EIGENVALUE PROBLEMS

Consider the wave equation in some domain $D$ for $u(x,t)$

$$ \frac{\partial^2 u}{\partial t^2} - \nabla \cdot (p(x) \nabla u) - q(x) u = 0 \quad \text{where} \quad x \in \mathbb{R}^n$$

and

$$ \frac{\partial u}{\partial n} + b u = f(x) \quad \text{on} \quad \partial D \quad \text{if} \quad f(x) \geq 0, \quad p(x) > 0, \quad q(x) > 0, \quad b > 0.$$ We also could have a similar problem for a heat equation of the form

$$(2) \quad \frac{\partial u}{\partial t} = \nabla \cdot (p(x) \nabla u) - q(x) u$$

$$ \frac{\partial u}{\partial n} + b u = f(x) \quad \text{on} \quad \partial D$$

Now in (2) we look for separation of variables for the homogeneous version of (2) in the form

$$ u(x,t) = e^{-\lambda t} \phi(x)$$

Setting $f = 0$ in (2) gives the eigenvalue problem

$$ \nabla \cdot (p(x) \nabla \phi) - q(x) \phi + \lambda g(x) \phi = 0 \quad \text{in} \quad D$$

$$ \frac{\partial \phi}{\partial n} + b \phi = 0 \quad \text{on} \quad \partial D$$

(3)

For (1), it is more natural to expect vibration so we set

$$ u(x,t) = e^{-i\omega t} \phi(x)$$

which gives

$$ \nabla \cdot (p(x) \nabla \phi) - q(x) \phi = -\omega^2 g(x) \phi \quad \text{in} \quad D$$

$$ \frac{\partial \phi}{\partial n} + b \phi = 0 \quad \text{on} \quad \partial D$$

(4)

Setting $\lambda = \omega^2$ in (4) we get (3). We take (3) as the basic eigenvalue problem.

Goal: For (3), we want to find all values of $\lambda$ (called eigenvalues) for which (3) has a non-trivial solution. The non-trivial solutions are called eigenfunctions.

The eigenvalue problem is (analogous to one-dimensional Sturm-Liouville)

$$ \nabla \cdot (p(x) \nabla \phi) - q(x) \phi + \lambda g(x) \phi = 0 \quad \text{in} \quad D \quad \text{if} \quad f(x) \geq 0, \quad p(x) > 0, \quad q(x) > 0, \quad b > 0$$

What are the properties of the eigenvalues $\lambda_j$ and eigenfunction $\phi_j$?

1. There exists a countably infinite number of $\lambda_j$ with

$$ \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots \quad \text{with} \quad \lambda_j \to +\infty \quad \text{as} \quad j \to \infty.$$

2. Eigenvectors corresponding to different eigenvalues are orthogonal in the sense that

$$ \int_D \phi_j(x) \phi_k(x) g(x) \, dx = 0 \quad \text{for} \quad \lambda_j \neq \lambda_k.$$
We can then normalize the eigenfunctions by \((\phi_j, \phi_j) = \int_D \phi_j \phi_j^* \, dx = 1.\)

(ii) If \(\lambda_j\) is an eigenvalue of degree \(n\), there exists \(n\) independent eigenfunctions \(\phi_j(x), j = 1, \ldots, n\) which can be made into an orthogonal set by the Gram-Schmidt process. Then normalize.

(iii) The eigenfunctions are complete in the sense that any reasonable \(u(x)\) can be expanded as

\[
u(x) = \sum_{j=0}^{\infty} c_j \phi_j(x)
\]

by orthornormality, if \((\phi_j, \phi_j) = \int_D \phi_j \phi_j^* \, dx = 1\), then \(c_j = (u, \phi_j) = \int_D u \phi_j \, dx.\)

Completeness means that

\[
\lim_{n \to \infty} \| R_n \| = 0 \quad \text{where} \quad R_n(x) = \nu(x) - \sum_{j=1}^{n} c_j \phi_j(x) \quad \text{and} \quad \| R_n \|^2 = (R_n, R_n).
\]

(iv) If \(q(x) > 0\), then \(\lambda_j > 0.\) \(\Rightarrow\) All eigenvalues are \(> 0.\)

Remark (iv) At the moment we will assume that properties (ii), (ii) b and the completeness property hold. We will derive these later.

(b) We now derive (ii) a), the formula for \(c_j\) and (iv).

We first derive two key identities. Let \(u, v\) be any two functions. Then,

\[
\nabla \cdot [u \nabla \phi_j \nabla v] = \nabla \cdot [v \nabla \phi_j \nabla u] + \nabla \cdot [u \nabla \phi_j \nabla v]
\]

and

\[
\nabla \cdot [u \nabla \phi_j \nabla v] = \nabla \cdot [v \nabla \phi_j \nabla u] + \nabla \cdot [u \nabla \phi_j \nabla v]
\]

Subtracting and using the divergence theorem gives

\[
\int_D \left( \nabla \cdot [v \nabla \phi_j \nabla u] - q(x) u \right) \phi_j \, dx = \int_D \phi_j \left( \nabla \cdot \left( \frac{\partial u}{\partial n} + b u \right) - u \left( \frac{\partial v}{\partial n} + b v \right) \right) \, ds
\]

Define the operator \(L u = \nabla \cdot [p(x) \nabla u] - q(x) u.\) Then if \((u, v) = \int_D u v \, dx\)

\[
\int_D (v L u - u L v) \, dx = \int_D \phi_j \left( \nabla \cdot \left( \frac{\partial u}{\partial n} + b u \right) - u \left( \frac{\partial v}{\partial n} + b v \right) \right) \, ds
\]

\[
\int_D (v (v L u - u L v)) \, dx = \int_D \phi_j \left( \nabla \cdot \left( \frac{\partial u}{\partial n} + b u \right) - u \left( \frac{\partial v}{\partial n} + b v \right) \right) \, ds
\]
Now the eigenvalue problem (5) is
\[ L \phi = \nabla \cdot (p(x) \nabla \phi) - q(x) \phi = -\lambda \phi \quad \text{in } \Omega, \]
\[ \partial_\Omega \phi + \beta \phi = 0 \quad \text{on } \partial \Omega. \]

Now if \( \lambda_j \neq \lambda_k \) and \( \lambda_j \neq \lambda_k \), and \( \lambda_j \neq \lambda_k \), then from (6)
\[ L \lambda_j \phi_j = \lambda_j \phi_j \quad \text{and} \quad L \phi_j = \lambda_k \phi_k, \]
then for \( \phi_j \neq \phi_k \), the following orthogonality holds:
\[ \int_D \phi_j^T L \phi_k \, d\Omega = 0. \]

Therefore, if \( \lambda_j \neq \lambda_k \), then we have orthogonality
\[ \int_D \phi_j \phi_k \, d\Omega = 0. \]

This proves (ii) a).

Now we derive another identity, start with any eigenvalue and eigenfunction satisfying:
\[ L \phi = \lambda \phi = \nabla \cdot (p(x) \nabla \phi) - q(x) \phi = 0 \quad \text{in } D; \quad \text{with } \partial_\Omega \phi + \beta \phi = 0 \quad \text{on } \partial D \]

Multiply by \( \phi \):
\[ L \lambda \phi = \lambda^2 \phi = \nabla \cdot (p(x) \nabla \phi) - q(x) \phi = 0. \]

Now
\[ \nabla \cdot (p(x) \nabla \phi) = \nabla \cdot (p(x) \nabla \phi) + \phi \nabla \cdot (p(x) \nabla \phi) \]

Integrate over \( D \) and use the divergence theorem to get:
\[ \int_D \lambda \phi^2 \, d\Omega = \int_D \left[ p(x) \phi^2 + q(x) \right] \, d\Omega - \int_{\partial D} \phi p(x) \phi \, d\Omega \]
\[ \frac{\partial \phi}{\partial n} + \beta \phi = 0 \quad \text{on } \partial D. \]

Thus
\[ \int_D \lambda \phi^2 \, d\Omega = \int_D \left[ p(x) \phi^2 + q(x) \right] \, d\Omega + b \int_{\partial D} \phi \phi \, d\Omega \]

The formula (7) holds for any eigenvalue-eigenfunction pair. By definition \( \phi \) is a non-trivial solution and by assumption \( p(x) > 0 \), \( q(x) > 0 \) and \( b > 0 \).

