LAPLACE AND POISSON'S EQUATION

LAPLACE AND POISSON'S EQUATION ARISES AS THE STEADY-STATE OF

$$u_t = \Delta u + F \quad \text{IN } \Omega$$

$$\partial_n u + K(u - u_b) = 0 \quad \text{ON } \partial\Omega$$

THE STEADY-STATE IS

$$\Delta u = -F \quad \text{IN } \Omega$$

$$\partial_n u + K(u - u_b) = 0 \quad \text{ON } \partial\Omega$$

- POISSON WHEN $F \neq 0$
- LAPLACE WHEN $F = 0$.

WE NOW CONSIDER A <u>CIRCULAR DOMAIN</u>, $0 \le r \le R$, $0 \le \varphi \le 2\pi$ AND DERIVE <u>POISSON'S INTEGRAL FORMULA</u> FOR THE SOLUTION TO

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = 0 \qquad 0 \le r \le R, \ 0 \le \varphi \le 2\pi$$

$$u(R, \varphi) = f(\varphi), \quad u \text{ bounded } r \to 0$$

$$u, u_\varphi \text{ ARE } 2\pi \text{ periodic in } \varphi.$$

WE SEPARATE VARIABLES $u(r, \varphi) = R(r)\,\Phi(\varphi)$ TO OBTAIN

$$\left(R'' + \frac{1}{r} R'\right)\Phi + \frac{1}{r^2} R \Phi'' = 0 \longrightarrow \frac{r^2\left(R'' + 1/r\, R'\right)}{R} = -\frac{\Phi''}{\Phi} = \lambda.$$

THEN $\Phi'' + \lambda\Phi = 0$ WITH $\Phi(0) = \Phi(2\pi)$ AND $\Phi'(0) = \Phi'(2\pi)$.

THEN $\lambda_n = n^2$ AND $\Phi_n(\varphi) = \begin{cases} A_n \cos n\varphi + B_n \sin n\varphi, & n \ge 1 \\ A_0, & n = 0 \end{cases}$

THEN $r^2 R'' + r R' + n^2 R = 0$ LET $R = r^\beta \to \beta(\beta - 1) + \beta + n^2 = 0$

SO THAT $\beta = \pm n$. WE CALCULATE

$$R_n(r) = \begin{cases} c_n r^n + d_n r^{-n}, & n \ge 1 \\ c_0 + d_0 \log r, & n = 0 \end{cases}$$

FOR BOUNDEDNESS WE TAKE $d_0 = d_n = 0$ FOR $n = 1, 2, \ldots$.

THEN SET $C_0 = C_n = 1$ $\forall n$ SO THAT $R_0 = 1$ AND $R_n = r^n$, $n = 1, 2, \ldots$

BY SUPERPOSITION $u_n = (A_n \cos(n\varphi) + B_n \sin(n\varphi))\, r^n$ IS A SOLUTION FOR $n = 1, 2, \ldots$

AND $u_0 = A_0$ IS A SOLUTION.

SO (*) $$u(r, \varphi) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos(n\varphi) + B_n \sin(n\varphi)\right) r^n$$

THEN $$f(\varphi) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos(n\varphi) + B_n \sin(n\varphi)\right) R^n$$

WE INTEGRATE $\int_0^{2\pi}$ TO OBTAIN $\int_0^{2\pi} f(\varphi)\, d\varphi = A_0 (2\pi) \rightarrow A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi)\, d\varphi$.

NOW WE MULTIPLY BY $\cos(m\varphi)$ AND INTEGRATE $\int_0^{2\pi}$ GIVES

$$A_n R^n \int_0^{2\pi} \cos^2(n\varphi)\, d\varphi = \int_0^{2\pi} f(\varphi) \cos(n\varphi)\, d\varphi \rightarrow A_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\varphi) \cos(n\varphi)\, d\varphi.$$

SIMILARLY, $$B_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\varphi) \sin(n\varphi)\, d\varphi.$$

REPLACE $\varphi \rightarrow w$ IN INTEGRALS AND SUBSTITUTE IN (*)

THIS GIVES:

$$u(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} f(w)\, dw + \sum_{n=1}^{\infty} \frac{1}{\pi} \left(\frac{r}{R}\right)^n \int_0^{2\pi} f(w) \left[\cos(n\varphi)\cos(nw) + \sin(n\varphi)\sin(nw)\right] dw$$

THIS CAN BE WRITTEN AS

$$u(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} f(w) \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \cos(n(w - \varphi))\right] dw.$$

NOW DEFINE $z = \frac{r}{R} e^{in(w-\varphi)}$ WITH $|z| < 1$ WHEN $r < R$.

NOW WE OBTAIN: NOTE $1+z+z^2+\dots = \frac{1}{1-z} \longrightarrow z+z^2+\dots = \frac{1}{1-z} - 1$.

$$1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \cos(n(w-\varphi)) = 1 + 2\,\mathrm{RE}\left(\sum_{n=1}^{\infty} z^n\right) = 1 + 2\left(\frac{1}{1-z} - 1\right)$$

$$= 1 + \frac{2z}{1-z} = \frac{1+z}{1-z} = \frac{(1+z)(1-\bar{z})}{|1-z|^2}$$

$$= \frac{1 - |z|^2 + (z-\bar{z})}{|1-z|^2}.$$

THEN $$1 + 2\,\mathrm{RE}\left(\sum_{n=1}^{\infty} z^n\right) = \mathrm{RE}\left(1 + 2\sum_{n=1}^{\infty} z^n\right) = \mathrm{RE}\left(\frac{1-|z|^2+z-\bar{z}}{|1-z|^2}\right) = \frac{1-|z|^2}{|1-z|^2}$$

$$= \frac{1 - r^2/R^2}{\left(1 - \frac{r}{R}\cos(w-\varphi)\right)^2 + \frac{r^2}{R^2}\sin^2(w-\varphi)} = \frac{1 - r^2/R^2}{1 + r^2/R^2 - \frac{2r}{R}\cos(w-\varphi)}$$

THEN $$1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \cos(n(w-\varphi)) = \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(w-\varphi)}$$

THIS YIELDS THAT

POISSON'S INTEGRAL FORMULA $$u(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(w)\left(\frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(w-\varphi)}\right) dw.$$

<u>REMARK</u> (i) $u\big|_{r=0} = \frac{1}{2\pi}\int_0^{2\pi} f(w)\,dw$ TEMPERATURE AT CENTER OF DISC IS THE AVERAGE OF THE TEMPERATURE OVER THE ENTIRE BOUNDARY OF THE DISC.

