PROBLEM 1

(i) Show that $x \delta'(x) + \delta(x) = 0$.

(ii) Let $F(x) = H(x-1) \sin x$ where $H(z)$ is the Heaviside function. Find the derivative of the generalized function $F(x)$.

(iii) Express $\delta(x^2 - a^2)$ as a sum of delta functions.

(iv) Show that $F(1) \delta'(x) = F(0) \delta'(x) - F'(0) \delta(x)$.

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PROBLEM 2

Define $L u = u'' - u^2$ on $0 < x < 1$ with $u > 0$.

(i) Find the Green's function for $L u$ with $u(0) = u(1) = 0$.

(ii) Find an integral representation for the solution to

$L u = f(x), \quad 0 < x < 1$

$u(0) = 1, \quad u(1) = 0 \quad$ (Notice inhomogeneous BC here).

(iii) Suppose that $f(x)$ depends on a parameter $\varepsilon$ as

$f(x) = \frac{1}{\varepsilon} \sech^2 \left( \frac{x-1/2}{\varepsilon} \right)$.

Find a good approximation when $\varepsilon \to 0^+$ for the solution to

$L u = f(x)$ on $0 < x < 1$

with $u(0) = 0, \quad u(1) = 0$.

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PROBLEM 3

Consider $L u = u'' - u^2$ with $u > 0$.

(i) Find the Green's function for $L u$ on $-\infty < x < \infty$ with $u \to 0$ as $|x| \to \infty$.

(ii) Find the Green's function for $L u$ on $0 < x < \infty$ with $u(0) = 0$ and $u \to 0$ as $x \to \infty$. 
PROBLEM 4

(i) Find the Green's function for
\[ Lu = \left( xu^2 \right)' - Lu = f(x) \text{ on } 0 < x < 1 \]
with \( u(0) \) finite, \( u(1) = 0 \).

(ii) Let \( u(\rho, \phi) \) with \( \rho, \phi \) being polar coordinates satisfying
\[ \rho \partial_{\rho} + \frac{1}{\rho} \partial_{\rho} + \frac{1}{\rho^2} \partial_{\phi} \rho = \delta(\rho - \rho_0) \cos \phi \text{ in } 0 < \rho < 1, 0 < \phi < 2\pi, \text{ with } 0 < \rho_0 < 1, \]
and with
\[ u = 0 \text{ on } \rho = 1; \text{ } u \text{ bounded as } \rho \to 0 \]
\[ u, \partial_\phi u \text{ } 2\pi \text{ periodic in } \phi. \]

Find an explicit solution for \( u(\rho, \phi) \) in terms of \( \cos \phi \) and your 1-D Green's function in part (i).

PROBLEM 5

Define \( Lu = u'' + \kappa^2 u \) on \( 0 < x < a \), with \( \kappa > 0 \) and \( \kappa > 0 \).

(i) Find the Green's function for \( Lu \) with \( u'(0) = u'(a) = 0 \).

(ii) Show that the Green's function in (i) does not exist when
\[ \kappa = \frac{n\pi}{a} \text{ with } n = 1, 2, \ldots \]

(iii) Suppose that \( Lu = f(x) \) on \( 0 < x < a \)
with \( u'(0) = 0 \) and \( u'(a) = 1 \).

Let \( \kappa = \frac{n\pi}{a} \). Find the condition on \( f(x) \) required for
this problem to have a solution.

(Hint: Notice that \( u \) has inhomogeneous boundary conditions.)
PROBLEM 1

(i) Show that \( X \delta'(x) + \delta(x) = 0 \).

