANGLE AS SHOWN.



RECALL - STOKES THEOREM IN THE PLANE

$$\iint_R [Q_\xi - P_\tau]\, d\xi\, d\tau = \oint_C (P\, d\xi + Q\, d\tau)$$

THEN

$$\iint_R [v Lu - u L^* v]\, d\xi\, d\tau = \iint_R [K^2 (v u_\xi - u v_\xi)_\xi - (vu)_\tau]\, d\xi\, d\tau$$

$$\iint_R [v Lu - u L^* v]\, d\xi\, d\tau = \oint_C [v u\, d\xi + K^2 (v u_\xi - u v_\xi)\, d\tau]$$

NOW INTEGRATE EXPLICITLY TO GET

$$(2) \qquad \iint_R [v Lu - u L^* v]\, d\xi\, d\tau = \int_{-b}^{c} v u \Big|_{\tau=0} d\xi - \int_{-b}^{c} v u \Big|_{\tau=T} d\xi$$
$$+ K^2 \int_0^T (v u_\xi - u v_\xi)\Big|_{\xi=c} d\tau - K^2 \int_0^T (v u_\xi - u v_\xi)\Big|_{\xi=-b} d\tau$$

NOW WE LET $Lu = g(\xi,\tau)$ AND $L^* v = \delta(\xi - x, \tau - t)$. THEN IF $T > t$ WE HAVE

$$(3) \qquad -u(x,t) + \iint_R v g\, d\xi\, d\tau = \int_{-b}^{c} v u \Big|_{\tau=0} d\xi - \int_{-b}^{c} v u \Big|_{\tau=T} d\xi + K^2 \int_0^T (v u_\xi - u v_\xi)\Big|_{\xi=c} d\tau$$
$$- K^2 \int_0^T (v u_\xi - u v_\xi)\Big|_{\xi=-b} d\tau$$

NOW $u(x,t_0)$ CANNOT DEPEND ON VALUES OF $t > t_0$ WHERE t_0 IS ANY FIXED TIME. THUS WE MUST IMPOSE A CAUSALITY CONDITION THAT

$$V(\xi,\tau) \equiv 0 \quad \text{FOR} \quad \tau > t.$$

FROM (3) WE GET

$$(4) \qquad -u(x,t) + \int_{-b}^{c}\int_0^t V g \, d\xi \, d\tau = \int_{-b}^{c} V u\Big|_{\tau=0} d\xi + K^2\int_0^t [V u_\xi - u V_\xi]\Big|_{\xi=c} d\tau - K^2\int_0^t [V u_\xi - u V_\xi]\Big|_{\xi=-b} d\tau$$

<u>CASE 1</u> SUPPOSE $b = c = \infty$ AND $u(\xi,0) = u_0(\xi)$. THEN WE IMPOSE $u, V \to 0$ AS $|\xi| \to \infty$ TO OBTAIN FROM (4)

$$(5a) \qquad u(x,t) = \int_{-\infty}^{\infty}\int_0^t V g \, d\xi \, d\tau - \int_{-\infty}^{\infty} V\Big|_{\tau=0} u_0(\xi)\, d\xi$$

HERE $V(\xi,\tau : x,t)$ SATISFIES

$$(5b) \qquad \begin{aligned} &K^2 V_{\xi\xi} + V_\tau = \delta(\xi - x, \tau - t) \quad \text{IN} \quad 0 < \tau < t, \ -\infty < \xi < \infty \\ &V \equiv 0 \quad \text{FOR} \quad \tau > t \\ &V \to 0 \quad \text{AS} \quad |\xi| \to \infty. \end{aligned}$$

<u>CASE 2</u> NOW SUPPOSE THAT u IS GIVEN ON $\xi = -b$ AND $\xi = c$. THEN WE TAKE $V = 0$ ON $\xi = -b, c$ SO THAT FROM (4)

$$(6a) \qquad u(x,t) = \int_{-b}^{c}\int_0^t V g \, d\xi \, d\tau - \int_{-b}^{c} V u\Big|_{\tau=0} d\xi + K^2\int_0^t u V_\xi\Big|_{\xi=c} d\tau - K^2\int_0^t u V_\xi\Big|_{\xi=-b} d\tau$$