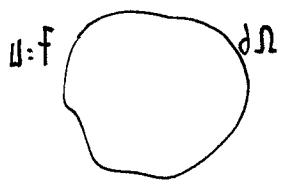
$$\min_{\varphi} f(\varphi) = f_{MIN} \le u\big|_{r=0} \le f_{MAX} = \max_{\varphi} f(\varphi)$$

(ii) $$\lim_{r \to R^-} \frac{1}{2\pi}\left(\frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(w-\varphi)}\right) = \delta(w-\varphi) \quad \text{so} \quad u(R,\varphi) = \int_0^{2\pi} f(w)\,\delta(w-\varphi)\,dw$$

GIVES $u(R,\varphi) = f(\varphi)$.

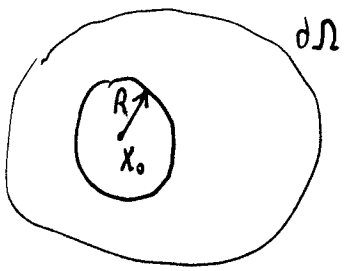
MAX - MIN PRINCIPLE

CONSIDER $\Delta u = 0$ IN Ω AND $u = f$ ON $\partial\Omega$ WITH Ω BEING A BOUNDED TWO-DIMENSIONAL DOMAIN.



THEN

$$\text{MIN } f \le u \le \text{MAX } f.$$



PROOF SUPPOSE, BY CONTRADICTION THAT u ACHIEVES ITS MAXIMUM AT SOME POINT $\underline{X}_0$ IN Ω. LET R BE ANY VALUE SO THAT THE DISC CENTERED AT $\underline{X}_0$ IS STRICTLY INSIDE Ω.

THEN BY POISSON'S INTEGRAL FORMULA $f(w) = u(R,w)$

$$u(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(w)\left(\frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(w-\varphi)}\right)dw$$

THEN WITH $u(\underline{X}_0) = u(0,\varphi)$ WE OBTAIN

$$u(\underline{X}_0) = \frac{1}{2\pi}\int_0^{2\pi} u(R,\varphi)\,d\varphi$$

SO $$\min_{r=R} u \le u(\underline{X}_0) \le \max_{r=R} u$$

THIS VIOLATES THE ASSUMPTION THAT u ATTAINS ITS MAXIMUM AT $\underline{X} = \underline{X}_0$ (UNLESS OF COURSE u = CONSTANT EVERYWHERE IN Ω).

EXAMPLE SOLVE THE FOLLOWING:

$$u_{xx} + u_{yy} = 0 \quad \text{IN } 0 \le x < \infty, \ 0 \le y \le 1$$

$$u(x,0) = u(x,1) = 0, \qquad u(0,y) = 1-y, \qquad u \to 0 \text{ AS } x \to \infty$$

WE SEPARATE VARIABLES TO OBTAIN $u = XY$ SO $-\dfrac{X''}{X} = \dfrac{Y''}{Y} = -\lambda$

THEN $\begin{array}{l} Y'' + \lambda Y = 0, \ 0<y<1 \\ Y(0) = Y(1) = 0 \end{array} \longrightarrow Y = \sin(n\pi y), \ \lambda = n^2\pi^2.$

THEN $X_n'' - n^2\pi^2 X_n = 0$ SO $X_n = e^{-n\pi x}$ BOUNDED AS $x \to +\infty$.

THIS YIELDS THAT $u(x,y) = \sum_{n=1}^{\infty} b_n e^{-n\pi x} \sin(n\pi y).$

NOW $u(0,y) = 1-y = \sum_{n=1}^{\infty} b_n \sin(n\pi y) \qquad b_n = 2\int_0^1 (1-y)\sin(n\pi y)\,dy.$

IF WE CALCULATE $\rightarrow b_n = \dfrac{2}{n\pi}, \ n = 1,2,3,\ldots$

THIS YIELDS $u(x,y) = \sum_{n=1}^{\infty} \dfrac{2}{n\pi} e^{-n\pi x} \sin(n\pi y).$

WE LET $z = e^{-\pi x + i\pi y}$ SO $u(x,y) = \dfrac{2}{\pi}\sum_{n=1}^{\infty} \mathrm{IM}\left(\dfrac{z^n}{n}\right) = \dfrac{2}{\pi}\mathrm{IM}\left(\sum_{n=1}^{\infty}\dfrac{z^n}{n}\right).$

NOW $z + \dfrac{z^2}{2} + \dfrac{z^3}{3} + \ldots = \sum_{n=1}^{\infty}\dfrac{z^n}{n} = -\log(1-z).$

THUS $u(x,y) = -\dfrac{2}{\pi}\mathrm{IM}\left[\log(1-z)\right] = -\mathrm{IM}\left[\dfrac{2}{\pi}\left(\ln|1-z| + i\varphi\right)\right]$

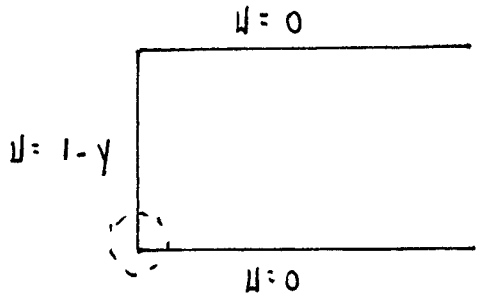
THUS $u(x,y) = -\dfrac{2}{\pi}\varphi \qquad \varphi = \tan^{-1}\left(\dfrac{\mathrm{IM}(1-z)}{\mathrm{RE}(1-z)}\right).$

$$1-z = 1 - e^{-\pi x}\cos(\pi y) - i e^{-\pi x}\sin(\pi y)$$

$$\mathrm{RE}(1-z) = 1 - e^{-\pi x}\cos(\pi y), \qquad \mathrm{IM}(1-z) = -e^{-\pi x}\sin(\pi y).$$

THEN $\varphi = \tan^{-1}\left[\dfrac{-e^{-\pi x}\sin(\pi y)}{1-e^{-\pi x}\cos(\pi y)}\right] = -\tan^{-1}\left(\dfrac{\sin(\pi y)}{e^{\pi x}-\cos(\pi y)}\right).$

THIS YIELDS THAT $u(x,y) = \dfrac{2}{\pi}\tan^{-1}\left(\dfrac{\sin(\pi y)}{e^{\pi x}-\cos(\pi y)}\right).$



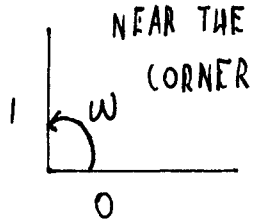
NOTICE THAT THERE IS A DISCONTINUITY IN THE BOUNDARY DATA NEAR $x=0$, $y=0$.