Let \( \phi \) be any test function. Then integrating by parts:
\[
(X \delta', \phi) = \int_{-\infty}^{\infty} \delta(x) \phi'(x) \, dx = \delta(0) \phi'(0) - \int_{-\infty}^{\infty} \delta(x) (x \phi)' \, dx.
\]

Since \( \phi \) has compact support, \( \phi(0) = 0 \) and
\[
(X \delta', \phi) = -\int_{0}^{\infty} [X \delta'(x) \phi(x) + \delta(x) \phi'(x)] \, dx = -\delta(0) \phi'(0) \text{ by shifting property}.
\]

Thus
\[
(X \delta', \phi) = -\delta(0) = \int_{-\infty}^{\infty} (-\delta(x)) \phi(x) \, dx = (\delta, \phi).
\]

We conclude that \( X \delta' = -\delta \) in sense of distributions.

(ii) Find generalized derivative of \( f(x) = H(x-1) \sin x \).

Let \( \phi \) be any test function. Then integrating by parts:
\[
f'(x) \phi(x) = \int_{0}^{x} f(y) \phi'(y) \, dy = \int_{0}^{\infty} f(x-y) \phi'(y) \, dy = \int_{0}^{\infty} f(x-y) \sin y \phi'(y) \, dy.
\]

Now integrating by parts:
\[
\int_{0}^{\infty} f(x-y) \sin y \phi'(y) \, dy = \int_{0}^{\infty} \sin x \phi'(x) \, dx = \int_{0}^{\infty} \sin x \phi'(x) \, dx = -\int_{0}^{\infty} \sin x \phi(x) \, dx.
\]

Hence
\[
f'(x) \phi(x) = \left( \int_{0}^{\infty} \sin x \phi(x) \, dx \right) \frac{\partial}{\partial x} + \int_{0}^{\infty} \phi(x) \, dx = \sin x \phi'(x)
\]

Hence
\[
f'(x) \phi(x) = \left( \int_{0}^{\infty} \delta(x-1) \sin x \phi(x) \right) + \left( \int_{0}^{\infty} f(x) \phi'(x) \right).
\]

We conclude that \( f'(x) = \delta(x-1) \sin x + f(x) \) in distribution sense.

(iii) Expand \( \delta(x^2 - a^2) \) as a sum of delta functions.

We recall that if \( f(x) = 0, f'(x) \neq 0 \) then \( \delta(f(x)) = \sum_{k=1}^{\infty} \delta(x-k) \).

Here \( n = 2 \) and \( f(x) = x^2 - a^2 \) with \( x_1 = a, x_2 = -a \) and
\[
|f'(x)| = 2a. \text{ Thus yield } \delta(x^2 - a^2) = \frac{1}{2a} \left[ \delta(x-a) + \delta(x+a) \right].
\]

(iv) Show that \( f'(x) \delta'(x) = f(0) \delta'(x) - f'(0) \delta(x) \).

Let \( \phi(x) \) be any test function. We integrate by parts and use shifting property
\[
(f(x) \delta'(x), \phi(x)) = \int_{-\infty}^{\infty} f(x) \delta'(x) \phi(x) \, dx = \int_{-\infty}^{\infty} f(x) \delta'(x) \phi(x) \, dx = \int_{-\infty}^{\infty} f(x) \delta'(x) \phi(x) \, dx = -f(0) \phi'(0) - f'(0) \phi(0).
\]

But
\[
\phi'(0) = \left[ \int_{-\infty}^{\infty} \delta(x) \phi'(x) \, dx \right] - \int_{-\infty}^{\infty} \delta(x) \phi(x) \, dx. \text{ Thus } (f(x), \delta'(x)) = (f(0), \phi) + (f'(0), \phi).
\]

Thus yield
\[
f(x) \delta'(x) = -f(0) \phi'(0) - f'(0) \phi(0).
\]
Problem 2

\[ L u = u'' + u^2 \text{ on } 0 < x < 1 \text{ with } u > 0. \]

(i) Find \( g \)-function for \( L u \) with \( u(0) = u(1) = 0 \).

Since \( L u \) is \( ll \) operator and BC are separated, we have that the \( g \)-function 
\[ \begin{cases} \sqrt{\mu} - \mu^2 v = \delta(1-x) \text{ on } 0 < t < 1 \\ v(0) = v(1) = 0. \end{cases} \]

The two solutions are \( \{ \cosh[k(1-x)], \sinh[k(1-x)] \} \).