Thus from (7) we conclude that
\[ \lambda_j > 0 \quad \text{for all } j = 1, \ldots \]

Thus for the heat conduction problem we expect that \( u(x,t) \to 0 \) as \( t \to \infty \).

Notice that from (7)
\[ \lambda_j > 0 \quad \text{if } b > 0 \text{ and } q(x) > 0 \]
we have \( \lambda_j > 0 \).
Now the expansion formula is clear:

\[ u(x) = \sum_{j=1}^{\infty} c_j \phi_j(x) \]

Multiply by \( \phi_k(x) \) \( g(x) \) and assume \( \langle \phi_j, \phi_j \rangle = \int_{\Omega} g(x) \phi_j^2(x) \, dx = 1 \), then

\[ \int_{\Omega} u(x) \phi_k(x) \, dx = \sum_{j=1}^{\infty} c_j \langle \phi_j, \phi_k \rangle = c_k \langle \phi_k, \phi_k \rangle = c_k \]

Therefore, \( c_k = \langle u, \phi_k \rangle \) and

\[ u(x) = \sum_{j=1}^{\infty} c_j \phi_j(x) \]

Remark a) Property (iiib) follows from the self-adjointness of the operator \( L \) which results in identity (6). Although, it is difficult to prove, we recall that if \( \lambda_j \) is a repeated eigenvalue for the matrix problem \( A \phi_j = \lambda_j \phi_j \), then \( \exists \) sufficient independent eigenvectors for \( \lambda_j \). These eigenvectors can be made into an orthonormal set using Gram-Schmidt process. (See homework).

b) We still need to show (ii), the existence of eigenvalue, and (iiii) completeness. This is done later.

Now consider the energy integral

\[ E(u) = \int_{D} \left( p(x) \left( \nabla u \right)^2 + q(x) u^2 \right) \, dx \]

Substitute \( u(x) = \sum_{j=1}^{\infty} c_j \phi_j(x) \) where \( \phi_j, \lambda_j \) is an eigenpair of

\[ L \phi_j = \lambda_j \phi_j \text{ in } D; \phi_j = 0 \text{ on } \partial D \text{ or } \partial_n \phi_j = 0 \text{ on } \partial D. \]

Assuming the existence of these eigenpairs and that the \( \phi_j \) are normalized by \( \langle \phi_j, \phi_j \rangle = \int_{D} \phi_j^2 \, dx = 1 \), we substitute to get

\[ E(u) = \int_{D} \left[ p(x) \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \lambda_j \lambda_k \phi_j \phi_k \, dx \right] \]

\[ E(u) = \int_{D} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \lambda_j \lambda_k \phi_j \phi_k \, dx = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \lambda_j \lambda_k \langle \phi_j, \phi_k \rangle \]

Now by orthonormality we have

\[ E(u) = \sum_{j=1}^{\infty} \lambda_j \left( \sum_{j} \phi_j \right)^2 \]
NOW ASSUME THAT \( \mathbf{U} \) IS NORMALIZED SO THAT
\[
\begin{align*}
(\mathbf{U}, \mathbf{U}) &= \int_D \sum_{j=1}^{N} \phi_j \phi_j \, dx = 1
\end{align*}
\]

THEN
\[
\begin{align*}
1 = (\mathbf{U}, \mathbf{U}) &= \sum_{j=1}^{N} \sum_{k=1}^{N} N_j N_k \int_D \phi_j \phi_k \, dx = \sum_{j=1}^{N} N_j^2
\end{align*}
\]

SUMMARIZING

LET'S ASSUME THAT THE EIGENVALUES \( \lambda_j \) AND \( \phi_j \) OF
\[
\begin{align*}
\int_D \phi_j \phi_j \, dx - \lambda_j \phi_j \phi_j &= 0
\end{align*}
\]
WITH \( \phi_j = 0 \) ON \( \partial D \) OR \( \phi_j = 0 \) ON \( \partial D \) EXIST.

ASSUME THAT \( \mathbf{U} = \sum_{j=1}^{N} N_j \phi_j \) IS ANY FUNCTION WITH \((\mathbf{U}, \mathbf{U}) = 1\). NOTE THAT \( \mathbf{U} \) SATISFIES THE B.C. \( \mathbf{U} = 0 \) OR \( \partial_0 \mathbf{U} = 0 \) ON \( \partial D \). THEN WE CALCULATE THAT
\[
E(\mathbf{U}) = \sum_{j=1}^{N} \lambda_j N_j^2 \quad \text{WITH} \quad \sum_{j=1}^{N} N_j^2 = 1
\]

NOTICE THAT BY COMPLETENESS \( E(\mathbf{U}) \) IS ANY FUNCTION SATISFYING \((\mathbf{U}, \mathbf{U}) = 1\).

IMPLICATION

1. \[
E(\mathbf{U}) = \sum_{j=1}^{N} \lambda_j N_j^2 = \sum_{j=1}^{N} \lambda_j \sum_{i=1}^{N} N_i^2 = \lambda_1
\]

Thus for all \( \mathbf{U}(\mathbf{X}) \) satisfying \((\mathbf{U}, \mathbf{U}) = 1 \) and \( \mathbf{U} = 0 \) on \( \partial D \) or \( \partial_0 \mathbf{U} = 0 \) on \( \partial D \) we have \( E(\mathbf{U}) \geq \lambda_1 \). Notice that if \( \mathbf{U} = \phi_1 \) then \( N_j = 0 \) for \( j > 1 \), \( N_1 = 1 \) and thus we have equality \( E(\phi_1) = \lambda_1 \).

2. \[
E(\mathbf{U}) = \sum_{j=1}^{N} \lambda_j N_j^2 = \sum_{j=1}^{N} \lambda_j \sum_{i=1}^{N} N_i^2 = \lambda_N
\]

Since \( \sum_{j=1}^{N} N_j^2 = 1 \). Thus \( E(\mathbf{U}) \geq \lambda_N \). Notice that we have equality when \( \mathbf{U} = \phi_N \).

REMARK

If \( \partial_0 \phi + b \phi = 0 \) on \( \partial D \), the energy integral should be replaced with
\[
E(\mathbf{U}) = \int_D \left( \rho_0 |\mathbf{U}|^2 + 
\int_{\partial D} \rho(\mathbf{X}) |\mathbf{U}|^2 \, dx + b \int_{\partial D} \rho(\mathbf{X}) \phi \mathbf{U} \, dx \right) dx.
\]
THEOREM ASSUME THE EXISTENCE OF THE EIGENPAIRS $\lambda_j, \phi_j$ OF \( L \phi_j = -\lambda_j \phi_j \)
with $\phi_j \neq 0$ on $\partial D$ and $\partial_n \phi_j = 0$ on $\partial D$ and assume that $\lambda_1 \leq \lambda_2 \leq \ldots$ and that the completeness property holds. Define

\[ E(\nu) = \int_0^1 \int_D (\text{tr}(\nu^2 + q \nu^2) \, \phi_j, \nu) \, \text{d}x, \]

and let $\nu$ be any function satisfying

(a) $\nu = 0$ or $\partial_n \nu = 0$ on $\partial D$.

(b) $\int_0^1 \int_D \nu^2 \, \text{d}x = 1$.

(c) $(\nu, \phi_j) = 0$ for $j = 1, \ldots, n - 1$.

Then

1) $E(\nu) \geq \lambda_n$.

If $n = 1$, the condition (c) is absent.

2) Equality holds if $\nu = \phi_n(x)$ and then $E(\nu) = 0$.

REMARK (1) We do not need to normalize $\nu$ by $(\nu, \nu) = 1$ if we modify the theorem appropriately. For instance, suppose $\nu$ satisfies (a) and (c) but not (b). Then $\nu = W/\|W\|$ with $\|W\| = (W, W)^{1/2}$ satisfies (a), (b) and (c). Thus

\[ E(\nu) = \frac{E(W)}{\|W\|^2} = \frac{E(W)}{(W, W)} \]

Then theorem applies if we replace $E(\nu)$ by $\frac{E(W)}{(W, W)}$.

i.e. the minima of $E(W)/(W, W)$ with $W$ satisfying (a) and (c) will lead to the eigenvalue $\lambda_n$. The function $W$ that minimizes $E(W)/(W, W)$ is a multiple of $\phi_n$. 
Notice that to get a bound on \( \lambda_n \) we must find the eigenfunctions \( \phi_1, \ldots, \phi_{n-1} \) so that we can find the trial jet for which \( \int_{D} D \phi_j \ dx = 0 \) for \( j = 1, \ldots, n-1 \).

