Problem 1

In 2-D find an integral representation for the solution \( u(x, y, t) \) to

\[ u_t = D \left( u_{xx} + u_{yy} \right) \text{ in } 0 < x < \infty, \quad -\infty < y < \infty, \quad t > 0 \]

with \( u_x = 0 \) on \( x = 0 \); \( u \to 0 \) as \( x \to \infty \)

and \( u(x, y, 0) = f(x, y) \). (Here \( D > 0 \) is constant)

Problem 2

In 1-D find an integral representation for the solution \( u(x, t) \) to

\[ u_t = u_{xx} + g(x, t) \text{ in } 0 < x < \infty, \quad t > 0 \]

with \( u_x = 0 \) on \( x = 0 \); \( u \to 0 \) as \( x \to \infty \)

and \( u(x, 0) = f(x) \). Assume \( g \to 0 \) and \( f \to 0 \) as \( x \to \infty \).

Problem 3

In 1-D find an integral representation for the solution \( u(x, t) \) to

\[ u_t = D u_{xx} \text{ in } 0 < x < \infty, \quad t > 0 \]

with \( u(0, t) = g(t) \); \( u \to 0 \) as \( x \to \infty \)

and \( u(x, 0) = f(x) \). Assume \( f \to 0 \) and \( g \to 0 \) as \( x \to \infty \) and \( t \to \infty \)

respectively.

Problem 4

Consider the 2-D problem for \( u(x, y, t) \):

\[ u_t = u_{xx} + u_{yy} - \delta(x - x_0) \delta(y - y_0) \delta(t - 0^+) \text{ in } 0 < x < a, \quad -\infty < y < \infty, \quad t > 0 \]

with \( u(0, y, t) = 0 \), \( u(a, y, t) = 0 \)

and \( u(x, y, 0) = 0 \).

Assume that \( x_0 \) is in the channel so that \( 0 < x_0 < a \).

Find an eigenfunction expansion solution of the form

\[ u(x, y, t) = \sum_{n=1}^{\infty} c_n(y, t) \sin \left( \frac{n\pi x}{a} \right) \]

(i) Derive the PDE for \( c_n(y, t) \)

(ii) Solve the PDE for \( c_n(y, t) \) either by taking a Fourier transform in \( y \), or else writing \( c_n(y, t) = e^{-n^2\pi^2 t/a^2} v_n(y, t) \), finding the PDE for \( v_n(y, t) \) and then using an expression from the notes.
Problem 5

Here we will consider the acoustic field $u(x, x_2, x_3, t)$ in 3-D due to a localized source moving in the $x_1$ axis. We want to find a solution to

$$u_{tt} - c^2 \Delta u = \delta(x_1 + vt) \delta(x_2) \delta(x_3)$$

where $v > 0$ is constant.

(i) Find $u(x, t)$ for the subsonic case $0 < v < c$ by starting with the change of variables:

$$y_1 = x_1 + vt, \quad y_2 = x_2, \quad y_3 = x_3$$

and then converting the equation (with a further scaling) to the Laplacian with a "static" time independent point.

Problem 6 (Bonus Problem worth additional 5 points).

In Problem 5 find $u(x, t)$ for the supersonic case where $v > c$. Plot the Mach cone which is the boundary of the set separating where $u \geq 0$ from where $u > 0$. Discuss briefly the difference between the subsonic and supersonic cases.