By inspection to satisfy the BC we take 
\[ v = \begin{cases} c_1 \sinh[k(t)], & 0 < t < 1 \\ c_2 \sinh[k(t-1)], & x < t < 1. \end{cases} \]

Now imposing continuity across \( t = x \) we have 
\[ v = \begin{cases} c_1 \sinh[k(t)] \sinh[k(x-t)], & 0 < t < x \\ c_2 \sinh[k(t)] \sinh[k(x-t)], & x < t < 1. \end{cases} \]

Now the jump condition is \( v^+(x) - v^-(x) = 1 \).

Thus \[ c_1 \left( \sinh[k(x)] \cosh[k(t-1)] \right]_{t=x}^{t=x} - \sinh[k(x-t)] \cosh[k(t)] \right]_{t=x}^{t=x} = 1. \]

Hence \[ c_1 \left( \sinh[k(x)] \cosh[k(x-t)] - \sinh[k(x-t)] \cosh[k(x)] \right) = 1. \]

Since \( \sinh(a-b) = \sinh(a) \cosh(b) - \sinh(b) \cosh(a) \) we have 
\[ c_1 \sinh[k(x)] - k(x-t)] = 1 \rightarrow c_1 \frac{1}{\sinh[k].} \]

We conclude that 
\[ v = \begin{cases} \frac{1}{\sinh[k]} \sinh[k(t)] \sinh[k(x-t)], & 0 < t < x \\ \sinh[k(t)] \sinh[k(t-1)], & x < t < 1. \end{cases} \]

(ii) Now find integral representation for solution to 
\[ L u = f(x), \quad 0 < x < 1 \]
\[ u(0) = 1, \quad u(1) = 0. \]

We have by Lagrange 
\[ (v, Lu) = (vL' - L v') \bigg|_0^1 + (u, L v). \]

Since \( v = 0 \) at \( t = 0, 1 \) and \( L v = \delta(1-x) \) we get 
\[ (v, f) = -u(x) v' (x) + L u(x). \]

Now 
\[ v'(0) = \frac{1}{\sinh[k]} \left( \cosh[k(t)] \right]_{t=0}^{t=x} - \sinh[k(x-1)] = \frac{\sinh[k(x-1)]}{\sinh[k].} \]

}\[ v'(1) = \frac{1}{\sinh[k]} \left( \cosh[k(t)] \right]_{t=0}^{t=x} - \sinh[k(x-1)] = \frac{\sinh[k(x-1)]}{\sinh[k].} \]
We conclude that
\[ U(x) = -\frac{\sinh \left( x(x-u) \right)}{\sinh u} + \int_0^1 V(t; x) \phi(t) \, dt \]

where \( V(t; x) \) is defined in (ii).

(iii) Suppose that \( f(x) = \frac{1}{\epsilon} \sech^2 \left( \frac{x - l/2}{\epsilon} \right) \).

Find a good approximation for \( \epsilon \to 0^+ \) for solution to
\[ L \phi = f(x) \] on \( 0 < x < 1 \), \( \phi(0) = 0 \), \( \phi(1) = 0 \).

Objective \( \lim_{\epsilon \to 0} f(x) = 0 \) and \( \int_0^1 f(x) \, dx = \frac{1}{\epsilon} \int_{-\infty}^{\infty} \sech \left( \frac{x - l/2}{\epsilon} \right) \, dx = \int_{-\infty}^{\infty} \sech^2 z \, dz \),

Now \( \int_{-\infty}^{\infty} \sech^2 z \, dz = \tanh z \bigg|_{-\infty}^{\infty} = 2 \).

Hence for \( \epsilon \to 0 \) we have \( f(x) \to 2 \delta(x - l/2) \).