We now formulate the constant maximum-minimum principle that yields \( \lambda_n \) without knowing \( \phi_1, \ldots, \phi_{n-1} \). To determine \( \lambda_n \) we consider a collection of functions \( \{ m_1(x), \ldots, m_{n-1}(x) \} \).

Theorem (Min-Max Principle)

Let \( w \) be any function satisfying \( w = 0 \) on \( \partial D \) or in \( D \) and the \( n-1 \) conditions

\[
(\nabla w, \nabla m_j) = 0 \quad \text{for} \quad j = 1, \ldots, n-1
\]

Then we claim that \( \lambda_n = \max \left[ \min_{w} \frac{E(w)}{(w, w)} \right] \max \text{-min principle} \).

Remark

In other words:

For fixed \( m_1, \ldots, m_{n-1} \), find \( w \) such that \( \frac{E(w)}{(w, w)} \) is minimized over all those functions \( w \) for which \( w = 0 \) (or \( \partial_{D} w = 0 \)) on \( \partial D \) and \( (w, \nabla m_j) = 0 \) for \( j = 1, \ldots, n-1 \). Then we vary all these minima over all possible sets of functions \( \{m_1, \ldots, m_{n-1}\} \) and extract the maximum value.

Proof

Let \( \{m_1(x), \ldots, m_{n-1}(x)\} \) be an arbitrary set of functions. Define \( w^k = \sum_{j=1}^{n} c_j \phi_j \).

Then \( w^k \) is admissible if \( w^k = 0 \) (or \( \partial_D w^k = 0 \)) on \( \partial D \) (which is trivial) and

\[
(\nabla w^k, \nabla m_j) = 0 \quad \text{for} \quad j = 1, \ldots, n-1
\]

Thus we need that \( \sum_{j=1}^{n} c_j (m_j, \phi_j) = 0 \) for \( k = 1, \ldots, n-1 \). This is an underdetermined system i.e. an \( n-1 \) by \( n \) matrix. Thus \( \exists \) a solution \( c_1, \ldots, c_n \) with at least one \( c_j \) arbitrary. Therefore we can satisfy the additional condition \( \sum_{j=1}^{n} c_j^2 = (w^k, w^k) = 1 \).

Therefore \( w^k \) is admissible and from the bottom of (E4) we have

\[
\frac{E(w^k)}{(w^k, w^k)} = \sum_{j=1}^{n} c_j^2 \lambda_j \leq \lambda_n \sum_{j=1}^{n} c_j^2 = \lambda_n.
\]

Therefore

\[
\frac{E(w^k)}{(w^k, w^k)} \leq \lambda_n
\]
Now since \( w \) is a \textit{specific} admissible function, then
\[
(\star) \quad \min_{w} \frac{E(w)}{(w, w)} \leq \frac{E(w^{*})}{(w^{*}, w^{*})} \leq \lambda_{0}
\]

Finally since \( \{ m_{1}, \ldots, m_{n-1} \} \) is arbitrary jet then (\( \star \)) must hold for the jet that maximizes the \( \min. \) Hence
\[
(+) \quad \max_{m_{1}, \ldots, m_{n-1}} \min_{w} \frac{E(w)}{(w, w)} \leq \lambda_{0}
\]

To conclude the proof we must show that we can achieve equality in (+).

This is clear if we take \( \{ m_{1}, \ldots, m_{n-1} \} = \{ \phi_{1}, \ldots, \phi_{n-1} \} \) and \( w = \phi_{n} \). Thus
\[
\max_{m_{1}, \ldots, m_{n-1}} \min_{w} \frac{E(w)}{(w, w)} = \lambda_{0}
\]
which completes the proof.

\textbf{Example: Consider}
\[
\phi_{xx} + \phi_{yy} + \lambda \phi = 0 \quad \text{in} \quad 0 < x < \pi, \quad 0 < y < \pi
\]
\[
\phi = 0 \quad \text{on boundary of square } D.
\]

The Rayleigh quotient with \( p(x) = 1, \quad q(x) = 0, \quad r(x) = 1 \) is
\[
E(w) = \frac{\int_{0}^{\pi} \int_{0}^{\pi} \left| \nabla w \right|^{2} \, dx \, dy}{\int_{0}^{\pi} \int_{0}^{\pi} w^{2} \, dx \, dy}
\]

We want to get an upper bound on the first eigenvalue \( \lambda_{1} \).
\[
\lambda_{1} = \min_{w} \frac{E(w)}{(w, w)} \leq \frac{E(w^{*})}{(w^{*}, w^{*})} \quad \text{for any } w^{*} \text{ that satisfies } \ w^{*} = 0 \text{ on boundary of } D.
\]

Now choose
\[
w^{*} = A xy (\pi - x) (\pi - y)
\]

Then
\[
\langle w^{*}, w^{*} \rangle = A^{2} \left[ \int_{0}^{\pi} y^{2} (\pi - y)^{2} \, dy \right]^{2} = 1 \quad \int_{0}^{\pi} y^{2} (\pi - y)^{2} \, dy = \pi^{5}/30
\]

Thus
\[
A = 30 \sqrt{\frac{5}{\pi}} \Rightarrow w^{*} = \frac{30}{\pi^{5}} \cdot x y (\pi - x) (\pi - y)
\]

Now
\[
E(w^{*}) = \frac{\pi^{4}}{100} \left[ \int_{0}^{\pi} \left[ x^{2} (\pi - x)^{2} (\pi - 2y)^{2} + y^{2} (\pi - y)^{2} (\pi - 2x)^{2} \right] \, dx \, dy = 2.03
\]

Thus \( \lambda_{1} < 2.03 \) and \( w^{*} \) approximates the first eigenfunction.
Now find eigenvalues explicitly.

Try \( \phi(x, y) = \sin(mx) \sin(ny) \) then \( \phi = 0 \) on square

\[ \phi_{xx} + \phi_{yy} = -\left(m^2 + n^2\right) \sin(mx) \sin(ny) = -\lambda \sin(mx) \sin(ny) \]

\[ \Rightarrow \lambda = m^2 + n^2 \quad m, n = 1, 2, 3, \ldots \]

The first eigenvalue is \( \lambda_1 = 2 \). The bound was \( \lambda_1 \leq 2.03 \). We claim that the function \( w^k \) does not approximate \( \phi_1 \) as well. This can be done by comparing in the \( || \cdot || \) norm.

In the derivation above we have assumed the existence of \( \lambda_j, \phi_j \) and the completeness of \( \phi_j \). We now show that the existence of eigenvalues arises from a variational problem. The eigenvalue problem is

\[ \nabla \cdot (p \nabla \phi) = \lambda r(x) \phi \quad \text{in} \quad D; \quad \partial_n \phi + b \phi = 0 \quad \text{on} \quad \partial D \]

**Theorem.** Consider the variational problem

\[ \min_{w \in H} \frac{E(w)}{(w, w)} \quad \text{where} \quad (w, w) = \int_D w^2 \, r(x) \, dx \quad p \geq 0 \quad q \geq 0 \quad b \leq 0 \]

\[ E(w) = \int_D \left( b \left| \nabla w \right|^2 + q w^2 \right) \, dx + \int_{\partial D} b p w^2 \, ds \]

The trial function space \( H \) are those \( w \) for which \( w \) is continuous with piecewise continous derivatives.

(i) \( (w, \phi_1) = (w, \phi_2) = \ldots = (w, \phi_{n-1}) = 0 \)

(ii) If \( b = 0 \), then we require that \( w = 0 \) on \( \partial D \)

The conclusion is that

\[ \lambda_n = \min_{w \in H} \frac{E(w)}{(w, w)} \quad \text{and that the minimizing function is} \quad w = \phi_{n}(x) \quad (\text{the } n\text{th eigenfunction}) \]

**Remark.**

(1) If we are to determine \( \lambda \), the condition (ii) is valuable.