THEN $\sin(\pi y) \approx \pi y$

$$e^{\pi x} - \cos(\pi y) \approx 1 + \pi x - \left[1 - \frac{\pi^2 y^2}{2}\right] \approx \pi x.$$

THIS YIELDS NEAR $x \approx 0$ AND $y \approx 0$ THAT

$$u(x,y) \approx \frac{2}{\pi}\tan^{-1}\left(y/x\right) = \frac{2}{\pi}\omega$$



WHEN $\omega = 0 \rightarrow u = 0$

$\omega = \pi/2 \rightarrow u = 1$

NOW NEAR THE CORNER

$$u_x \sim \frac{2}{\pi}\,\frac{-y/x^2}{1+y^2/x^2} \approx -\frac{2}{\pi}\,\frac{y}{x^2+y^2}$$

which looks like $u_x \approx -\dfrac{2}{\pi}\left(\dfrac{\sin\varphi}{r}\right)$

SIMILARLY, $u_y \approx \dfrac{2}{\pi}\left(\dfrac{\cos\varphi}{r}\right)$

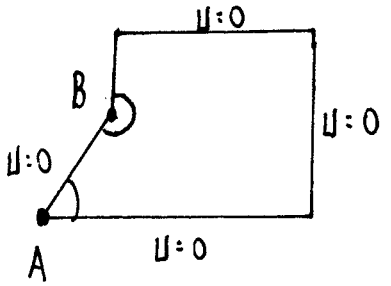
AND SO THE DERIVATIVES BLOW UP AS $(x,y) \rightarrow (0,0)$.

CORNER SINGULARITIES

SUPPOSE THAT WE WANT TO SOLVE

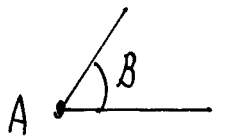
$$\Delta u = F$$

IN THE DOMAIN AS SHOWN BELOW WITH $u = 0$ ON ALL SIDES.



WE WANT TO SEE WHAT IS THE BEHAVIOR OF THE SOLUTION NEAR THE CORNERS AT POINTS A AND B.

WE ZOOM IN A NEIGHBORHOOD NEAR A



AND INTRODUCE A LOCAL COORDINATE SYSTEM NEAR A.

NEAR A WE INTRODUCE POLAR COORDINATES r AND φ

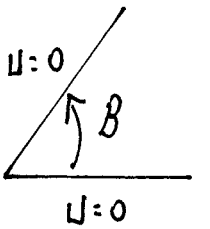
TO GET $$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = F(r, \varphi).$$ $r = 0 \longrightarrow$ point A

WE THEN LET $r = \varepsilon \rho$ WHERE ε IS SMALL TO LOCALIZE THE REGION NEAR A. THEN $u_r = u_\rho \, 1/\varepsilon$ $\quad u_{rr} = \frac{1}{\varepsilon^2} u_{\rho\rho}$

AND SO $$\frac{1}{\varepsilon^2}\left(u_{\rho\rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\varphi\varphi}\right) = F(\varepsilon\rho, \varphi).$$

SO IF F IS BOUNDED AS $r \to 0$ THEN IN A SMALL NEIGHBOURHOOD OF POINT A WE MUST SOLVE

$$\begin{cases} u_{\rho\rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\varphi\varphi} = 0 \\ u = 0 \text{ ON } \varphi = 0 \\ u = 0 \text{ ON } \varphi = B \end{cases}$$



WE SEPARATE VARIABLES $u = P(\rho)\,\Phi(\varphi)$ $\qquad \dfrac{\rho^2\left(P'' + \frac{1}{\rho}P'\right)}{P} = -\dfrac{\Phi''}{\Phi} = \lambda$

THIS YIELDS THAT

$$\left.\begin{array}{l}\Phi'' + \lambda \Phi = 0 \\ \Phi(0) = \Phi(B) = 0\end{array}\right\} \longrightarrow \Phi = \sin(\sqrt{\lambda}\,\varphi) \quad \text{so} \quad \sqrt{\lambda}\, B = \pi$$

OR $\lambda = \dfrac{\pi^2}{B^2}$ IS <u>smallest</u> λ.

THEN $\rho^2 P'' + \rho P' - \dfrac{\pi^2}{B^2} P = 0$ SO $P = \rho^{\pi/B}$ IS BOUNDED SOLUTION.

THEREFORE NEAR THE CORNER WE HAVE

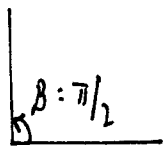
$$u(\rho, \varphi) \approx C \rho^{\pi/B} \sin\left(\frac{\pi \varphi}{B}\right) \qquad \text{for some } C.$$

WE CALCULATE THE ELECTRIC FIELD

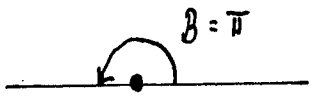
$$\nabla u \approx \left(\partial u/\partial \rho,\ 1/\rho\, \partial u/\partial \varphi \right) = C\left(\frac{\pi}{B} \rho^{\pi/B - 1} \sin\left(\frac{\pi\varphi}{B}\right), \frac{\pi}{B} \rho^{\pi/B - 1} \cos\left(\frac{\pi\varphi}{B}\right)\right)$$

THIS YIELDS $\nabla u \approx C \dfrac{\pi}{B} \rho^{\pi/B - 1} \left(\sin(\pi\varphi/B), \cos(\pi\varphi/B) \right).$

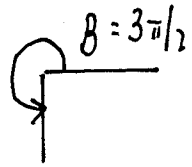
WE CONSIDER A FEW CASES:



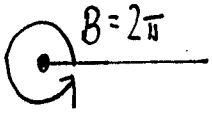
$\longrightarrow \nabla u = O(\rho) \longrightarrow$ DOES NOT BLOW UP AS $\rho \to 0$.



$\longrightarrow \nabla u = O(1) \longrightarrow$ NO BLOW-UP AS $\rho \to 0$



$\longrightarrow \nabla u = O(\rho^{-1/3})$ BLOWS UP AS $\rho \to 0$



$\longrightarrow \nabla u = O(\rho^{-1/2})$ BLOWS UP AS $\rho \to 0$

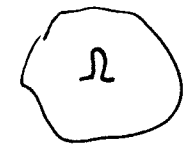
THIS LAST EXAMPLE IS A "LIGHTNING ROD" THAT CAN STORE LOTS OF CHARGE AT TIP.

UNIQUENESS AND EXISTENCE

WE WANT TO PROVE THAT THERE IS A UNIQUE SOLUTION TO

$$\Delta u = F \text{ IN } \Omega$$

$$\partial_n u + K(u-g) = 0 \text{ ON } \partial\Omega \text{ WITH } K>0 \text{ AND CONSTANT.}$$



HERE Ω IS AN ARBITRARY BOUNDED DOMAIN.

SUPPOSE THAT u_1, u_2 ARE TWO SOLUTIONS AND LET $V = u_1 - u_2$.

THEN V SATISFIES

$$\Delta V = 0 \text{ IN } \Omega$$

$$\partial_n V + KV = 0 \text{ ON } \partial\Omega, \quad K>0.$$

WE WANT TO PROVE THAT $V \equiv 0$ IN Ω SO THAT $u_1 \equiv u_2$ IN Ω.