We conclude that for \( \epsilon \to 0 \), \( \phi \) satisfies
\[ L \phi = \phi'' - \phi \phi = 2 \delta(x - l/2), \quad 0 < x < 1 \]
\[ \phi = 0 \text{ at } x = 0, 1. \]

This is precisely the G-function problem multiplied by 2 with source point at \( x = l/2 \).

Let \( x = l/2 \) in (ii) and multiply by 2.

\[ V = \frac{2}{K \sinh x} \begin{cases} \sinh (x+1) \sinh (-x/2), & 0 < x < 1/2 \\ \sinh (x+1) \sinh (y/2), & 1/2 < x < 1. \end{cases} \]

Replace \( x \mapsto x \). The solution to (iii) is
\[ U(x) = \frac{2}{K \sinh x} \begin{cases} -\sinh (x/2) \sinh (yX), & 0 < X < 1/2 \\ \sinh (y/2) \sinh (y(X+1)), & 1/2 < X < 1. \end{cases} \]
PROBLEM 3: Consider \( u = u^+ - u^- \) with \( \eta > 0 \).

(i) Find the \( \psi \)-function for \( Lu \) on \( -\infty < x < \infty \) with \( \eta \rightarrow 0 \) as \( |x| \rightarrow \infty \).

Since \( L \) is a quadratic operator, the \( \psi \)-function \( \psi (x, t) \) satisfies
\[
\psi'' - \eta^2 \psi = 0 \text{ for } t \geq 0, \quad \psi = 0 \text{ at } t = 0.
\]

For \( \psi'' - \eta^2 \psi = 0 \) we have \( \psi = c_1 e^{\pm \eta t} + c_2 \), to have \( \psi \) decay as \( t \rightarrow \infty \), we have
\[
\psi = \left\{ \begin{array}{ll}
    c_1 e^{\eta t}, & -\infty < t < 0 \\
    c_2 e^{-\eta t}, & 0 < t < \infty
  \end{array} \right.
\]

Now to have continuity at \( t = x \), we put
\[
\psi = \left\{ \begin{array}{ll}
    c_1 e^{\eta t} e^{-\eta x}, & -\infty < t < x \\
    c_2 e^{-\eta t} e^{\eta x}, & x < t < \infty
  \end{array} \right.
\]

where \( c_1 \) is found from jump condition \( \psi'(x^-) = \psi'(x^+) \).

Thus
\[
\left. \frac{\partial \psi}{\partial x} \right|_{t = x^-} - \left. \frac{\partial \psi}{\partial x} \right|_{t = x^+} = 0 \implies c_1 = -\frac{1}{2\eta}
\]

Thus
\[
\psi = -\frac{1}{2\eta} \left( e^{\eta (t-x)} - e^{-\eta (t-x)} \right), \quad -\infty < t < x
\]

\( \psi(x, t) \) is continuous.

We conclude that \( \psi (t; x) = \frac{-1}{2\eta} e^{-\eta |t-x|} \).

(ii) Now find \( \psi \)-function for \( Lu \) with \( 0 < x < \infty \), \( \eta (0) = 0 \) and \( \psi \rightarrow 0 \) as \( |x| \rightarrow \infty \).

The \( \psi \)-function satisfies
\[
\psi'' - \eta^2 \psi = 0 \quad \text{for } t \geq 0, \quad \psi(0) = 0, \quad \psi \rightarrow 0 \text{ as } t \rightarrow \infty.
\]

So
\[
\psi = \left\{ \begin{array}{ll}
    c_1 e^{\eta t} - c_2 e^{-\eta t}, & 0 < t < x \\
    c_2 e^{-\eta t}, & t > x
  \end{array} \right.
\]

Imposing continuity
\[
\psi = \left\{ \begin{array}{ll}
    c_1 (e^{\eta t} - e^{-\eta t}) e^{-\eta x}, & 0 < t < x \\
    c_2 e^{-\eta t} (e^{\eta x} - e^{-\eta x}), & t > x
  \end{array} \right.
\]

Now \( \psi'(x^-) - \psi'(x^+) = 0 \) determines \( c_1 \).