WHERE $V(\xi,\tau; x,t)$ SATISFIES

$$(6b) \qquad \begin{aligned} &K^2 V_{\xi\xi} + V_\tau = \delta(\xi - x, \tau - t) \quad 0 < \tau < t \quad -b < \xi < c \\ &V \equiv 0 \quad \text{FOR} \quad \tau > t \\ &V = 0 \quad \text{ON} \quad \xi = -b, c \end{aligned}$$

IT IS CLEAR THAT A SIMILAR FORMULA FOR u CAN BE WRITTEN WITH MANY COMBINATIONS OF BOUNDARY CONDITIONS ON $\xi = -b, c$.

BEFORE SOLVING FOR V IN (5b) AND (6b) WE MAKE SOME REMARKS ABOUT BACKWARD HEAT EQUATIONS.

BACKWARD HEAT EQUATION (ASIDE)

CONSIDER $u_t = D\, u_{xx}$ $\quad -\infty < x < \infty$, $t > 0$ WITH D EITHER > 0 OR $D < 0$.

SUBSTITUTE A PLANE WAVE $u(x,t) = e^{i(kx - \omega t)}$ $\quad k = 2\pi/\lambda$ $\quad \lambda$ WAVELENGTH IN SPACE. k WAVENUMBER

THEN, $-i\omega = -k^2 D$ $\quad \omega = -ik^2 D$

$$u(x,t) = e^{-k^2 D t}\, e^{ikx}$$

NOTICE (i) AS t INCREASES, A HIGH WAVENUMBER (LOW WAVELENGTH) DISTURBANCE GROWS EXPONENTIALLY IF $D<0$ (BACKWARD HEAT EQUATION). IT DECAYS EXPONENTIALLY IF $D>0$.

(ii) THE GROWTH OR DECAY RATE BECOMES LARGER AS $k \uparrow$ (i.e. $\lambda \downarrow$)

(iii) NEGATIVE DIFFUSION $D<0$ CAN BE REALIZED BY LETTING $D>0$ BUT BY INTEGRATING BACKWARDS IN TIME.

NOW CONSIDER (5b). ALTHOUGH THE DIFFUSION COEFFICIENT IS NEGATIVE WE ARE INTEGRATING FOR DECREASING VALUES OF τ. $\Rightarrow$ WELL POSED.

THUS LET $\sigma = t - \tau$, AND $G(\xi,\sigma) = -V(\xi, t-\sigma)$, WHERE $G(\xi,\sigma)$ IS THE GREEN'S FUNCTION. IT SATISFIES FOR CASE 1 (RECALL $\delta(z) = \delta(-z)$)

$$(7) \qquad G_\sigma - k^2 G_{\xi\xi} = \delta(\xi - x, \sigma) \qquad \sigma > 0 \quad -\infty < \xi < \infty$$

$$G = 0 \quad \text{FOR } \sigma \leq 0. \qquad G \to 0 \text{ AS } |\xi| \to \infty.$$

THE WAY TO INTERPRET (7) IS THE FOLLOWING. LET G^ε SOLVE

$$G^\varepsilon_\sigma - k^2 G^\varepsilon_{\xi\xi} = \delta(\xi - x, \sigma - \varepsilon) \qquad \varepsilon > 0 \text{ FIXED}$$

$$G^\varepsilon = 0 \quad \text{FOR } \sigma = 0$$

THEN $$G = \lim_{\varepsilon \to 0} G^\varepsilon .$$

REMARK (i) WE WANT TO SOLVE (7) FOR $G(\xi,\sigma)$ (WITH THIS INTERPRETATION IN MIND). THEN

$$V(\xi,\tau) = -G(\xi, t-\tau)$$

AND WE CAN SUBSTITUTE IN (5a) TO SOLVE THE CAUCHY PROBLEM.