WE RECALL A VECTOR IDENTITY

$$\nabla\cdot[\underline{F}\,\phi] = \phi\,\nabla\cdot\underline{F} + \underline{F}\cdot\nabla\phi$$

SO THAT $\nabla\cdot[V\nabla V] = \nabla V\cdot\nabla V + V\Delta V = |\nabla V|^2 + V\Delta V.$

THEN $$0 = \int_\Omega V\Delta V\,d\underline{x} = \int_\Omega V\,\nabla\cdot[\nabla V]\,d\underline{x} = \int_\Omega \nabla\cdot[V\nabla V]\,d\underline{x} - \int_\Omega |\nabla V|^2\,d\underline{x}$$

NOW USING DIVERGENCE THEOREM,

$$\int_{\partial\Omega} V\nabla V\cdot\hat{n}\,ds = \int_\Omega |\nabla V|^2\,d\underline{x}$$

BUT $\nabla V\cdot\hat{n} = \partial_n V = -KV$ ON $\partial\Omega$. THEN,

$$-\int_{\partial\Omega} KV^2\,ds = \int_\Omega |\nabla V|^2\,d\underline{x} \longrightarrow \int_\Omega |\nabla V|^2\,d\underline{x} + \int_{\partial\Omega} KV^2\,ds = 0.$$

SINCE $K>0$ THIS IMPLIES THAT $V \equiv 0$ IN Ω SO $u_1 \equiv u_2$.

IF $K=0$ (NO FLUX) THEN $V =$ CONSTANT IN Ω AND SO ANY TWO SOLUTIONS DIFFER BY A CONSTANT.

UNBOUNDED REGIONS - UNIQUENESS THEOREMS

CONSIDER Ω TO BE A DOMAIN IN $\mathbb{R}^3$ WITH

$$\Delta u = 0 \quad \text{outside } \Omega$$

$$u = 0 \quad \text{ON } \partial\Omega$$

AND LET'S IMPOSE u IS BOUNDED AS $r = |\underline{x}| \to \infty$.

DOES THIS MEAN THAT $u \equiv 0$ AT EACH x OUTSIDE Ω?

NO! LET Ω BE A SPHERE OF RADIUS 1 SO THAT

$$u_{rr} + \frac{2}{r} u_r = 0 \quad \text{WITH } r \geq 1$$

$$u = 0 \quad \text{ON } r = 1$$

$$u \text{ BOUNDED AS } r \to \infty$$

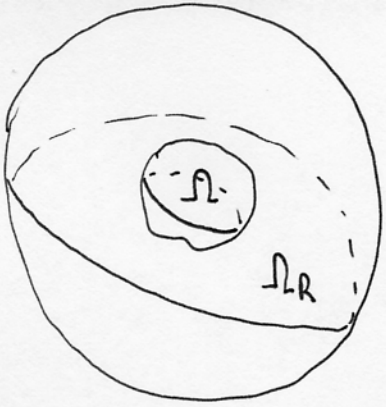
THEN THIS PROBLEM HAS AN INFINITE NUMBER OF SOLUTIONS GIVEN BY $u = A\left(1 - 1/r\right)$ FOR ANY A.

IN ORDER TO GET THE UNIQUE SOLUTION $u = 0$ WE MUST IMPOSE THE STRONGER CONDITION THAT

$$u = O\left(1/r\right) \quad \text{AS } r \to \infty.$$

i.e. $u \to 0$ AS $r \to \infty$ LIKE $u = C/r$.

TO SEE THAT THIS CONDITION IS SUFFICIENT TO GUARANTEE THAT $u \equiv 0$ WE "SOLVE" THE FOLLOWING PROBLEM IN THE DOMAIN WHERE Ω IS SURROUNDED BY A LARGE SPHERE OF RADIUS R.



LET $\Omega_R = \{ |\underline{x}| \leq R / \Omega \}$

REGION OUTSIDE Ω BUT INSIDE SPHERE.

THEN RECALL $u \Delta u = u \nabla \cdot (\nabla u)$

$$= \nabla \cdot [u \nabla u] - |\nabla u|^2 .$$

THUS $$0 = \int_{\Omega_R} u \Delta u \, d\underline{x} = \int_{\Omega_R} \nabla \cdot [u \nabla u] \, d\underline{x} - \int_{\Omega_R} |\nabla u|^2 \, d\underline{x} .$$

NOW USE DIVERGENCE THEOREM

$$\int_{\partial\Omega_R} u \nabla u \cdot \hat{n}_1 \, ds + \int_{\partial\Omega} u \nabla u \cdot \hat{n}_2 \, ds = \int_{\Omega_R} |\nabla u|^2 \, d\underline{x}$$

BUT $u = 0$ ON $\partial\Omega$ AND $\partial\Omega_R$ IS THE BOUNDARY OF A SPHERE OF RADIUS R.

$$\longrightarrow \int_0^{2\pi} \int_0^{\pi} u \frac{\partial u}{\partial r}\Big|_{r=R} R^2 \sin\phi \, d\phi \, d\varphi = \int_{\Omega_R} |\nabla u|^2 \, d\underline{x}$$

NOW IF WE CAN SHOW THAT LHS $\to 0$ AS $R \to \infty$

THEN $\displaystyle\int_{\text{outside } \Omega} |\nabla u|^2 \, d\underline{x} = 0 \longrightarrow u = \text{CONSTANT}$ AND SINCE $u = 0$ ON $\partial\Omega \longrightarrow u \equiv 0$.

SO WE NEED $$u \frac{\partial u}{\partial r}\Big|_{r=R} \lesssim \frac{1}{R^{2+\delta}}$$ FOR ANY $\delta > 0$.

THIS IS CLEARLY SATISFIED IF $u \lesssim C/r$ AS $r \to \infty$

FOR THEN $$u \frac{\partial u}{\partial r}\Big|_{r=R} = -\frac{C^2}{R^3}$$ WITH $\delta = 1$.

THEREFORE IN $\mathbb{R}^3$ THE FOLLOWING PROBLEM HAS A <u>UNIQUE</u> <u>SOLUTION</u>:

$$\begin{cases} \Delta u = 0 & \text{outside } \Omega \\ u = 1 & \text{ON } \partial\Omega \\ u \approx C/r & \text{AS } r = |\underline{x}| \to \infty \end{cases}$$

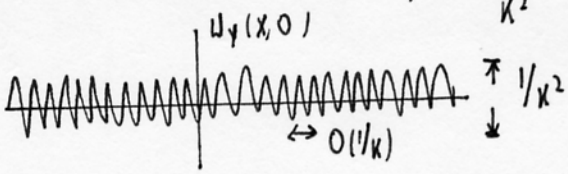
THE CONSTANT C IS CALLED THE "CAPACITANCE" OF THE "BODY" Ω. IT MEASURES HOW MUCH CHARGE CAN BE STORED ON THE SURFACE OF Ω (HINT: USE THE DIVERGENCE THEOREM).