(2) Notice that for any boundary condition other than \( w = 0 \) on \( \partial D \), we do not require that the trial functions satisfy \( \partial_n w + b w = 0 \). It will be guaranteed that the minimizing function will.
PROOF 

Assume that \( w^* \) is the minimizing function. Then \( w^* \in H \). Consider an \( \varepsilon \) neighborhood of \( w^* \) and \( w = w^* + \varepsilon W \) where \( W \in H \). Thus \( w \in H \).

A necessary condition for \( E(w)/\{w, w\} \) to be minimized at \( w^* \) is that

\[
\frac{d}{d\varepsilon} \left. I(\varepsilon) \right|_{\varepsilon = 0} = 0
\]

where

\[
I(\varepsilon) = \int_D \left[ \left( p \nabla (w^* + \varepsilon W) \right)^2 + b \left( w^* + \varepsilon W \right)^2 \right] \, dx + \int_D \frac{b}{||w^*||^2} \, ds
\]

Now

\[
\frac{d}{d\varepsilon} \left. I(\varepsilon) \right|_{\varepsilon = 0} = \frac{\partial E}{\partial \varepsilon} - \frac{\partial E}{\partial ||W||^2} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon = 0}
\]

\[
G = \int_D \left( w^* + \varepsilon W \right)^2 \, dx, \quad G^{\prime}\left|_{\varepsilon = 0} = 2 \int_D w^* W \, dx, \quad G\left|_{\varepsilon = 0} = \left( w^*, w^* \right) = ||w^*||^2, \right.
\]

\[
E = \int_D \left[ \left( p \nabla (w^* + \varepsilon W) \cdot \nabla (w^* + \varepsilon W) + b \left( w^* + \varepsilon W \right)^2 \right] \, dx + \int_{\partial D} b \left( w^* + \varepsilon W \right)^2 \, ds
\]

\[
E^{\prime}\left|_{\varepsilon = 0} = \int_D \left( 2 p \nabla w^* \cdot \nabla W + 2 b w^* W \right) \, dx
\]

Substitute in (1) to get

\[
I^{\prime}(\varepsilon) \left|_{\varepsilon = 0} = \frac{2}{||w^*||^2} \left[ \int_D \left( p \nabla w^* \cdot \nabla W + b w^* W \right) \, dx - \frac{E(w^*)}{||w^*||^2} \int_D w^* W \, dx + \int_{\partial D} b \left( w^* \right)^2 \, ds \right]
\]

Rewriting gives

\[
I^{\prime}(\varepsilon) \left|_{\varepsilon = 0} = \frac{2}{||w^*||^2} \left[ \int_D \left( p \nabla w^* \cdot \nabla W + b w^* W - \frac{E(w^*)}{||w^*||^2} \left( \nabla w^* \cdot W \right) \right) \, dx \right.
\]

Now

\[
\nabla \cdot \left( p \nabla \cdot w^* \right) = \nabla \cdot \left( p \nabla w^* \right) + p \nabla \cdot \nabla w^*
\]

This gives

\[
\int_D p \nabla w^* \cdot \nabla W \, dx = - \int_D W \cdot \nabla \left( p \nabla w^* \right) \, dx + \int_{\partial D} p \nabla \cdot w^* \, ds
\]

so

\[
I^{\prime}(\varepsilon) \left|_{\varepsilon = 0} = \frac{2}{||w^*||^2} \left[ \int_D \left[ \nabla \cdot \left( p \nabla w^* \right) + b w^* - \frac{E(w^*)}{||w^*||^2} \right] W \, dx + \int_{\partial D} p \left( \frac{\partial w^*}{\partial n} + b w^* \right) \, ds \right]
\]

Now a necessary condition for \( E(w)/\{w, w\} \) to be a minimizer is that \( I^{\prime}(\varepsilon) \left|_{\varepsilon = 0} = 0 \)

For all \( W \in H \):

\[
\int_D \left[ \nabla \cdot \left( p \nabla w^* \right) + b w^* - \frac{E(w^*)}{||w^*||^2} \right] W \, dx + \int_{\partial D} p \left( \frac{\partial w^*}{\partial n} + b w^* \right) \, ds = 0
\]
**Lemma** Suppose that \( \int_D F \varphi \, dx = 0 \) for a continuous function \( F \) and for arbitrary functions \( \varphi \). Then \( \varphi \) must be zero on \( \Omega \).

**Proof:** Assume that \( \exists x_0 \in \Omega \) s.t. \( F(x_0) \neq 0 \). Then, w.l.o.g., \( F(x_0) > 0 \) and by continuity \( \exists \) a neighborhood near \( x_0 \) where \( F(x) > 0 \). Let \( \varphi \) be a function which is \( 1 \) inside the neighborhood and zero outside. Then \( \int_D F \varphi \, dx > 0 \) which is a contradiction.

Now in (2) let \( \varphi \) be a function in \( \mathcal{H} \) with \( \varphi = 0 \) on \( \partial D \). Then assuming continuity of \( \varphi^k \) and its first derivatives, the lemma gives

\[
- \nabla \cdot (p \nabla \varphi^k) + q \varphi^k - \frac{E(\varphi^k)}{||\varphi^k||^2} \varphi^k = 0 \quad \text{in} \quad D
\]

The integral in (2) over \( D \) then vanishes.

Now we are left with

\[
\int_D p \left( \frac{\partial \varphi^k}{\partial n} + b \varphi^k \right) \varphi \, dx = 0
\]

Since this must hold for all \( \varphi \in \mathcal{H} \) (without \( \varphi \) satisfying any condition on \( \partial D \)), we must have that \( \frac{\partial \varphi^k}{\partial n} + b \varphi^k = 0 \). Therefore, the minimizer satisfies

\[
- \nabla \cdot (p \nabla \varphi^k) + q \varphi^k = -\frac{E(\varphi^k)}{||\varphi^k||^2} \varphi^k \quad \text{in} \quad D
\]

\[
\frac{\partial \varphi^k}{\partial n} + b \varphi^k = 0 \quad \text{on} \quad \partial D
\]

In view of the constraints (i), \( A_0 = E(\varphi^k)/||\varphi^k||^2 \) (see Eq.), and \( \varphi^k = \varphi_0 \).

**Remark:** Suppose \( b = \infty \) then the appropriate functional \( E(\varphi) \) is

\[
E(\varphi) = \int_D \left( p \left| \nabla \varphi \right|^2 + q \varphi^2 \right) \, dx
\]

\( b = \infty \Rightarrow \text{Dirichlet boundary condition} \).

Repeating the proof, we obtain in place of (2)

\[
\int_D \left[ - \nabla \cdot (p \nabla \varphi^k) + q \varphi^k - \frac{E(\varphi^k)}{||\varphi^k||^2} \varphi^k \right] \varphi \, dx + \int_{\partial D} p \frac{\partial \varphi^k}{\partial n} \varphi \, ds = 0
\]

Now \( \varphi = 0 \) on \( \partial D \) (the condition (iii) for admissibility) eliminates the surface integral. Thus

\[
- \nabla \cdot (p \nabla \varphi^k) + q \varphi^k = -\frac{E(\varphi^k)}{||\varphi^k||^2} \varphi^k \quad \text{in} \quad D \quad \text{if} \quad \varphi^k = 0 \quad \text{on} \quad \partial D.
\]
If we did not impose in $H$ that $w = 0$ on $\partial D$ when $b = 0$ then we would conclude that $\partial w/\partial n = 0$ on $\partial D$, which would lead to the wrong problem.

**Unsought** the eigenvalues for

$$
\nabla \cdot [p \nabla \phi] - b \phi = -\lambda \phi \quad \text{in } D
$$

$$
\frac{\partial \phi}{\partial n} + b \phi = 0 \quad \text{on } \partial D
$$

are given by the variational principle. If $b = 0$ then in the admissibility condition must impose $w = 0$ on $\partial D$. If $0 \leq b \leq \alpha$ we do not need to impose that $w$ satisfy the condition $\partial w/\partial n + b w = 0$. The minimizing function $w^x$ will automatically (or naturally) satisfy this condition. Thus we say that $\partial w/\partial n + b w = 0$ on $\partial D$ with $b \rightarrow 0$ is a natural boundary condition.

