LAPLACE AND POISSON'S EQUATION

LAPLACE AND POISSON'S EQUATION Arises AS THE STEADY-STATE OF

\[ u_t = \Delta u + F \text{ in } \Omega \]

\[ \partial_\nu u + k(u - u_b) = 0 \text{ on } \partial\Omega \]

The steady-state is

\[ \Delta u = -F \text{ in } \Omega \]

\[ \partial_\nu u + k(u - u_b) = 0 \text{ on } \partial\Omega \]

* Poisson when \( F \neq 0 \)
* Laplace when \( F = 0 \).

We now consider a circular domain, \( 0 \leq \varphi \leq R, 0 \leq \varphi \leq 2\pi \) and derive Poisson's integral formula for the solution to

\[ \Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi \varphi} = 0 \quad 0 \leq r \leq R, 0 \leq \varphi \leq 2\pi \]

\[ u(r, \varphi) = f(\varphi), \text{ } u \text{ bounded } r \to 0 \]

\( u, u_\varphi \) are \( 2\pi \) periodic in \( \varphi \).

We separate variables, \( u(r, \varphi) = R(r) \Phi(\varphi) \) to obtain

\[ (r'' + \frac{1}{r} r') \Phi + \frac{1}{r^2} R \Phi'' = 0 \implies \frac{r^2 R'' + r R'}{R} = -\frac{\Phi''}{\Phi} = \lambda \]

Then \( \Phi'' + \lambda \Phi = 0 \) with \( \Phi(0) = \Phi(2\pi) \) and \( \Phi'(0) = \Phi'(2\pi) \).

Then \( \lambda = \pi^2 \) and \( \Phi_n(\varphi) = \begin{cases} A_n \cos n\varphi + B_n \sin n\varphi, & n \geq 1 \\ A_0, & n = 0 \end{cases} \]

Then \( r^2 R'' + r R' + \pi^2 R = 0 \) let \( R = r^B \to B(8-1) + B + \pi^2 = 0 \)

so that \( B = 2 \pi \). We calculate

\[ R_n(r) = \begin{cases} C_n r^n + d_n r^{-n}, & n \geq 1 \\ c_0 + d_0 \log r, & n = 0 \end{cases} \]
For boundedness we take \(d_0 = d_n = 0\) for \(n = 1, 2, \ldots\).

Then set \(c_0 = c_n = 1\) \(\forall n\), so that \(R_0 = 1\) and \(R_n = R^n\), \(n = 1, 2, \ldots\)

By superposition \(U_n = (A_n \cos(n \varphi) + B_n \sin(n \varphi)) R^n\) is a solution for \(n = 1, 2, \ldots\)

And \(U_0 = A_0\) is a solution.

So \((\ast)\) \(U(\varphi, \varphi) = A_0 + \sum_{n=1}^{\infty} (A_n \cos(n \varphi) + B_n \sin(n \varphi)) R^n\)

Then \(F(\varphi) = A_0 + \sum_{n=1}^{\infty} (A_n \cos(n \varphi) + B_n \sin(n \varphi)) R^n\)

We integrate \(\int_0^{2\pi} F(\varphi) \, d\varphi = A_0 \int_0^{2\pi} \) \(\rightarrow A_0 = \frac{1}{2\pi} \int_0^{2\pi} F(\varphi) \, d\varphi\).

Now we multiply by \(\cos(m \varphi)\) and integrate \(\int_0^{2\pi} \) gives

\[A_n R^n \int_0^{2\pi} \cos^2(n \varphi) \, d\varphi = \int_0^{2\pi} F(\varphi) \cos(n \varphi) \, d\varphi \rightarrow A_n = \frac{1}{\frac{1}{R^n} \int_0^{2\pi} F(\varphi) \cos(n \varphi) \, d\varphi.\]

Similarly, \(B_n = \frac{1}{\frac{1}{R^n} \int_0^{2\pi} F(\varphi) \sin(n \varphi) \, d\varphi.\)

Replace \(\varphi \rightarrow \omega\) in integrals and substitute in \((\ast)\)

This gives:

\[U(\varphi, \omega) = \frac{1}{2\pi} \int_0^{2\pi} F(\omega) \, d\omega + \sum_{n=1}^{\infty} \frac{1}{R^n} \int_0^{2\pi} F(\omega) \left[ \cos(n \varphi) \cos(n \omega) \right.\]

\(\left. + \sin(n \varphi) \sin(n \omega) \right] \, d\omega\)

This can be written as

\[U(\varphi, \omega) = \frac{1}{2\pi} \int_0^{2\pi} F(\omega) \left[ 1 + 2 \sum_{n=1}^{\infty} \left( \frac{R}{R} \right)^n \cos(n(\omega - \varphi)) \right] \, d\omega,\]