<u>ILL-POSED PROBLEM</u> CONSIDER THE FOLLOWING PROBLEM WHERE WE GIVE BOTH u AND THE FLUX u_y AT $y=0$ FOR

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y \geq 0$$

$$u(x,0) = 0, \quad u_y(x,0) = \frac{1}{K^2}\sin(Kx)$$

IF $K >> 1$ IS LARGE THEN $|u_y| \leq \frac{1}{K^2} \to 0$ BUT u_y HIGHLY OSCILLATORY SO THAT



THE EXACT SOLUTION IS $u = f(y)\sin(Kx) \longrightarrow f'' - K^2 f = 0$ WITH $f(0) = 0$ AND $f'(0) = 1/K^2$. THE SOLUTION IS $f(y) = \frac{1}{K^3}\sinh(Ky)$

AND SO $u(x,y) = \frac{1}{K^3}\sin(Kx)\sinh(Ky)$.

NOTICE THAT FOR EACH FIXED $y>0$ WE GET THAT $|u| \leq \frac{C}{K^3}e^{Ky}$ WHICH HAS UNBOUNDED GROWTH FOR HIGH FREQUENCY $K \to \infty$.

THE SOLUTION <u>DOES NOT HAVE CONTINUOUS DEPENDENCE ON DATA</u> $\to$ ILL-POSED

UNIQUENESS OF SOLUTION TO HEAT EQUATION

WE CONSIDER THE FOLLOWING PDE:

$$u_t = \nabla \cdot [p(\underline{x}) \nabla u] - q u - F \quad \text{IN } \Omega$$

$$\partial_n u + K(u - g) = 0 \quad \text{ON } \partial\Omega$$

$$u(\underline{x}, 0) = u_0(\underline{x}).$$

WE ASSUME THAT $p(\underline{x}) > 0$, $q(\underline{x}) > 0$ AND $K > 0$ FOR X IN Ω.

NOW SHOW THERE IS A UNIQUE SOLUTION.

LET u_1, u_2 BE TWO SOLUTIONS AND DEFINE $V \equiv u_1 - u_2$.

WE WANT TO SHOW THAT $V \equiv 0$ IN Ω AND FOR $t \geq 0$.

BY SUBTRACTION, WE OBTAIN THAT

$$V_t = \nabla \cdot [p(\underline{x}) \nabla V] - q V \quad \text{IN } \Omega$$

$$\partial_n V + K V = 0 \quad \text{ON } \partial\Omega$$

$$V = 0 \quad \text{AT } t = 0.$$

WE WANT TO SHOW $V \equiv 0$. WE MULTIPLY BY V AND INTEGRATE OVER Ω TO OBTAIN:

$$V V_t = V \nabla \cdot [p(\underline{x}) \nabla V] - q V^2.$$

THEN, $$\frac{1}{2} \frac{d}{dt} \int_\Omega V^2 \, d\underline{x} = \int_\Omega V \nabla \cdot [p(\underline{x}) \nabla V] \, d\underline{x} - \int_\Omega q V^2 \, d\underline{x}.$$

NOW $$\nabla \cdot [p V \nabla V] = V \nabla \cdot [p \nabla V] + p \nabla V \cdot \nabla V$$

THIS GIVES $$\frac{d}{dt} \frac{1}{2} \int_\Omega V^2 \, d\underline{x} = \int_\Omega \left(\nabla \cdot [p V \nabla V] - p |\nabla V|^2 \right) d\underline{x} - \int_\Omega q V^2 \, d\underline{x}$$

WE USE THE DIVERGENCE THEOREM NEXT TO OBTAIN,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2\,d\underline{x} = \int_{\partial\Omega} p\,v\,\nabla v\cdot\hat{n}\,ds - \int_\Omega \left(p\,|\nabla v|^2 + q\,v^2\right)d\underline{x}$$

$$(*)\qquad \frac{1}{2}\frac{d}{dt}\int_\Omega v^2\,d\underline{x} = -\int_{\partial\Omega} p\,K\,v^2\,ds - \int_\Omega \left(p\,|\nabla v|^2 + q\,v^2\right)d\underline{x}.$$

NOW DEFINE $\quad E(t) = \frac{1}{2}\int_\Omega v^2(\underline{x},t)\,d\underline{x}.$

THEN $E(t)$ IS CONTINUOUS, $E(0)=0$ SINCE $v(\underline{x},0)=0$ FOR $\underline{x}\in\Omega$, AND $E(t)\geq 0$ SINCE $v^2\geq 0$ IN Ω.

BUT SINCE $p\geq 0$, $q\geq 0$ AND $K>0$, $(*)$ YIELDS $\frac{dE}{dt}\leq 0$.

THEREFORE, BY CALCULUS $E(t)\equiv 0$ FOR ALL t.

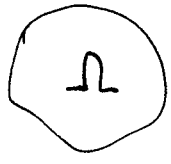
THIS IMPLIES THAT $v\equiv 0$ IN Ω AND FOR $t\geq 0$,

WHICH YIELDS $u_1 = u_2$.

<u>REMARK</u>

(i) THE CONDITION FOR THE <u>EXISTENCE</u> OF A SOLUTION TO

$$\Delta u = F \quad \text{IN } \Omega$$
$$\partial_n u = g \quad \text{ON } \partial\Omega$$



IS THAT $\displaystyle\int_\Omega F\,d\underline{x} = \int_{\partial\Omega} g\,ds.$

<u>PROOF</u> USE DIVERGENCE THEOREM $\displaystyle\int_\Omega \nabla\cdot(\nabla u)\,d\underline{x} = \int_{\partial\Omega}\nabla u\cdot\hat{n}\,ds.$

HENCE $\displaystyle\int_\Omega F\,d\underline{x} = \int_{\partial\Omega}\partial_n u\,ds = \int_{\partial\Omega} g\,ds$ IS NEEDED.

REMARKS ON UNIQUENESS

NONLINEAR PROBLEMS CAN HAVE MORE THAN ONE SOLUTION.