(ii) FOR <u>CASE 2</u> LET $\sigma = t - \tau$ AND $V = -G$ SO THAT

$$(8) \qquad G_\sigma - K^2 G_{\xi\xi} = \delta(\xi - x, \sigma) \qquad -b < \xi < c, \ \sigma > 0$$

$$G(\xi, \sigma) = 0 \qquad G(-b, \sigma) = G(c, \sigma) = 0$$

THEN $V(\xi, \tau) = -G(\xi, t - \tau)$ IS TO BE SUBSTITUTED IN (6a).

CALCULATING THE GREEN'S FUNCTION G EXPLICITLY

WE NOW WANT TO SOLVE (7) EXPLICITLY. BEFORE DOING SO WE SOLVE THE RELATED PROBLEM

$$(9) \qquad W_\sigma = D \Delta W \quad \text{IN } \mathbb{R}^N \qquad \Delta W = W_{\xi_1 \xi_1} + \cdots + W_{\xi_N \xi_N}$$

$$W(\xi_1, \ldots, \xi_N, 0) = \delta(\xi_1)\delta(\xi_2) \cdots \delta(\xi_N)$$

WE THEN RELATE THE SOLUTION TO (9) TO THAT OF (7). TO SOLVE (9) INTRODUCE THE N-DIMENSIONAL FOURIER TRANSFORM

$$\hat{W}(K_1, \ldots, K_N, \ ) = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-i \underline{K} \cdot \underline{\xi}} \, W(\xi_1, \ldots, \xi_N, \ ) \, d\xi_1 \ldots d\xi_N$$

$$W(\xi_1, \ldots, \xi_N, \ ) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i \underline{K} \cdot \underline{\xi}} \, \hat{W}(K_1, \ldots, K_N, \ ) \, dK_1 dK_2 \ldots dK_N$$

WHERE $\underline{K} \cdot \underline{\xi} = (K_1, \ldots K_N) \cdot (\xi_1, \ldots, \xi_N) = K_1 \xi_1 + \ldots + K_N \xi_N$. TAKING FOURIER TRANSFORMS GIVES

$$\hat{W}_\sigma = -D |\underline{K}|^2 \hat{W} \qquad \longrightarrow \qquad \hat{W} = \left(\frac{1}{2\pi}\right)^N e^{-D|K|^2 \sigma} \qquad |K|^2 = \underline{K} \cdot \underline{K} = K_1^2 + \ldots + K_N^2$$

$$\hat{W}(K_1, \ldots, K_N, 0) = 1/(2\pi)^N$$

NOW $$W(\xi_1, \ldots \xi_N, \sigma) = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i \underline{K} \cdot \underline{\xi} - D|K|^2 \sigma} \, dK_1 \ldots dK_N$$

THIS CAN BE WRITTEN AS THE PRODUCT OF N INTEGRALS OF THE FORM

$$\int_{-\infty}^{\infty} e^{i K_i \xi_i - K_i^2 D \sigma} \, dK_i = \left(\pi / D\sigma\right)^{1/2} e^{-\xi_i^2 / 4\sigma D}.$$

THUS $$W(\xi_1, \ldots, \xi_N, \sigma) = \frac{1}{(2\pi)^N} \left(\frac{\pi}{D\sigma}\right)^{N/2} e^{-(\xi_1^2 + \ldots + \xi_N^2)/4\sigma D}$$

RE-WRITING $$W(\xi_1, \xi_2, \ldots, \xi_N, t) = \frac{1}{2^N (\pi D \sigma)^{N/2}} e^{-(\xi_1^2 + \ldots + \xi_N^2)/4\sigma D}$$