Remark by showing that the minimizer exist we can conclude the existence of the eigenvalues and eigenfunctions.

Since the trial space $H$ becomes more restrictive as $\alpha$ increases the inequalities $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ are clearly true.

What we will be able to show from this variational principle are

(i) $\lambda_n \rightarrow \alpha$ as $n \rightarrow \infty$

(ii) Eigenfunction form a complete set.

We do this below after looking at some more practical things.

**Bounds on Eigenvalues**

Consider the eigenvalue problem

$$
\nabla \cdot [p \nabla \phi] - b \phi = -\lambda \phi \quad \text{in } D
$$

$$
\phi = 0 \quad \text{on } \partial D
$$

Then

$$
\lambda_n = \max \left[ \min_{m_1, \ldots, m_n} \frac{E(w)}{||w||^2} \right] \quad \text{in the set of functions for which}
$$

$$
H \quad \text{in } D, \quad \frac{\partial w}{\partial n} + b w = 0 \quad \text{on } \partial D
$$

$$
\rho > 0, \quad \gamma > 0, \quad \beta > 0 \quad \text{in } D
$$

(iii) $(w, m) = (w, m_1) = \ldots = (w, m_n) = 0$. 

Suppose that the \( \lambda_j \) and \( \phi_j \) are calculated from the variational principle on P. E. Q. Then we want to show that, for \( C_j = (\phi_j, u) \),
\[
R_n(x) = u(x) - \sum_{j=1}^{n} C_j \phi_j(x) \to 0 \quad \text{as} \quad n \to \infty
\]
in the norm \( \| \cdot \| \). In other words,
\[
\| R_n(x) \| \to 0 \quad \text{as} \quad n \to \infty.
\]
This is the property of completeness.

Now calculate
\[
( R_n, \phi_j ) = ( u - \sum_{k=1}^{n} c_k \phi_k, \phi_j ) = ( u, \phi_j ) - c_j = 0 \quad \text{for} \quad j = 1, \ldots, n
\]
and since \( ( R_n, \phi_j ) = 0 \) for \( j = 1, \ldots, n \), \( R_n \) can be used to give a bound on \( \lambda_{n+1} \).

\[
\lambda_{n+1} \leq \frac{E(R_n)}{\| R_n \|^2}
\]

\[
\to \quad \| R_n \|^2 \leq \frac{E(R_n)}{\lambda_{n+1}}
\]

We already know from (15) that \( \lambda_{n+1} \to 0 \) as \( n \to \infty \) by bounding the geometry. Therefore to show that \( \| R_n \| \to 0 \) we need to show that \( E(R_n) \) is bounded as \( n \to \infty \).

Now (15) \( E(u) = E[ R_n + \sum_{k=1}^{n} c_k \phi_k ] = E(R_n) + E[ \sum_{k=1}^{n} c_k \phi_k ] + 2 \sum_{k=1}^{n} c_k \int_{D} (p \nabla R_n \cdot \nabla \phi_k + q R_n \phi_k) dx
\]

This equality follows from
\[
E(w + v) = \int_{D} \left[ p \left( \nabla (w + v) \cdot \nabla (w + v) + q (w + v)^2 \right) \right] dx
\]
\[
= \int_{D} \left[ p \nabla w \cdot \nabla u + q w^2 \right] dx + \int_{D} \left[ p \nabla v \cdot \nabla v + q v^2 \right] dx + \int_{D} \left[ 2 p \nabla w \cdot \nabla v + 2 q w v \right] dx
\]

\[
E(w + v) = E(w) + E(v) + 2 \int_{D} \left[ p \nabla w \cdot \nabla v + q w v \right] dx
\]
Choosing \( \mathbf{w} = \mathbf{R}_n \) and \( \mathbf{v} = \sum_{j=1}^{\infty} c_j \phi_j \) gives (†).

Now \( E \left( \sum_{k=1}^{n} c_k \phi_k \right) = \sum_{k=1}^{n} c_k^2 \lambda_k \) as shown on page 45.

Now \( \int_D \left( p \nabla \phi_k + q \phi_k \right) \cdot \nabla \mathbf{v} \, d\mathbf{x} = \int_D \left( -\nabla \cdot \left( p \nabla \phi_k \right) + q \phi_k \right) \mathbf{R}_n \cdot \nabla \mathbf{v} \, d\mathbf{x} + \int_{\partial D} \mathbf{R}_n \cdot \frac{\partial \phi_k}{\partial n} \, d\mathbf{s} \)

By \( \mathbf{w} = \mathbf{R}_n \mathbf{v} \cdot \nabla \phi_k = \mathbf{R}_n \mathbf{v} \cdot \left( p \nabla \phi_k \right) + p \nabla \mathbf{R}_n \cdot \nabla \phi_k \)

The boundary term cancels since either \( \mathbf{R}_n = 0 \) on \( \partial D \) (Dirichlet) or \( \partial \phi_k / \partial n = 0 \) on \( \partial D \) (Neumann).

Then \( \int_D \left( p \nabla \phi_k \cdot \nabla \mathbf{v} + q \phi_k \mathbf{v} \cdot \nabla \phi_k \right) \, d\mathbf{x} = \int_D \lambda_k \phi_k \mathbf{R}_n \cdot \nabla \mathbf{v} \, d\mathbf{x} = 0 \) for \( k = 1, \ldots, n \).

Putting this information in (†) gives

\[
E(\mathbf{u}) = E(\mathbf{R}_n) + \sum_{k=1}^{n} \lambda_k c_k^2
\]

Or \( -\sum_{k=1}^{n} \lambda_k c_k^2 + E(\mathbf{u}) = E(\mathbf{R}_n) \)

Thus \( E(\mathbf{R}_n) \leq E(\mathbf{u}) \) since \( \lambda_k \geq 0 \).

Since \( \mathbf{u} \) is fixed, this means \( E(\mathbf{R}_n) \) is bounded

\[
\| \mathbf{R}_n \|^2 \leq \frac{E(\mathbf{R}_n)}{\lambda_{n+1}} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
\]

Completeness.
A) Suppose that in \( D \)
\[ 0 < r_{\text{min}} \leq r \leq r_{\text{max}} \]
\[ 0 < \varphi_{\text{min}} \leq \varphi \leq \varphi_{\text{max}} \]

Then if
\[ E_{\text{max}}(w) \equiv \int_D (p_{\text{max}}|\nabla w|^2 + q_{\text{max}}w^2) \, dx, \quad \|w\|_{\text{max}}^2 \equiv \int_D r_{\text{max}}w^2 \, dx \]
\[ E_{\text{min}}(w) \equiv \int_D (p_{\text{min}}|\nabla w|^2 + q_{\text{min}}w^2) \, dx, \quad \|w\|_{\text{min}}^2 \equiv \int_D r_{\text{min}}w^2 \, dx \]

It is clear that for any function \( w \) in \( H \) that (see \( \psi \) below)
\[ \frac{E_{\text{max}}(w)}{\|w\|_{\text{min}}^2} \geq \frac{E(w)}{\|w\|_{\text{min}}^2} \geq \frac{E_{\text{min}}(w)}{\|w\|_{\text{max}}^2} \]

Thus if \( \lambda_{n,\text{max}} \) is the \( n \)-th eigenvalue for
\[ (1) \quad \nabla \cdot [p_{\text{max}} \nabla \psi] - q_{\text{max}} \psi = -\lambda_{n,\text{max}} r_{\text{min}} \psi \]
\[ \psi = 0 \quad \text{on} \quad \partial D \]

and if \( \lambda_{n,\text{min}} \) is the \( n \)-th eigenvalue for
\[ (2) \quad \nabla \cdot [p_{\text{min}} \nabla \phi] - q_{\text{min}} \phi = -\lambda_{n,\text{min}} r_{\text{max}} \phi \]
\[ \phi = 0 \quad \text{on} \quad \partial D \]

The min-max principle gives that
\[ (3) \quad \lambda_{n,\text{min}} \leq \lambda_n \leq \lambda_{n,\text{max}} \]

(3) Notice that if \( r \) is replaced by \( r_{\text{max}} \) the admissibility criteria seemingly should be changed to \( \int_D r_{\text{max}} w_m \psi \, dx = 0 \) for \( j = 1, \ldots, n-1 \).