FOR INSTANCE CONSIDER THE NONLINEAR PROBLEM IN A DISK:

$$\Delta u + B e^{u} = 0, \quad 0 \le r \le 1, \ 0 \le \varphi \le 2\pi$$

$$u = 0 \text{ ON } r = 1, \quad u \text{ BOUNDED AS } r \to 0.$$

WE LOOK FOR RADIALLY SYMMETRIC SOLUTIONS $u = u(r)$ SATISFYING

$$u_{rr} + \frac{1}{r} u_r + B e^{u} = 0, \quad 0 \le r \le 1$$

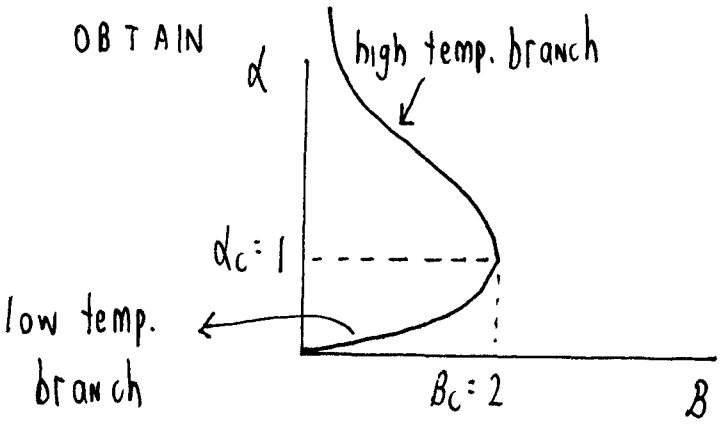
$$u = 0 \text{ ON } r = 1, \quad u \text{ BOUNDED AS } r \to 0$$

THE SOLUTIONS HAVE THE FORM

$$(*) \quad \left\{ u = 2 \log\left(\frac{1+\alpha}{1+\alpha r^2} \right) \right. \quad \text{WITH} \quad B = \frac{8\alpha}{(1+\alpha)^2}, \quad \alpha > 0.$$

(WE VERIFY THIS BELOW) NOTICE $u(0) = 2\log(1+\alpha)$ SO THAT α IS A MEASURE OF THE TEMPERATURE AT THE CENTER. NOTICE THAT $u(0)$ IS THE MAXIMUM TEMPERATURE.

IF WE PLOT α VERSUS B, THEN BY CALCULUS WE OBTAIN



NOTICE $dB/d\alpha = 0$ AT $\alpha_c = 1$.

THEN $B(\alpha_c) = \dfrac{8}{(1+1)^2} = 2 = B_c$

- FOR $0 < B < B_c = 2$, THERE ARE TWO RADIALLY SYMMETRIC SOLUTIONS.
- NO RADIALLY SYMMETRIC SOLUTIONS FOR $B > B_c = 2$.

VERIFICATION OF (*) AS SOLUTION

WE WRITE $u = -2\log(1+\alpha r^2) + 2\log(1+\alpha)$.

THEN $u_r = -4\alpha r(1+\alpha r^2)^{-1}$

$$u_{rr} = -4\alpha(1+\alpha r^2)^{-1} + 8\alpha^2 r^2(1+\alpha r^2)^{-2}$$

NOW $u_{rr} + \frac{1}{r}u_r = 8\alpha^2 r^2(1+\alpha r^2)^{-2} - 8\alpha(1+\alpha r^2)^{-1} = 8\alpha(1+\alpha r^2)^{-2}[\alpha r^2 - (1+\alpha r^2)]$

SO $u_{rr} + \frac{1}{r}u_r = -8\alpha(1+\alpha r^2)^{-2}$.

NOW $e^u = \dfrac{(1+\alpha)^2}{(1+\alpha r^2)^2}$ SO $u_{rr} + \frac{1}{r}u_r + Be^u$

$$= \frac{-8\alpha}{(1+\alpha r^2)^2} + \frac{B(1+\alpha)^2}{(1+\alpha r^2)^2} = 0$$

THUS $8\alpha = B(1+\alpha)^2$ OR $B = 8\alpha/(1+\alpha)^2$.

STABILITY AND STURM-LIOUVILLE THEORY

CONSIDER $u_t = u_{rr} + \frac{1}{r}u_r + Be^u$ IN $0 \le r \le 1$, $t > 0$

$u(1,t) = 0$, u BOUNDED AS $r \to 0$.

LET $u_s(r)$ BE STEADY-STATE SOLUTION GIVEN BY

$$u_s(r) = 2\log\left(\frac{1+\alpha}{1+\alpha r^2}\right) \qquad B = \frac{8\alpha}{(1+\alpha)^2}.$$

IS THIS SOLUTION STABLE? IF WE START WITH INITIAL CONDITION NEAR $u_s(r)$ DO WE REMAIN CLOSE AS $t \to \infty$?

TO STUDY THIS WE LINEARIZE THE PDE AROUND $u_s(r)$. WE WRITE $u(r,t) = u_s(r) + \delta e^{\lambda t}\phi(r)$

WITH $\delta \ll 1$ (δ SMALL).

THIS YIELDS THAT

$$\lambda \delta e^{\lambda t} \phi = u_s'' + \frac{1}{r} u_s' + \delta e^{\lambda t}\left(\phi'' + \frac{1}{r}\phi'\right) + B e^{u_s + \delta e^{\lambda t}\phi}$$

$$= u_s'' + \frac{1}{r} u_s' + \delta e^{\lambda t}\left(\phi'' + \frac{1}{r}\phi'\right) + B e^{u_s}\left(1 + \delta e^{\lambda t}\phi + \dots\right)$$

UPON USING $e^h \approx 1 + h + \dots$ AS $h \to 0$.

THEN SINCE $u_s'' + \frac{1}{r} u_s' + B e^{u_s} = 0$ WE OBTAIN THAT

$$\lambda \phi = \phi'' + \frac{1}{r}\phi' + B e^{u_s}\phi$$

NOW $$B e^{u_s} = \frac{8\alpha}{(1+\alpha)^2} e^{2\log\left(\frac{1+\alpha}{1+\alpha r^2}\right)} = \frac{8\alpha}{(1+\alpha)^2}\frac{(1+\alpha)^2}{(1+\alpha r^2)^2} = \frac{8\alpha}{(1+\alpha r^2)^2}.$$

THE EIGENVALUE PROBLEM BECOMES A SL PROBLEM

$$\phi'' + \frac{1}{r}\phi' + q(r)\phi = \lambda\phi$$

$$\phi(1) = 0, \quad \phi(0) \text{ BOUNDED}$$

WITH $q(r) = \dfrac{8\alpha}{(1+\alpha r^2)^2}$. $\exists$ INFINITE NUMBER OF EIGENVALUES $\lambda_1, \lambda_2, \dots$

IF ONE CAN SHOW THAT ANY EIGENVALUE λ SATISFIES $\lambda_K < 0$, $K = 1, 2, \dots$ $\longrightarrow$ THEN WE HAVE STABILITY OF $u_s(r)$.

IT TURNS OUT TO BE TRUE ONLY FOR $0 < \alpha < 1$ (LOWER BRANCH).