NOW IF $N = 1$, THEN

$$W(\xi, \sigma) = \frac{1}{2\sqrt{\pi D \sigma}} e^{-\xi^2/4\sigma D}$$

<u>REMARK</u> (i) BY TRANSLATING SPACE IT IS CLEAR THAT THE SOLUTION TO

$$(10) \qquad \begin{aligned} &W_\sigma = K^2 W_{\xi\xi} \qquad -\infty < \xi < \infty \\ &W(\xi, 0) = \delta(\xi - X) \quad \text{IS} \end{aligned}$$

$$W(\xi, \sigma) = \frac{1}{2\sqrt{\pi D \sigma}} e^{-(\xi - X)^2/4\sigma D} \qquad \text{FOR} \quad \sigma > 0.$$

NOW WHAT DOES THE SOLUTION TO (10) HAVE TO DO WITH THE SOLUTION TO (7). WHAT WE NOW SHOW IS THAT

$$G(\xi, \sigma) = W(\xi, \sigma). \qquad (\text{WITH} \quad D = K^2)$$

<u>THEOREM</u> THE SOLUTION TO (7) AND (10) ARE THE SAME.

FOR (10) $\quad W = \dfrac{1}{2\sqrt{\pi D \sigma}} H(\sigma) e^{-(\xi - X)^2/4\sigma D}$ WITH $D = K^2$ $\quad$ H IS HEAVY-SIDE FUNCTION.

THEN $\quad W_\sigma - K^2 W_{\xi\xi} = \dfrac{H'(\sigma)}{2\sqrt{\pi D \sigma}} e^{-(\xi - X)^2/4\sigma D} + H(\sigma)(\partial_\sigma - K^2 \partial_{\xi\xi}) \left[\dfrac{1}{2\sqrt{\pi D \sigma}} e^{-(\xi - X)^2/4\sigma D} \right]$

THUS $\quad W_\sigma - K^2 W_{\xi\xi} = \delta(\sigma) \dfrac{1}{2\sqrt{\pi D \sigma}} e^{-(\xi - X)^2/4\sigma D}$

NOW AS $\sigma \to 0$ WE HAVE $\quad \dfrac{1}{2\sqrt{\pi D \sigma}} e^{-(\xi - X)^2/4\sigma D} \longrightarrow \delta(\xi - X)$

THUS $\quad W_\sigma - K^2 W_{\xi\xi} = \delta(\sigma)\, \delta(\xi - X) \quad$ WHICH IS (7).

THUS FOR (7), $\quad G(\xi, \sigma) = \dfrac{1}{2\sqrt{\pi D \sigma}} e^{-(\xi - X)^2/4\sigma D} \qquad D = K^2$

AND SO IN (5b) WE SET

$$\boxed{V(\xi, \tau) = \frac{-1}{2\sqrt{\pi D (t - \tau)}} e^{-(\xi - X)^2/4(t - \tau)K^2}}$$

THEREFORE THE SOLUTION TO

$$K^2 u_{xx} - u_t = g(x,t) \quad \text{IN} \quad -\infty < x < \infty, \quad t > 0$$

$u(x,0) = u_0(x)$ IS, FROM (5a),

$$(11) \qquad u(x,t) = \int_{-\infty}^{\infty}\int_0^t \frac{-g(\xi,\tau)}{2\sqrt{\pi D(t-\tau)}}\, e^{-(\xi-x)^2/4(t-\tau)K^2}\, d\xi\, d\tau + \int_{-\infty}^{\infty} \frac{u_0(\xi)}{2\sqrt{\pi D t}}\, e^{-(\xi-x)^2/4tK^2}\, d\xi$$

<u>EXAMPLE</u> FIND THE SOLUTION TO

$$(12) \qquad \begin{aligned} &u_{xx} - u_t = g(x,t) \quad \text{IN} \quad 0 < x < a, \quad t > 0 \\ &u(x,0) = u_0(x) \quad \text{AND} \quad u \text{ GIVEN ON } x = 0 \text{ AND } x = a. \end{aligned}$$

FROM (6a, b) WE HAVE THE INTEGRAL REPRESENTATION

$$(13) \qquad \begin{aligned} u(x,t) = &-\int_0^a\int_0^t G(\xi, t-\tau; x)\, g(\xi,\tau)\, d\xi\, d\tau + \int_0^a G(\xi, t; x)\, u_0(\xi)\, d\xi \\ &- \int_0^t \partial_\xi G\, u\Big|_{\xi=0} d\tau + \int_0^t \partial_\xi G\, u\Big|_{\xi=a} d\tau \end{aligned}$$

