Problem 1. By bounding the coefficient $xy$ in the eigenvalue problem below, find upper and lower bounds for the lowest eigenvalue of
\[
\Delta \phi - xy \phi = -\lambda \phi \quad \text{in} \quad \Omega \times \Omega, \quad 0 \leq x, y \leq \pi
\]
\(\phi = 0\) on all sides of the square.

Problem 2. Consider the lowest eigenvalue $\lambda_1$ for the Laplacian in the unit sphere in 3-D with Dirichlet BC, where
\[
\begin{align*}
\Delta \phi + \Delta \phi &= 0 \quad \text{in} \quad 0 \leq r \leq 1 \\
\phi &= 0 \quad \text{on} \quad r = 1, \quad \phi \text{ bounded as } r \to 0.
\end{align*}
\]

(i) Formulate a 1-D variational problem for the lowest eigenvalue $\lambda_1$ of (x).

(ii) Compute the Rayleigh quotient with the trial function $1 - r$ to get an upper bound for $\lambda_1$.

(iii) Repeat (ii) with the trial function $\cos \left( \frac{\pi r}{2} \right)$.

(iv) By setting $\phi(r) = \frac{\pi r}{2}$ in (x) find the lowest eigenvalue explicitly and compare with the bounds in (iii) and (ii).

Problem 3. Consider the lowest eigenvalue $\lambda_1$ of an equilateral triangle $\Omega$ of side length $1$, for the Laplacian with Dirichlet boundary condition on all sides, so that
\[
\begin{align*}
\Delta \phi + \Delta \phi &= 0 \quad \text{in} \quad \Omega \\
\phi &= 0 \quad \text{on} \quad \partial \Omega
\end{align*}
\]

(i) By introducing a coordinate system, and by calculating the radii of the largest inscribed disk and the radius of the smallest disk that contains $\Omega$, find upper and lower bounds for $\lambda_1$.

(ii) Amazingly all the eigenvalues for (x) are $\lambda_{m,n} = \frac{16\pi^2}{9} \left( \frac{m^2 + n^2 + mn}{m^2 n^2} \right)$, $m, n = 1, 2, 3, \ldots$ Find lowest eigenvalue and compare with result in (i).

(iii) Give a simple trial function $w(x,y)$ at most quadratic in $(x,y)$ and positive in $\Omega$ that can be used in Rayleigh quotient to get an upper bound.
DO NOT EVALUATE THIS UPPER BOUND ANALYTICALLY.

**PROBLEM 4**

Let \( \Omega \) be the ellipse
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \quad \text{with} \quad a < b \]
and consider the eigenvalue problem
\[
\left\{ \begin{array}{l}
\Delta \phi + \lambda \phi = 0 \quad \text{in} \quad \Omega \\
\phi = 0 \quad \text{on} \quad \partial \Omega
\end{array} \right.
\]

(i) Find a simple upper bound on the lowest eigenvalue \( \lambda_1 \) of \((x)\) by using the largest disk that can be inscribed in \( \Omega \).

(ii) By using the trial function \( w = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \)

directly in Rayleigh's quotient find another explicit upper bound on the lowest eigenvalue. (Your bound here will depend on \( a \) and \( b \).)

(HINT: in performing the integration needed you will find use for \( \int_0^{\pi/2} \sin^2 \varphi \cos^3 \varphi \, d\varphi = \frac{\pi}{16} \), \( \int_0^{\pi/2} \cos^4 \varphi \, d\varphi = \frac{3\pi}{16} \), \( \int_0^{\pi/2} \cos^6 \varphi \, d\varphi = \frac{5\pi}{32} \).)

**PROBLEM 5**

Consider the eigenvalue problem in the unit disk
\[
\left\{ \begin{array}{l}
\Delta \phi + \lambda \phi = 0 \quad \text{in} \quad 0 < r < 1 \\
\phi = 0 \quad \text{on} \quad r = 1, \quad \phi \text{ bounded as} \quad r \to 0.
\end{array} \right.
\]

Using the Rayleigh-Ritz method with the two trial functions
\( w_1 = 1 - r^2 \) and \( w_2 = (1 - r^2)^3 \)
show that the minimum eigenvalue \( \lambda_1 \) of the matrix problem
\[
(\mathbf{A} - \mu \mathbf{B}) \mathbf{C} = 0
\]

where
\[
\mathbf{A} = \begin{pmatrix}
\int_0^1 4 r^3 \, dr & \int_0^1 8 r^2 (1 - r^3) \, r \, dr \\
\int_0^1 8 r^2 (1 - r^3) \, r \, dr & \int_0^1 16 r^2 (1 - r^2) \, r \, dr
\end{pmatrix}
\]

and
\[
\mathbf{B} = \begin{pmatrix}
\int_0^1 (1 - r^2)^2 r \, dr & \int_0^1 (1 - r^2)^3 r \, dr \\
\int_0^1 (1 - r^2)^3 r \, dr & \int_0^1 (1 - r^2)^4 r \, dr
\end{pmatrix}
\]

Calculate \( \lambda_1 \) analytically and compare with the exact result for \( \lambda_1 \).