Thus \( \psi(x, t) = c_2 \) if \( c_1 = -1/2\eta \). Hence
\[
\psi = -\frac{1}{2\eta} \left( e^{\eta (t+x)} - e^{-\eta (t+x)} \right), \quad t > x.
\]

Now \( \psi'(x^-) - \psi'(x^+) = 0 \) determines \( c_1 \).

We conclude that \( \psi(t; x) = \frac{-1}{2\eta} \left( e^{-\eta |t-x|} - e^{-\eta (t+x)} \right) \) on \( t > 0 \).
(i) Find the G-function for
\( LU = (xu')' - \frac{1}{x} u = f(x) \) on \( 0 < x < 1 \) with \( u(0) \) finite and \( u(1) = 0 \).

**Solution** Since \( L \) is a self-adjoint operator with separated BC we have that the G-function satisfies
\[
\left\{ \begin{array}{l}
\left( \frac{d}{dx} \right)^2 + \frac{d}{dx} - \frac{1}{x} \nu = 0 \quad \text{on} \quad 0 < x < 1 \\
\nu: 0 \text{ at } x = 0, 1
\end{array} \right.
\]
with \( \nu: 0 \) at \( x = 0, 1 \).

For \( x \neq 1 \) we have
\[
\frac{d^2}{dx^2} \nu + \frac{d}{dx} \nu - \frac{1}{x} \nu = 0 \Rightarrow \frac{d}{dx} \nu + \frac{1}{x} \nu = 0
\]
Euler's equation.

We put \( \nu = \Gamma \) so that \( \Gamma (x-1) + \Gamma - 1 = 0 \rightarrow \Gamma^2 = 1 \) so \( \Gamma = \pm 1 \).

Thus \( \nu = \Gamma \) satisfy the BC we write
\[
\nu: \begin{cases}
C_1, & 0 < x < 1 \\
C_2 (x-1/\Gamma), & x < 1
\end{cases}
\]

Next imposing continuity at \( x = 0 \),
\[
\nu: \begin{cases}
C + (x-1/\Gamma), & 0 < x < 1 \\
C + (1-1/\Gamma)x, & x < 1
\end{cases}
\]

Where \( C \) is found from jump condition \( \nu' (x^+) - \nu' (x^-) = \frac{1}{x} \).

Now this gives
\[
C \left[ \left( 1 + \frac{1}{x} \right)x - \left( x - \frac{1}{x} \right) \right] = \frac{1}{x} \rightarrow C = \frac{1}{2}.
\]

Thus \( \nu = \frac{1}{2} \begin{cases}
C + (x-1/\Gamma), & 0 < x < 1 \\
C + (1-1/\Gamma)x, & x < 1
\end{cases}
\)

(ii) Now let \( \overline{u}(\Gamma, \Phi) \) satisfy, for \( 0 < \Gamma, \Phi < 1 \),
\[
\overline{u}_{\Gamma} + \frac{1}{\Gamma} \overline{u}_{\Gamma} + \frac{1}{\Gamma} \overline{u}_{\Phi} = \delta (\Gamma - \Gamma_0) \cos \Phi \quad \text{in} \quad 0 < \Gamma < 1, \quad 0 < \Phi < 2\pi
\]
\( \overline{u} = 0 \) on \( \Gamma = 1; \overline{u} \) bounded at \( \Gamma \to 0; \overline{u}, \overline{u}_{\Phi} \) \( 2\pi \)-periodic.

Solve for \( \overline{u}(\Gamma, \Phi) \) in terms of solution in (i).

Write \( \overline{u}(\Gamma, \Phi) = u(\Gamma) \cos \Phi \). We will obtain \( U(\Gamma) \). Upon substituting we obtain that
\[
\overline{u}_{\Gamma} + \frac{1}{\Gamma} \overline{u} = 0
\]

Multiplying by \( \Gamma \):
\[
\overline{u}_{\Gamma} + \frac{1}{\Gamma} \overline{u} = \delta (\Gamma - \Gamma_0)
\]
\( \overline{u} = 0 \) at \( \Gamma = 1, \overline{u} \) bounded at \( \Gamma \to 0 \).