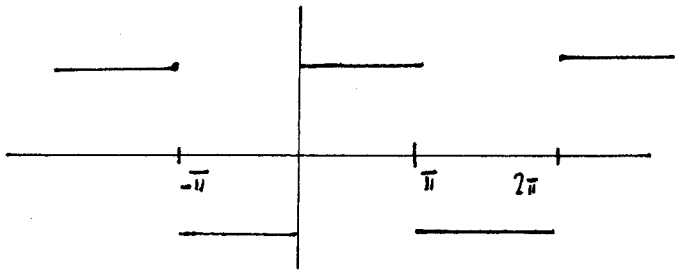
IF $\exists\, \lambda_1 > 0$, THEN $u_s(r)$ IS UNSTABLE. THIS OCCURS ON THE HIGH TEMPERATURE BRANCH WHERE $\alpha > 1$.

GIBBS PHENOMENA

WE CONSIDER THE FOURIER SINE SERIES OF THE FUNCTION

$$f(x) = 1, \quad 0 \le x \le \pi$$

THE PERIODIC ODD-EXTENSION IS



NOW WE WANT TO ESTABLISH CONVERGENCE BEHAVIOR NEAR $x=0$. WE WRITE

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

NOW
$$b_n = \frac{2}{\pi}\int_0^{\pi} (1)\sin(nx)\,dx = \frac{2}{\pi n}\left[-\cos(nx)\right]\Big|_0^{\pi} = \begin{cases} 4/n\pi, & n \text{ ODD} \\ 0, & n \text{ EVEN.} \end{cases}$$

THIS YIELDS
$$f(x) = \sum_{\substack{n=1 \\ n \text{ ODD}}}^{\infty} \frac{4}{n\pi}\sin(nx)$$

THIS CAN BE WRITTEN AS
$$f(x) = \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\sin((2m+1)x)}{(2m+1)}$$

WE WRITE THE PARTIAL SUM AS

$$S_{N+1}(x) = \frac{4}{\pi}\sum_{k=0}^{N} \frac{\sin((2k+1)x)}{2k+1} = \frac{4}{\pi}\,\mathrm{IM}\left(\sum_{k=0}^{N} \frac{e^{i(2k+1)x}}{2k+1}\right).$$

THEN
$$S_{N+1}'(x) = \frac{4}{\pi}\,\mathrm{IM}\left(i\sum_{k=0}^{N} e^{i(2k+1)x}\right) = \frac{4}{\pi}\,\mathrm{IM}\left(ie^{ix}\sum_{k=0}^{N} e^{i2kx}\right)$$

NOW WE LET $Z = e^{2iX}$

THEN $$\sum_{K=0}^{N} Z^K = \frac{1-Z^{N+1}}{1-Z} = \frac{1-e^{2i(N+1)X}}{1-e^{2iX}}$$

THEN $$S_{N+1}'(X) = \frac{4}{\pi} \, \mathrm{IM}\left(i e^{iX} \left(\frac{1-e^{2i(N+1)X}}{1-e^{2iX}} \right) \right)$$

SO $$S_{N+1}'(X) = \frac{4}{\pi} \, \mathrm{IM}\left[\frac{i\left(1-e^{2i(N+1)X}\right)}{e^{-iX}-e^{iX}} \right]$$

SO $$S_{N+1}'(X) = \frac{2}{\pi} \, \mathrm{IM}\left[\frac{e^{2i(N+1)X}-1}{\sin X} \right]$$

THIS YIELDS THAT $$S_{N+1}'(X) = \frac{2}{\pi} \frac{\sin[2(N+1)X]}{\sin(X)}$$

THIS CAN BE INTEGRATED TO GET

$$S_{N+1}(X) = \frac{2}{\pi} \int_0^X \frac{\sin[2(N+1)\xi]}{\sin \xi} \, d\xi \quad \text{SINCE } S_{N+1}(0) = 0.$$

NOW LET $t = 2(N+1)\xi$. THEN

$$S_{N+1} = \frac{2}{\pi} \int_0^{2(N+1)X} \frac{\sin t}{\sin[t/2(N+1)]} \frac{1}{2(N+1)} \, dt$$

NOW FOR $N >> 1$ WE USE $\sin\left[\frac{t}{2(N+1)}\right] \sim \frac{t}{2(N+1)}$.

THEN, WE OBTAIN FOR $N >> 1$

$$S_{N+1}(X) \approx \frac{2}{\pi} \int_0^{2(N+1)X} \frac{\sin t}{t} \, dt$$

NOW NOTICE THAT

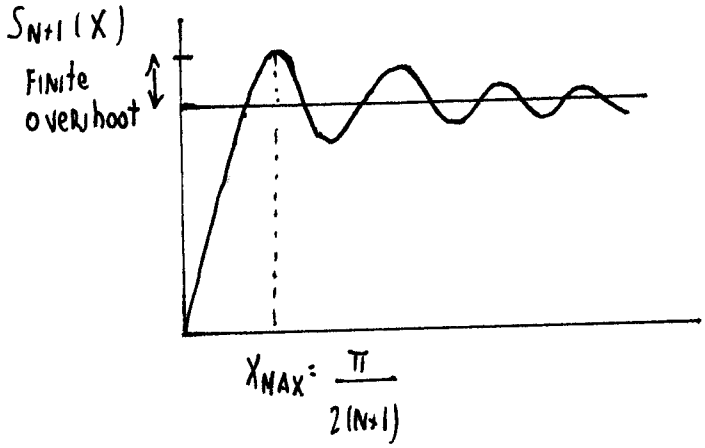
$$S_{N+1}'(x) \cong \frac{4}{\pi}(N+1)\frac{\sin\left[2(N+1)x\right]}{2(N+1)} \approx \frac{2}{\pi}\sin\left[2(N+1)x\right]$$

WE SET $\sin[2(N+1)x] = 0$ TO OBTAIN $2(N+1)x = \pi$

OR $x_{MAX} = \dfrac{\pi}{2(N+1)}$

THIS YIELDS $S_{N+1}\left(\dfrac{\pi}{2(N+1)}\right) \approx \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2}\frac{\sin t}{t}\,dt$ = FINITE AND > 0 AS $N \to \infty$.

THIS IS THE FINITE OVERSHOOT ASSOCIATED WITH GIBBS PHENOMENON. IT IS EASY TO SEE THAT $x_{MAX} = \dfrac{\pi}{2(N+1)}$ IS A MAXIMUM OF $S_{N+1}(x)$ AND HAS ALTERNATING LOCAL MINIMUM AND MAXIMUM OF $S_{N+1}(x)$ AT $x_k = \dfrac{\pi k}{2(N+1)}$, $k = 1, 2, 3, \ldots$



GIBBS PHENOMENA

SINCE $f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$

WITH $a_n = O\left(\dfrac{1}{n}\right)$

AS $n \to \infty$

<u>REMARK</u> (i) THE SAME QUALITATIVE OVERSHOOT BEHAVIOR OCCURS FOR BESSEL EXPANSIONS. FOR INSTANCE,

$$f(r) = \sum_{k=1}^{\infty} c_k J_0(\sqrt{\lambda_k}\, r) \quad \text{WITH } J_0(\sqrt{\lambda_k}) = 0$$

IF $f(r) = 1$, THEN $c_k = \dfrac{\int_0^1 r J_0(\sqrt{\lambda_k}\, r)\,dr}{\int_0^1 r J_0^2(\sqrt{\lambda_k}\, r)\,dr}$, $c_k = O(1/k)$ $k \to \infty$

DISPERSION RELATIONS AND FOURIER TRANSFORMS

RECALL THAT $\mathcal{F}(f(x)) = \hat{F}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$ IS THE FOURIER TRANSFORM. THEN $f(x) = \int_{-\infty}^{\infty} \hat{F}(k)e^{ikx}\,dk$ IS $\mathcal{F}^{-1}(\hat{F}(k))$.