This can be re-written as \( \int_D \frac{r_{\text{max}}}{r} w \left( \frac{r_{\text{max}}}{r} m_j \right) \psi \, dx = 0 \), since \( m_j \)

are arbitrary functions the admissibility criteria are the same!

This would not have been the case if we did not use the max-min principle and instead required that \( \int_D w \psi_j \, dx = 0 \) for \( j = 1, \ldots, n-1 \).

As we will show, \( (3) \) is useful since the problems determining \( \lambda_{n,\text{min}} \) and \( \lambda_{n,\text{max}} \) have constant coefficients and hence may be explicitly solvable.
B) Let \( \tilde{D} \) be a subregion of \( D \), where part of the boundary of \( D \) and \( \tilde{D} \) may coincide.

Consider \( E(w) \) for the max-min variational problem in \( D \).

where \( w \) satisfies

1. \( w = 0 \) on \( \partial D \)
2. \( \int_{\partial \tilde{D}} w \partial_{n_j} \phi = 0 \) for \( j = 1, \ldots, n-1 \).

Suppose that in addition to satisfying 1, \( \tilde{w} \) also satisfies

3. \( \tilde{w} = 0 \) on \( \partial \tilde{D} \)
4. \( \tilde{w} \neq 0 \) in the region exterior to \( \tilde{D} \) but inside \( D \).

then if \( \tilde{w} \) satisfies 1-4, it is clear that \( \tilde{w} \) is admissible for the variational problem in \( \tilde{D} \). However, since the set of functions satisfying 1-4 are more restrictive than the set of functions satisfying only 1 and 2, it follows that the max-min for the problem in \( \tilde{D} \) cannot be smaller than for the max-min problem in \( D \).

Thus

\[ \lambda_n \leq \tilde{\lambda}_0 \]

Thus if the area \( \lambda_n \) the \( n \)th eigenvalue cannot decrease.

Application consider \( \Delta \phi + \lambda \phi = 0 \) in a rectangular domain \( D_0 \) with a notch as shown

\[ b \]

\[ o \]

\[ a+h \]

Let \( D_{\text{max}} \) be the rectangle \( 0 < x < a+h, 0 < y < b \)

Let \( D_{\text{min}} \) be the rectangle \( 0 < x < a, 0 < y < b \)

Then applying B) we get

\[ \lambda_n(D_{\text{max}}) \leq \lambda_n(D_0) \leq \lambda_n(D_{\text{min}}) \]

We then can calculate \( \lambda_n(D_{\text{min}}) \) and \( \lambda_n(D_{\text{max}}) \) explicitly.
C) We now want to combine A) and B) to show that for the eigenvalue problem
\[ \nabla \cdot [ \rho(x) \nabla \phi ] - \eta(x) \phi = -\lambda \Gamma(x) \phi \quad \text{in } D \]
\[ \rho(x) \phi > 0 \]
\[ \phi = 0 \quad \text{on } \partial D \]
\[ \Gamma(x) > 0 \]
that \( \lambda_n \to \infty \) as \( n \to \infty \).

By combining A) and B) it is clear that for a region \( D \) containing \( D \), \( \lambda_{n, \min} \) satisfies (recall \( \min \) mean \( \rho \to \rho_{\min}, \eta \to \eta_{\min}, \Gamma \to \Gamma_{\max} \))
\[ \lambda_{n, \min} (D) \to \infty \quad \text{as } n \to \infty \] by explicit calculation.

Thus \( \lambda_n \to \infty \) as \( n \to \infty \).

Very slick!!

General Principles

1) By strengthening the condition in a minimum problem (restricting the trial set) we do not diminish the value of the minimum. Conversely by weakening the condition the minimum does not increase.

2) Given two minimum problems, with the same class \( H \) of admissible functions, such that \( \forall \omega \in H \) the functional to be minimized is no smaller in the first problem than in the second, then the minimum for the first problem is also no smaller than the minimum for the second problem.
**Examples on Bounds on Eigenvalues**

**Example 1**

\[ \Delta u + \lambda u = 0 \quad 0 < x < a \]

\[ u(1, q) = 0 \]

If we separate variables \( a \) in HW 8 & Prob. 4 we get

\[ u(r, \varphi) = J_0(\lambda \sqrt{a}) J_0(z_0, \varphi) \quad \lambda \geq \frac{\pi}{2} \]

\[ u(r, \varphi) = J_0(\lambda \sqrt{a}) \cos \varphi \quad \text{or} \quad J_0(\lambda \sqrt{a}) \sin \varphi \]

\[ J_0(z_0, \varphi) = 0 \quad \Rightarrow \lambda \geq \frac{n \pi}{2} \quad n = 0, 1, \ldots \]

\[ J_0(\lambda \sqrt{a}) = 0 \quad \Rightarrow \lambda = \frac{n \pi}{2} \quad \text{or} \quad \lambda = \frac{(2n + 1) \pi}{2} \]

It is a general fact that the eigenfunction for the first eigenvalue is of one sign inside the domain. Thus the lowest eigenvalue \( \lambda_0 \) is

\[ \lambda_0 = \frac{(z_0, D)^2}{\pi^2} \]

where \( z_0, 0 \) is the smallest root of \( J_0(z) = 0 \). It turns out that \( z_0, 0 \approx 2.4 \).

Thus \( \lambda_0 \approx \frac{5.76}{\pi^2} \).

Now consider getting an approximation to \( \lambda_0 \) by bounding the geometry.

Let \( D_{\text{small}} \) and \( D_{\text{big}} \) be the domains \( a \) shown. Let \( a = 1 \).

Then \( D_{\text{big}} \) is \(-1 < x < 1, -1 < y < 1\)

\[ \lambda_0(D_{\text{big}}) = \frac{\pi^2}{4} + \frac{\pi^2}{4} = \frac{\pi^2}{2} \approx 4.93. \]

Now \( D_{\text{small}} \) is \(-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} < y < \frac{1}{\sqrt{2}}\)

\[ \lambda_0(D_{\text{small}}) = \frac{\pi^2}{2} + \frac{\pi^2}{2} = 9.87. \]

Thus \( \lambda_0(D_{\text{big}}) < \lambda_0(D) < \lambda_0(D_{\text{small}}) \).

\[ 4.93 < \lambda_0(D) < 9.87 \]

whereas \( \lambda_0(D) \approx 5.76 \).

Now can we get a better bound for an upper bound?

Try the Rayleigh Quotient

\[ \lambda_0 = \min \left[ \frac{\int_D \| \nabla w \|^2 \, dx}{\int_D w^2 \, dx} \right] \quad w \in \mathcal{H} \quad \text{subject to} \quad w = 0 \quad \text{on} \quad \partial D. \]
Now since we expect that $\phi_0$ is radial and of one sign we can try either

(i) \[ \psi = 1 - r^2 \]  

Notice $\psi = 0$ on $r = 1$.

(ii) \[ \psi = \cos \left( \frac{\pi r}{2} \right) \]

Then \[ \lambda_0 \leq \frac{\int_0^1 \psi r^2 \, dr}{\int_0^1 \psi^2 \, dr} \]

Now for (i) we have \[ \psi = 1 - r^2, \quad \psi_r = -2r, \quad \psi_r^2 = 4r^2 \] so that

\[ \lambda_0 \leq \frac{\int_0^1 4r^3 \, dr}{\int_0^1 (1 - r^2)^2 \, dr} = 6 \]

Now for (ii) we have \[ \psi = \cos \left( \frac{\pi r}{2} \right) \sin \left( \frac{\pi r}{2} \right) \]

\[ \lambda_0 \leq \frac{\int_0^1 \frac{\pi^2}{4} r^2 \sin^2 \left( \frac{\pi r}{2} \right) \, dr}{\int_0^1 \frac{\pi^2}{4} r \cos^2 \left( \frac{\pi r}{2} \right) \, dr} = \frac{\pi^2}{4} \left[ \frac{1}{4} + \frac{1}{\pi^2} \right] = 5.8303 \]

Thus \[ \lambda_0 \leq 5.8303 \]

Thus the best bounds so far are \[ 4.93 \leq \lambda_0 \leq 5.8303 \]

The upper bound is good but the lower bound is difficult to improve.

