**Problem 1**

In 3-D, find the free-space Green's function for

\[ \Delta G - \kappa^2 G = \delta(\mathbf{r}) \quad \text{with} \quad \Gamma^2 = x^2 + y^2 + z^2 \quad \text{and} \quad G \to 0 \quad \text{as} \quad \Gamma \to \infty, \]

where \( \kappa > 0 \).

(Hint: Recall in 3-D that if \( G \) is radially symmetric \( \Delta G = G_{rr} + \frac{2}{r} G_r \).)

**Problem 2**

Using the method of images to find the Green's function, find an integral representation for the solution to each of the following:

(i) 2-D \( \Delta u - \kappa^2 u = f(x, y) \) in \(-\infty < x < \infty, y > 0 \) with \( \kappa > 0 \)

\[ u(x, 0) = h(x) \quad \text{and} \quad u \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty. \]

(ii) 3-D \( \Delta u = f(x, y, z) \) in \(-\infty < x < \infty, -\infty < y < \infty, z > 0 \)

\[ u(x, y, 0) = h(x, y) \quad \text{and} \quad u \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty. \]

**Problem 3**

Using the method of images to find the Green's function, find an integral representation for the solution to

\[ \Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\phi\phi} = f(r, \phi) \quad \text{in} \quad 0 < r < 1, \quad 0 < \phi < \pi \]

with

\[ u = 0 \quad \text{on} \quad r = 1 \quad \text{bounded as} \quad r \to 0 \]

\[ u = 0 \quad \text{on} \quad \phi = 0, \pi. \]

**Problem 4**

Consider \( \Delta u = 0 \) in a sphere of radius \( a \) with

\[ u(r, \theta, \phi) = f(r, \theta) \]

where in spherical coordinates \((r, \theta, \phi)\) are defined by

\[ x = r \cos \theta \sin \phi \]
\[ y = r \sin \theta \sin \phi \]
\[ z = r \cos \phi. \]

By finding the Green's function using the method of images, show that

\[ u(r, \theta, \phi) = \int_0^{2\pi} \int_0^\pi \frac{a \left( a^2 - r^2 \right) F\left( \hat{\theta}, \hat{\phi} \right)}{4\pi \left[ a^2 + r^2 - 2ar \cos \beta \right]^{3/2}} \sin \hat{\theta} \sin \hat{\phi} \, d\theta \, d\phi \]

where

\[ \cos \beta = \cos \theta \cos \hat{\theta} + \sin \theta \sin \hat{\theta} \cos (\phi - \hat{\phi}) \]
Problem 5

Suppose that \( x \in \mathbb{R}^3 \) and that \( F(x) \) has compact support in \( \mathbb{R}^3 \). Let \( u(x) \) satisfy

\[
\Delta u = F(x) \quad \text{in} \quad \mathbb{R}^3 \quad \text{with} \quad u = o \left( \frac{1}{|x|} \right) \quad \text{as} \quad |x| \to \infty.
\]

(i) Show that

\[
u(x) = -\frac{1}{4\pi} \int_{\Omega_F} \frac{F(x')}{|x'-x|} \, dx' \quad (1)
\]

where the 3-D integral \( \int_{\Omega_F} \) is taken over the support of \( F \), i.e., the region where \( F \neq 0 \).

(ii) Show from (1) that for \( |x| \to \infty \) we have the asymptotic behavior

\[u(x) \sim C + \frac{p \cdot x}{|x|} + \ldots\]

where \( C \) (the capacitance) and \( p \) (the dipole vector) are to be found.