WHERE $G(\xi, \sigma; x)$ SOLVES

$$G_\sigma - G_{\xi\xi} = \delta(\xi - x, \sigma) \qquad 0 < \xi < a, \quad \sigma > 0$$

$$G = 0 \quad \text{ON} \quad \sigma = 0 \quad \text{AND ON} \quad \xi = 0 \text{ AND } \xi = a.$$

WE EXPAND $\quad G(\xi,\sigma) = \sum_{n=1}^{\infty} T_n(\sigma) \sin(n\pi\xi/a) \longrightarrow T_n(\sigma) = \frac{2}{a}\int_0^a G(\xi,\sigma)\sin(n\pi\xi/a)\, d\xi$

THUS $\quad \frac{2}{a}\int_0^a [G_\sigma - G_{\xi\xi}] \sin(n\pi\xi/a)\, d\xi = \frac{2}{a}\delta(\sigma)\sin\left(\frac{n\pi x}{a}\right)$

$$\Rightarrow \qquad T_n'(\sigma) + \frac{n^2\pi^2}{a^2} T_n(\sigma) = \frac{2}{a}\delta(\sigma)\sin(n\pi x/a)$$

$$T_n(0) = 0$$

NOW TAKE LAPLACE TRANSFORMS IN σ, KEEPING IN MIND THAT $\delta(\sigma) \to \delta(\sigma - \varepsilon)$ $\varepsilon > 0$.

THUS $\quad \hat{T}_n(s)(s + n^2\pi^2/a^2) = \frac{2}{a} e^{-s\varepsilon} \sin(n\pi x/a)$

$$\hat{T}_n(s) = \frac{2}{a} e^{-s\varepsilon} \sin(n\pi x/a) \frac{1}{s + n^2\pi^2/a^2}$$

INVERTING GIVES $\quad T_n(\sigma) = \frac{2}{a} H(\sigma - \varepsilon) \sin\left(\frac{n\pi x}{a}\right) e^{-n^2\pi^2\sigma/a^2}$

LETTING $\varepsilon \to 0$ GIVES $\quad T_n(\sigma) = \frac{2}{a} H(\sigma) \sin\left(n\pi x/a\right) e^{-n^2\pi^2\sigma/a^2}$

SO THAT

$$G(\xi, \sigma; x) = \sum_{n=1}^{\infty} \frac{2}{a} H(\sigma) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi \xi}{a}\right) e^{-n^2\pi^2\sigma/a^2}$$

FINALLY, IN (13) WE SUBSTITUTE

$$G(\xi, t-\tau; x) = \sum_{n=1}^{\infty} \frac{2}{a} H(t-\tau) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi \xi}{a}\right) e^{-n^2\pi^2(t-\tau)/a^2}.$$

<u>EXAMPLE</u> (METHOD OF IMAGES)

FIND AN INTEGRAL REPRESENTATION FOR THE SOLUTION TO

$$u_{xx} - u_t = g(x,t) \qquad 0 < x < \infty, \; t > 0$$

$$u(0,t) = 0 \qquad u(x,0) = u_0(x) \qquad u \to 0 \text{ AS } x \to \infty$$

FROM (4) PAGE H2 WE CAN DERIVE

$$-u(x,t) + \int_0^\infty \int_0^t G(\xi, t-\tau; x)\, g(\xi,\tau)\, d\xi\, d\tau = -\int_0^\infty G(\xi, t; x)\, u_0(\xi)\, d\xi$$

OR $\quad u(x,t) = -\int_0^\infty \int_0^t G(\xi, t-\tau; x)\, g(\xi,\tau)\, d\xi\, d\tau + \int_0^\infty G(\xi, t; x)\, u_0(\xi)\, d\xi$