All we need \( \overline{u} \) to replace \( \Gamma \to \Gamma \)

\[
\Gamma = \frac{\Gamma_0}{2}
\]

And \( x \to \Gamma_0 \) in (ii) and multiply by \( \Gamma_0 \).
Define $L = \frac{\partial^2}{\partial x} + \frac{\partial}{\partial x}$ on $0 < x < a$ with $\alpha > 0$ and $\eta > 0$.

Find Green's function for $L$ with $u(0) = u(a) = 0$.

**Solution:** Since $L$ is a SL operator and we have well-separated BC, we have that the C-function $V$ satisfies

$$L V = \frac{\partial^2}{\partial x} + \frac{\partial}{\partial x} V = \delta(t - x) \text{ on } 0 < x < a,$$

$$V = 0 \text{ at } t = 0, a$$

For $t \neq x$, $V = \text{span}\{\cos(\eta(t-x)), \sin(\eta(t-x))\}$ and so satisfying the BC

$$V = \left\{ \begin{array}{ll} c_1 \cos(\eta t), & 0 < t < x \\ c_2 \cos(\eta(t-a)), & x < t < a. \end{array} \right.$$ 

Now imposing continuity at $t = x$ we have

$$V = \left\{ \begin{array}{ll} c_1 \cos(\eta t) \cos(\eta(x-a)), & 0 < t < x \\ c_2 \cos(\eta(t-a)) \cos(\eta x), & x < t < a. \end{array} \right.$$ 

Where $c_1$ is found by jump condition $V(0) - V(x) = 1$.

We obtain

$$c_1 \sin(\eta(x-a)) \cos(\eta x) + c_2 \sin(\eta x) \cos(\eta(x-a)) = 1.$$ 

Since $\sin(A - B) = \sin A \cos B - \sin B \cos A$ we get

$$c_1 \sin \left[ \eta x - (\eta(x-a)) \right] = 1 \rightarrow c_1 \sin(\eta x) = 1.$$ 

We conclude that $c_1 = \frac{1}{\sin(\eta x)}$.

Hence

$$V(0) = \frac{1}{\sin(\eta x)} \left( \cos(\eta t) \cos(\eta(x-a)), \ 0 < t < x \right) \cos(\eta(t-a)) \cos(\eta x), \ x < t < a.$$ 

(i) When $\eta = \frac{n\pi}{a}$ with $n = 1, 2, \ldots$ we have that $V$ is undefined at these values, so a nontrivial homogeneous solution to (8) given by

$$\phi = \cos \left[ \eta \frac{t-a}{a} \right]$$ 

by Lagrange if $V$ exists we have $\langle \phi, L \phi \rangle = \langle V, \phi \rangle = 0$ which is a contradiction. No C-function when $\eta = \frac{n\pi}{a}$.

(iii) Let $L = \frac{\partial^2}{\partial x} + \frac{\partial}{\partial x} = f(x)$ on $0 < x < a$ with $u(0) = 0$ and $u(a) = 0$, with $u' = \frac{\partial}{\partial x} u$.

The homogeneous problem has a nontrivial solution $\phi = \cos(\eta(x-a))$.

Let Lagrange: $\langle \phi, L \phi \rangle = \langle \phi' - \phi' \phi' \rangle + \langle u, L \phi \rangle \rightarrow (\phi, \phi') = \phi(a)$, so since $\phi = 0$ with $\phi(0) = \phi'(a) = 0$ and $u(0) = 0$. For a solution to exist we thus require $\langle \phi, \phi' \rangle = \phi(a). \rightarrow \int_0^a \cos(\eta(x-a)) f(x) dx = \cos(\eta a) = (-1)^\eta$.