**Example 2** Our theory applies equally well to 1-D problems

\[ (\rho u')' - \rho u = -\lambda r u \quad 0 < x < \pi \]

\[ \lambda_0 = \min \frac{\int_0^\pi \left( \rho u'^2 + \rho u^2 \right) \, dx}{\int_0^\pi r u^2 \, dx} \]

$\rho > 0$, $\rho \geq 0$, $r > 0$ on $[0, \pi]$. $\mathcal{H}$ is the set of functions with $u = 0$ on $x = 0, \pi$.

Find bounds on the first eigenvalue for

\[ u'' - \varepsilon x u = -\lambda u \quad 0 < x < \pi \]

Thus \[ \lambda_0 = \min \frac{\int_0^\pi (u'^2 + \varepsilon x u^2) \, dx}{\int_0^\pi u^2 \, dx} \]

Let's replace $\varepsilon x$ by its minimum value in $(0, \pi)$. Then \[ \lambda_0 > \lambda_{\text{min}} \]

where \[ \lambda_{\text{min}} \] is the first eigenvalue of

\[ u'' = -\lambda u \quad u(0) = u(\pi) = 0 \rightarrow u \equiv \sin x \]

$\lambda_{\text{min}} = 1$
A simple upper bound is to replace \( \cos \) by its max value on \([0, \pi]\). Then \( \lambda_0 \leq \lambda_{\text{max}} \), where \( \lambda_{\text{max}} \) is the first eigenvalue of

\[
\begin{align*}
\lambda'' - \cos \pi \pi = \lambda \pi \quad \pi (0) = \pi (\pi) = 0 \\
\to \quad \lambda_{\text{max}} = 1 + \cos \pi \pi
\end{align*}
\]

Thus \( 1 \leq \lambda_0 \leq 1 + \cos \pi \pi \).

Can we improve the upper bound? Try the trial function \( \pi = \sin \pi \), then

\[
\lambda_0 \leq \frac{\int_0^\pi \left[ (\cos \pi \pi + \pi \sin \pi \pi \right) \pi dx}{\int_0^\pi \pi \sin^2 \pi \pi dx} = 1 + \frac{\cos \pi \pi}{2}
\]

which is a better bound.

Therefore we let

\[
1 \leq \lambda_0 \leq 1 + \frac{\cos \pi \pi}{2}
\]

**Example 3** Find an upper bound on the lowest eigenvalue of

\[
\begin{align*}
\left[ (1 + x) \pi' \right]' + \lambda \pi = 0 \\
\pi (0) = \pi (1) = 0
\end{align*}
\]

\[
\lambda_0 = \min_{\pi \in H} \frac{\int_0^1 \left[ (1 + x) \pi' \right]' \pi dx}{\int_0^1 \pi^2 \pi dx}
\]

Let \( \pi = (x - x^2) \). Then

\[
\lambda_0 \leq \frac{\int_0^1 \left[ (1 + x)(1 - 2x)^2 \right] dx}{\int_0^1 (x - x^2)^2 dx} \leq \frac{\int_0^1 (1 + x)(1 + 4x - 4x^2) dx}{\int_0^1 (1 - 2x + x^2)^2 dx}
\]

Now we calculate \( \lambda_0 \leq 15 \).

There is another eigenvalue problem with variable coefficients that can be solved explicitly.

\[
\left( x^2 \pi'' \right)' + \lambda \pi = 0 \\
\pi (1) = 0 \\
\pi (b) = 0
\]

This is Euler's equation:

\[
x^2 \pi'' + 2x \pi' + \lambda \pi = 0
\]

\[
\Rightarrow x^2 \pi' + \lambda \pi = 0
\]

\[
\Rightarrow \pi = \frac{1 + i(4 \lambda - 1)^{\frac{1}{4}}}{2}
\]

\[
\pi = x^{\frac{1}{4}} \sin \left[ \frac{(4 \lambda - 1)^{\frac{1}{4}}}{2} \log x \right]
\]

And the equation
Example: Consider a "star-like" domain with boundary \( D < r < F(q) \) where \( F(q) > 0 \) on \( q \in [0, 2\pi] \). Then for \( \Delta u = \lambda u \) in \( 0 \leq r \leq F(q) \) \( 0 \leq q \leq 2\pi \), we have \( \lambda_0 = \min_{u \in H} \frac{\int \int \int_D |V||u|^2 \, d\sigma \, d\theta \, d\phi}{\int \int \int_D u^2 \, d\sigma \, d\theta \, d\phi} \).

Choose \( u = v \left( \frac{r}{F(q)} \right) \) with \( v(1) = 0 \) as a trial function. Notice that \( v \) is constant on curve \( r = dF(q) \) \( 0 < d < 1 \).

Now \( \lambda_0 \leq \int_0^{2\pi} \int_0^p \int_0^{F(q)} \left( u_r^2 + u_q^2 \right) \, r \, dr \, dq \),

\[
\int_0^{2\pi} \int_0^{F(q)} u^2 \, r \, dr \, dq = \int_0^{2\pi} \int_0^{F(q)} v^2 (r/F(q)) \, r \, dr \, dq = \int_0^{2\pi} \left( F(q) \right)^2 dq \int_0^1 p \, v^2 (p) \, dp
\]

Let's calculate the denominator.

\[
\int_0^{2\pi} \int_0^{F(q)} u^2 \, r \, dr \, dq = 2 \pi \int_0^1 p \, v^2 (p) \, dp
\]

Now the area of the domain \( A \) is
\[
A = \frac{1}{2} \int_0^{2\pi} \left( F(q) \right)^2 dq.
\]

So \( \int_0^{2\pi} \int_0^{F(q)} u^2 \, r \, dr \, dq = 2 A \int_0^1 p \, v^2 (p) \, dp \)

Now \( \int_D \left( u_x^2 + u_y^2 \right) \, dx \, dy = \int_D \left( u_r^2 + \frac{1}{r^2} u_q^2 \right) \, r \, dr \, dq \)

\( u_r = \frac{v'}{F} \quad u_q = -\frac{v'}{F} \frac{r}{F^2} \int_F' \)

So \( \int_D \left( u_x^2 + u_y^2 \right) \, dx \, dy = \int_0^{2\pi} \int_0^{F(q)} \left( \frac{1}{F^2} \left[ v'(r/F) \right]^2 + \frac{1}{r^2} \int_F' \right) \, r \, r^2 \left[ v'(r/F)^2 \right] \, r \, dr \, dq \)

Now let \( p = r/F \) \( dr = F \, dp \)

\( \int_D \left( u_x^2 + u_y^2 \right) \, dx \, dy = \int_0^{2\pi} \int_0^1 \left( 1 + \frac{F^2}{F^2} \right) \, dq \int_0^1 p \, (v'(p))^2 \, dp \)

Now let \( s \) be arclength along \( CD \): then
\[
ds^2 = dr^2 + r^2 \, dq^2 \quad \text{so that} \quad \left( \frac{ds}{dq} \right)^2 = \left( \frac{F'(q)}{F(q)} \right)^2 + \frac{r^2}{F^2} (q) \)
so \( \frac{1}{r^2} \left( \frac{ds}{dq} \right)^2 = 1 + \frac{r'^2}{r^2} \) on \( r = f(q) \)

Rewrite this as

\[
\frac{ds}{r \ dq} \frac{ds}{r} = \left[ 1 + \left( \frac{f'(q)}{f(q)} \right)^2 \right] dq
\]

From the picture
\( rdq = \cos \alpha \ ds \)

But it is also clear that \( \alpha \) is the angle between \( r \) and \( h \) where \( h \) is the distance to the tangent line to \( dd \) at point \( P \).