WHERE $G(\xi, \sigma; x)$ SOLVES

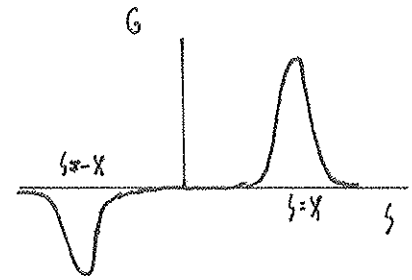
$$G_\sigma - G_{\xi\xi} = 0 \qquad 0 < \xi < \infty \qquad \sigma > 0$$

$$G = 0 \text{ ON } \xi = 0 \qquad G(\xi, 0; x) = \delta(\xi - x)$$

BY THE METHOD OF IMAGES WE PUT AN IMAGE CHARGE OF OPPOSITE SIGN AT $\xi = -x$ SO THAT WE WANT THE SOLUTION TO

$$G_\sigma - G_{\xi\xi} = 0 \qquad -\infty < \xi < \infty \qquad \sigma > 0$$

$$G(\xi, 0; x) = \delta(\xi - x) - \delta(\xi + x)$$



THIS WILL AUTOMATICALLY SATISFY $G(0, \sigma; x) = 0$

THUS BY SUPERPOSITION WE GET

$$G(\xi, \sigma; x) = \frac{1}{2\sqrt{\pi\sigma}} \left[e^{-(\xi - x)^2/4\sigma} - e^{-(\xi + x)^2/4\sigma} \right]$$

OR $\quad G(\xi, t-\tau; x) = \frac{1}{2\sqrt{\pi(t-\tau)}} \left[e^{-(\xi - x)^2/4(t-\tau)} - e^{-(\xi + x)^2/4(t-\tau)} \right]$

WHICH IS TO BE SUBSTITUTED IN THE EQUATION ABOVE.

PROBLEM

MORE SPACE DIMENSIONS

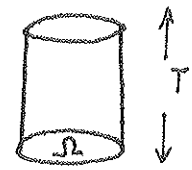
WE CONSIDER $D\Delta u - u_t = g(\underline{x},t)$ FOR $\underline{x} \in \Omega$

u GIVEN ON $\partial\Omega$

$u(\underline{x},0) = u_0(\underline{x})$

HERE Ω IS A DOMAIN IN $\mathbb{R}^N$. WE DEFINE Lu AND $L^x u$ BY

$$Lu \equiv D\Delta u - u_t \quad \text{AND} \quad L^x u = D\Delta u + u_t .$$



WE LET $C_T = \Omega \times [0,T]$ BE THE SPACE-TIME CYLINDER AS SHOWN.

WE LET $\underline{\xi}, \tau$ BE DUMMY VARIABLES FOR $\underline{x}, t$ AND WE INTEGRATE OVER C_T AS

$$\int_{C_T} v\,(D\Delta u - u_\tau)\,d\underline{\xi}\,d\tau = \int_0^T \left(\int_\Omega v D\Delta u \, d\underline{\xi} \right) d\tau - \int_\Omega \left(\int_0^T v u_\tau \, d\tau \right) d\underline{\xi}$$

NOW USE GREEN'S IDENTITY ON FIRST TERM AND $v u_\tau = (uv)_\tau - v_\tau u$.

WE GET

$$\int_{C_T} v\,Lu\,d\underline{\xi}\,d\tau = \int_0^T \left(\int_\Omega D u \Delta v \, d\underline{\xi} + \int_{\partial\Omega} D\left[v\partial_n u - u \partial_n v \right] ds \right) d\tau$$
$$- \int_\Omega v u \Big|_{\tau=0}^{T} d\underline{\xi} + \int_\Omega \left(\int_0^T u v_\tau \, d\underline{\xi} \right) d\tau .$$