THE IDEA OF A "DISPERSION RELATION" FOR A PDE WHICH IS LINEAR AND HAS CONSTANT COEFFICIENTS ON $-\infty < x < \infty$ IS TO LOOK FOR A SOLUTION OF THE FORM

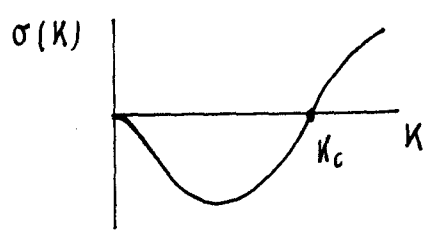
$$u(x,t) = e^{ikx + \sigma t}$$

THE CLASS OF PDE'S ARE, FOR c_0, b_0, b_1, b_2, b_3 CONSTANTS

$$u_{tt} + c_0 u_t = b_0 u_x + b_1 u_{xx} + b_2 u_{xxx} + b_3 u_{xxxx} + \cdots$$

THE RELATION $\sigma = \sigma(k)$ IS CALLED THE DISPERSION RELATION. IT MAY HAVE MORE THAN ONE BRANCH WHEN THERE ARE TWO OR MORE TIME DERIVATIVES.

REMARK THAT IF $\sigma(k)$ HAS THE FORM AS SHOWN, THEN



ILL-POSED

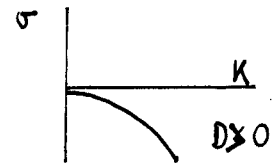
- $0 < k < k_c \rightarrow$ long wavelength
  these disturbances decay in time
  since $\sigma < 0$
- $k > k_c \rightarrow$ short wavelength
  these disturbances grow in time
  since $\sigma > 0$

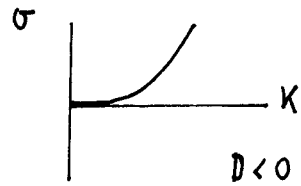
IF $\sigma > 0$ FOR ~~SMALL~~ LARGE k THEN THE PROBLEM IS ILL-POSED IN THE SENSE THAT AN ARBITRARY SMALL WAVELENGTH INITIAL PERTURBATION CAN GROW WITHOUT BOUND AS $k \rightarrow \infty$ AT EACH FIXED t.

ALTERNATIVELY IF σ IS IMAGINARY THEN THIS SIGNIFIES WAVE PROPAGATION.

(i) $u_t = D u_{xx}$ LET $u = e^{ikx + \sigma t}$.

THEN $\sigma = -Dk^2$



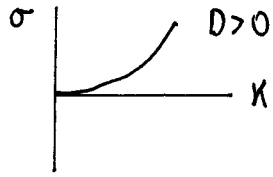


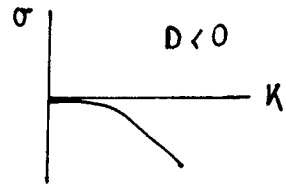
FOR $D > 0$ — USUAL HEAT EQUATION. DECAY OF INITIAL WAVE WITH FASTER DECAY FOR SHORTER SPATIAL WAVELENGTH

FOR $D < 0$ — BACKWARD HEAT EQUATION $\rightarrow$ ILL-POSED.

(ii) $u_t = D u_{xxxx}$ LET $u = e^{ikx + \sigma t}$

THEN $\sigma = Dk^4$





- $D > 0$ IS ILL-POSED
- $D < 0$ BEHAVES LIKE HEAT EQUATION.

CONSIDER THE $D < 0$ CASE AND SET $D = -1$ FOR SIMPLICITY.

THEN $u(x,t) = e^{ikx - k^4 t}$ IS A SOLUTION.

A MORE GENERAL SOLUTION IS

$$u(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{ikx - k^4 t}\, dk.$$

TO SATISFY $u(x,0) = f(x) = \int_{-\infty}^{\infty} A(k) e^{ikx}\, dk$

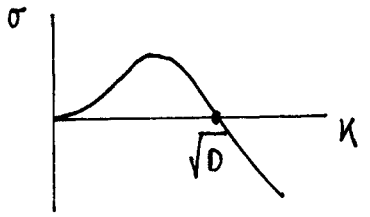
THEN $A(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{-ik\xi}\, d\xi.$

HENCE $u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) \left(\int_{-\infty}^{\infty} e^{ik(x-\xi) - k^4 t}\, dk \right) d\xi.$

iii) LET $u_t = -D u_{xx} - u_{xxxx}$, AND SUBSTITUTE $u = e^{ikx + \sigma t}$.

WE SUBSTITUTE TO OBTAIN

$$\sigma = Dk^2 - k^4.$$



- FOR $0 < k < 1/\sqrt{D}$ $\longrightarrow$ long spatial wavelengths lead to $\sigma > 0$ and so they grow IN TIME.
- FOR $k > 1/\sqrt{D}$ $\longrightarrow$ short wavelengths lead to $\sigma < 0$ AND SO THEY DECAY IN TIME.

(iv) CONSIDER THE LINEARIZED KDV EQUATION

$$u_t + u_x + u_{xxx} = 0, \quad -\infty < x < \infty, \quad t > 0$$

THEN WITH $u(x,t) = e^{ikx + \sigma t}$ $\longrightarrow$ $\sigma + ik + (ik)^3 = 0$.

THIS YIELDS THAT $\sigma = -ik + ik^3$.

OR $u = e^{ikx - ikt + ik^3 t} = e^{ik(x - t + k^2 t)}$

THIS IS DISPERSION; THE "SPEED" OF INDIVIDUAL PARTS OF THE WAVE DEPEND ON THEIR LOCAL WAVELENGTH.

NOW THE SUPERIMPOSED SOLUTION IS

$$u(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{ikx - ikt + ik^3 t}\, dk.$$

WE LET $t = 0$ AND $u(x,0) = f(x) = \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk$

SO THAT $A(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi)\, e^{-ik\xi}\, d\xi.$

THIS YIELDS THAT $u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) \left(\int_{-\infty}^{\infty} e^{ik(x-\xi-t) + ik^3 t}\, dk \right) d\xi$