Thus \( h = \cos \alpha \ \ r \)

so
\[
\frac{ds}{r \ dq} = \frac{r}{h}
\]

Combining (3) and (4) gives

\[
\frac{ds}{h} = \left[ 1 + \left( \frac{f'(q)}{f(q)} \right)^2 \right] dq
\]

Therefore from (3) \( \int_D \left( u_x^2 + u_y^2 \right) \ dx \ dy = \int_{dd} \frac{ds}{h} \ | p (y'(p))^2 | \ dp \)

Now substitute (5) and (1) into (6) to get

\[
\lambda_0 \leq \frac{\int_{dd} \frac{ds}{h} \ | p (y'(p))^2 | \ dp}{2 A \ \int_{a}^{b} (p v^2 (p)) \ dp}
\]

Now choose \( v(p) = J_0 (k_0 p) \) where \( k_0 \) is the smallest root of \( J_0 (z) = 0 \).

Then \( J_0 (k_0) = 0 \) so \( v(p) \) is admissible. Also \( v(p) \) satisfies

\[
p v'' + v' + \frac{k_0^2}{p} v = 0
\]

\[
o = \int_{a}^{b} \left( p v'' + v' + \frac{k_0^2}{p} v \right) \ dp = \int_{a}^{b} \left[ (p y')' + \frac{k_0^2}{p} v \right] \ dp = \int_{a}^{b} \left[ \frac{k_0^2}{p} v^2 - \frac{2}{p} \right] \ dp = 0
\]
This gives
\[ \lambda_0 \leq \frac{K_0}{2A} \int_\partial D \frac{ds}{h}, \quad K_0 = 2.4... \]

Now suppose that \( \partial D \) is convex, then the tangent line at any point on \( \partial D \) lies outside the domain as shown below.

Let \( R_{\text{in}} \) denote the radius of the largest circle that can be completely inscribed within \( D \).

Then \( h \geq R_{\text{in}} \)

and \( \lambda_0 \leq \frac{K_0}{2A} \frac{L}{R_{\text{in}}} \)

where \( R_{\text{in}} \) is assumed known and \( L \) = length of \( \partial D \)

\( A \) = area of \( \partial D \)

There is a famous lower bound for \( \lambda_0 \) due to Szegö (1930).

It states that for (\( \Delta u = -\lambda u \) with \( u = 0 \) on \( \partial D \)) we have

\[ \lambda_0 A \geq \pi \lambda_0 \]

for a 2-D domain (not necessarily convex).

Notice that equality is obtained if \( A = \pi \) (i.e., a circle of radius 1).

Therefore, it follows that of all domains of the same 2-dimensional area, the circle has the smallest principal eigenvalue \( \lambda_0 \).

This is intuitively clear since for

\[ u_0 = u \quad \text{in} \quad D \]

\[ u = 0 \quad \text{on} \quad \partial D \]

\[ u(x,0) = f(x) \]

the eigenvalue \( \lambda_0 \) measures the time needed to approach equilibrium \( u \equiv 0 \).

(i) If \( A \) is fixed, clearly \( \lambda_0 \) will be larger for domains with a longer perimeter — more surface area to remove heat — faster approach to equilibrium.

(ii) Of all domains with \( A \) fixed, the perimeter is shortest for a circular domain (we will show this later).

(iii) Therefore \( \lambda_0 \) should be smallest when \( L \) is smallest — circle.
RAYLEIGH-RITZ METHOD

Suppose we want to minimize

\[
\frac{E(w)}{\| w \|^2} = \frac{\int_D (p \, |\nabla w|^2 + q \, w^2) \, dx}{\int_D q \, w^2 \, dx} \quad \text{for } w \in H^1_0 \quad p > 0, \quad q > 0.
\]

Choose a set of \( m \) functions \( \psi_1(x), \ldots, \psi_m(x) \) that are admissible. Thus

(i) If \( w = 0 \) on \( \partial D \) \( \Rightarrow \) require \( \psi_1 = 0, \ldots, \psi_m = 0 \) on \( \partial D \)

(ii) If \( w = 0 \) on \( \partial D \) \( \Rightarrow \) natural boundary conditions, so no condition needed for \( \psi_1, \ldots, \psi_m \)

Now take

\[
w(x) = \sum_{k=1}^m c_k \psi_k(x) \quad \text{for some } c_k.
\]

In matrix notation

\[
w = C^T \psi \quad \text{where } \psi = (\psi_1, \ldots, \psi_m)\]

Now

\[
p \cdot \nabla w \cdot \nabla w + q \, w^2 = p \, C^T \nabla \psi \cdot C \, \nabla \psi + q \, C^T \psi \, C
\]

\[
\int_D (p \, \nabla w \cdot \nabla w + q \, w^2) \, dx = \int_D p \, C^T \nabla \psi \cdot C \, \nabla \psi + q \, C^T \psi \, C \, \psi \, \psi^T \, C \, dx = C^T \, AC
\]

Therefore, we have

\[
\frac{E(w)}{\| w \|^2} = \frac{C^T \, AC}{C^T \, BC}
\]

\[
A = \int_D (p \, \nabla \psi \cdot \nabla \psi^T + q \, \psi \, \psi^T) \, dx
\]

\[
B = \int_D q \, \psi \, \psi^T \, dx
\]

The matrices \( A \) and \( B \) are clearly symmetric. Also notice that \( B \) is positive definite and \( A \) is positive definite if \( q > 0 \).

Recall: A matrix \( M \) is positive definite if for all \( x \neq 0 \) vector, that

\[
x^T \, M \, x > 0 \quad \Rightarrow \quad \text{eigenvalues of } M \text{ are real and positive}.
\]

Now we try to minimize \((x)\) by varying the vector \( C \).

Assume that \((x)\) is minimized when \( C = C^* \), we then take

\[
C = C^* + \sum_{k=1}^m \varepsilon_k \psi_k \quad \text{where } \varepsilon_k = \text{a scalar with } \varepsilon_k \geq 1
\]

\[
\varepsilon_k = (0, 0, \ldots, 1, 0, 0, \ldots, 0)^T
\]

where \( \psi_k \) is in the position

Since \((x)\) is minimized at \( C = C^* \), we must have

\[
\frac{\partial}{\partial \varepsilon_k} \left[ \frac{E(w)}{\| w \|^2} \right]_{\varepsilon_k = 0} = 0
\]

The Taylor expansion of \( E(w) = C^T \, AC + \sum_{k=1}^m \varepsilon_k (e_k \, AC + C^T \, A e_k) \)

\[
+ O(\varepsilon_k^2)
\]
Now \[ c^T B c = c^{xT} B c^x + \sum_{k=1}^{m} E_k \left( e_k^T B c^x + c^{xT} B e_k \right) + \cdots. \]

Thus \[ \frac{c^T A c}{c^T B c} = \left( \frac{c^x A c^x + \sum_{k=1}^{m} E_k \left( e_k^T A C + c^{xT} A e_k \right)}{c^{xT} B c^x} \right) \left( \frac{1 - \sum_{k=1}^{m} E_k \left( e_k^T B c^x + c^{xT} B e_k \right)}{c^{xT} B c^x} \right) \]

We must set the coefficient of \( E_k \) to zero for \( c^x \) to be a minimum.  

Note that \( e_k^T A c^x = c^x A e_k = c^x A e_k \) since \( A = A^T \)

\( e_k^T B c^x = c^{xT} B e_k \).

Therefore, \( e_k \left[ A c^x - \left( \frac{c^{xT} A c^x}{c^{xT} B c^x} \right) B c^x \right] = 0 \) for \( k = 1, \ldots, m \)

is a necessary condition for \( c^x \) to be a minimizer.  Since all the \( \{e_k\} \)

for \( k = 1, \ldots, m \) form a basis for \( \mathbb{R}^m \), we require that

\( A c^x = \lambda^x B c^x \) \( \lambda^x = \frac{c^{xT} A c^x}{c^{xT} B c^x} = \frac{E_{1 \cdot \cdot \cdot m}}{||w||^2} \)

Therefore \( c^x \) is an eigenvector of the generalized eigenvalue problem \((+)\) with eigenvalue \( \lambda^x \).

Remarks

(i) Given a set of functions \( V_1, \ldots, V_m \) we can form the matrices \( A \) and \( B \):

\[ A_{ij} = \int_D (p \cdot V_i \cdot V_j + q \cdot V_i \cdot V_j) \, dx \]

\[ B_{ij} = \int_D q \cdot V_i \cdot V_j \, dx \]

(ii) Then we can calculate the eigenvalues and eigenvectors of \((+)\).