THIS YIELDS,

$$(1) \quad \int_{C_T} v\,Lu\,d\underline{\xi}\,d\tau = \int_{C_T} u L^x v \, d\xi\, d\tau + D\int_0^T \int_{\partial\Omega} (v\partial_n u - u\partial_n v)\,ds\,d\tau$$
$$- \int_\Omega v u \Big|_{\tau=T} d\underline{\xi} + \int_\Omega v u \Big|_{\tau=0} d\underline{\xi} .$$

NOW WE CHOOSE v TO SATISFY

$$(2) \quad \begin{cases} L^x v = \delta(\xi - x, \tau - t) & \text{WITH } 0 < t < T \text{ AND } \underline{x} \text{ IN } \Omega . \\ v \equiv 0 \quad \text{FOR} \quad \tau > t & \text{CAUSALITY CONDITION} \\ v = 0 \quad \text{ON} \quad \partial\Omega . \end{cases}$$

THEN (1) BECOMES

$$\int_{C_t} v\, g(\underline{\xi},\tau)\,d\underline{\xi}\,d\tau = u(\underline{x},t) - D\int_0^T \left(\int_{\partial\Omega} u \partial_n v \, ds \right) d\tau + \int_\Omega v\, u_0(\underline{\xi})\, d\underline{\xi}$$

THIS YIELDS

$$(3) \qquad u(\underline{x}, t) = \int_0^t \int_\Omega v\, g(\underline{\xi}, \tau)\, d\underline{\xi}\, d\tau + D \int_0^t \left(\int_{\partial\Omega} u\, \partial_n v\, ds \right) d\tau - \int_\Omega v\big|_{\tau=0} u_0(\underline{\xi})\, d\underline{\xi}.$$

AS THE REPRESENTATION OF THE SOLUTION.

NOW THE PROBLEM FOR V IS

$$(4) \quad \begin{cases} V_\tau = -D \Delta V + \delta(\xi - x, \tau - t) & \text{IN } \Omega \times [\tau \le t] \\ V \equiv 0 \quad \text{FOR } \tau > t \\ V = 0 \quad \text{ON } \partial\Omega \end{cases}$$

NOW DEFINE $\sigma \equiv -\tau + t$ AND $G(\xi, \sigma) = -V[\xi, -\sigma + t]$

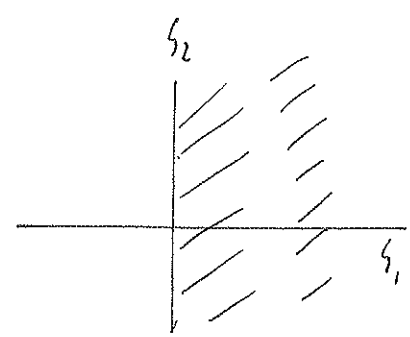
WE OBTAIN THAT

$$(5) \quad \begin{cases} G_\sigma = D \Delta G + \delta(\xi - x, \sigma) & \text{IN } \sigma \ge \sigma,\ \underline{\xi} \text{ IN } \Omega \\ G = 0 \quad \text{ON } \partial\Omega \\ G = 0 \quad \text{FOR } \sigma < 0. \end{cases}$$

THEN $V(\underline{\xi}, \tau) = -G[\underline{\xi}, t - \tau]$ IS TO BE USED IN (3).

WE SHOWED EARLIER THAT THIS PROBLEM IS <u>EQUIVALENT</u> TO THE FOLLOWING:

$$(6) \quad \begin{cases} G_\sigma = D \Delta G & \text{IN } \sigma > 0,\ \underline{\xi} \text{ IN } \Omega \\ G = 0 \quad \text{ON } \partial\Omega \\ G(\underline{\xi}, 0) = \delta(\underline{\xi} - \underline{x}) \end{cases}$$



NOW IF Ω IS THE HALF-SPACE $\xi_1 \ge 0,\ -\infty < \xi_2 < \infty$

THEN WE CAN USE THE METHOD OF IMAGES. WE THEN LET $\tilde{G}$ SATISFY THE FOLLOWING PROBLEM ON $\mathbb{R}^2$ WITH IMAGE - "CHARGE" AT $\underline{x}_i = (-x, y)$.