For Problems 1, 2, and 3 below, find whether a solvability condition is needed for $f(x)$, and if so find this solvability condition. Assuming that this condition is satisfied, calculate the modified Green's function and find an integral representation for the solution.

**Problem 1**

$L u = u'' + u = f(x)$, on $0 < x < 1$

With $u(0) = 0$, $u'(1) = u(1)$.

**Problem 2**

$L u = u'' + \pi^2 u = f(x)$, on $0 < x < 1$

With $u(0) = 0$, $u(1) = 0$.

**Problem 3**

$L u = [(1 - x^2) u']' = f(x)$, on $-1 < x < 1$

With $u(-1)$ and $u(1)$ bounded.

**Problem 4**

Consider the non-self-adjoint problem

$L u = u'' + 2u' + u = f(x)$, on $0 < x < 1$ \( \forall \)

With $u(0) = 0$ and $u(1) = 1$.

Find an integral representation for the solution to \( \forall \) in two ways.

(i) by finding the Green's function for $L$.

(ii) by first multiplying \( \forall \) by some function to make the resulting problem self-adjoint, then find the Green's function for the new problem.

**Problem 5**

Let $k > 0$ be arbitrary and consider the non-self-adjoint problem

$L u = u'' + u^2 = f(x)$, on $0 < x < 1$

With $u(0) = u(1)$ and $u'(0) = -u'(1)$.

Find a condition on $f(x)$ for this problem to have a solution.

(Hint: you will need to find a nontrivial solution to the adjoint problem).