Now define \(Z = \frac{R}{R} e^{i n(\omega - \varphi)}\) with \(|Z| < 1\) when \(\varphi < R\).
Now we obtain:

\[
1 + Z + Z^2 + \ldots = \frac{1}{1-Z} \quad \Rightarrow \quad Z + Z^2 + \ldots = \frac{1}{1-Z} - 1.
\]

\[
1 + 2 \sum_{n=1}^{\infty} \left( \frac{r}{R} \right)^n \cos(n(w-\phi)) = 1 + 2 \text{RE} \left( \sum_{n=1}^{\infty} Z^n \right) = 1 + 2 \left( \frac{1}{1-Z} - 1 \right)
\]

\[
= 1 + \frac{2Z}{1-Z} = \frac{1+Z}{1-Z} = \frac{(1+Z)(1-Z)}{|1-Z|^2}
\]

\[
= 1 - |Z|^2 + (Z-\bar{Z}) \frac{1-|Z|^2}{|1-Z|^2}.
\]

Then

\[
1 + 2 \text{RE} \left( \sum_{n=1}^{\infty} Z^n \right) = \text{RE} \left( 1 + 2 \sum_{n=1}^{\infty} Z^n \right) = \text{RE} \left( \frac{1-|Z|^2 + Z - \bar{Z}}{1-|Z|^2} \right) = \frac{1-|Z|^2}{|1-Z|^2}
\]

\[
= \frac{1 - \frac{r^2}{R^2}}{\left( 1 - \frac{r}{R} \cos(w-\phi) \right)^2 + \frac{r^2}{R^2} \sin^2(w-\phi)}
\]

\[
= \frac{1 - \frac{r^2}{R^2}}{1 + \frac{r^2}{R^2} - \frac{2r}{R} \cos(w-\phi)}
\]

Then

\[
1 + 2 \sum_{n=1}^{\infty} \left( \frac{r}{R} \right)^n \cos(n(w-\phi)) = \frac{R^2 - r^2}{R^2 + r^2 - 2rR \cos(w-\phi)}
\]

This yields that

Poisson's integral formula

\[
U(r,\phi) = \frac{1}{2\pi} \int_0^{2\pi} f(\omega) \left( \frac{R^2 - r^2}{R^2 + r^2 - 2rR \cos(w-\phi)} \right) d\omega.
\]

Remark (i): \( U \big|_{r=0} = \frac{1}{2\pi} \int_0^{2\pi} f(\omega) \, d\omega \) is the temperature at center of disc is the average of the temperature over the entire boundary of the disc.

\[
\min_{\phi} f(\phi) = f_{\text{MIN}} \leq U \big|_{r=0} \leq f_{\text{MAX}} \left( \frac{R^2 - r^2}{R^2 + r^2 - 2rR \cos(w-\phi)} \right)
\]
\[(ii) \lim_{r \to R^-} \frac{1}{2\pi} \left( \frac{R^2 - r^2}{R^2 + r^2 - 2r R \cos(\omega - \phi)} \right) = \delta(\omega - \phi) \text{ so } u(r, \phi) = \int_0^{2\pi} F(w) \delta(\omega - \phi) \, dw \]

\[\text{gives } u(R, \phi) = F(\phi).\]

**Max-Min Principle**

Consider \( \Delta u = 0 \) in \( \Omega \) and \( u = f \) on \( \partial \Omega \) with \( \Omega \) being a bounded two-dimensional domain. Then

\[\min f \leq u \leq \max f.\]

**Proof** Suppose, by contradiction, that \( u \) achieves its maximum at some point \( x_0 \) in \( \Omega \). Let \( R \) be any value so that the disc centered at \( x_0 \) is strictly inside \( \Omega \).

Then by Poisson's integral formula

\[u(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} F(w) \left( \frac{R^2 - r^2}{R^2 + r^2 - 2r R \cos(\omega - \phi)} \right) \, dw\]

Then with \( u(x_0) = u(0, \phi) \) we obtain

\[u(x_0) = \frac{1}{2\pi} \int_0^{2\pi} u(r, \phi) \, d\phi\]

so

\[\min_{r=R} u \leq u(x_0) \leq \max_{r=R} u\]

This violates the assumption that \( u \) attains its maximum at \( x = x_0 \) unless of course \( u \) is constant everywhere in \( \Omega \).
EXAMPLE

\[ u_{xx} + u_{yy} = 0 \quad \text{in} \quad 0 < x < \infty, \quad 0 < y < 1 \]
\[ u(x,0) = u_x(x,1) = 0, \quad u(0,y) = 1 - y, \quad u \to 0 \quad \text{as} \quad x \to \infty \]

We separate variables to obtain \( u = X \, Y \) so
\[ \frac{X''}{X} = \frac{Y''}{Y} = -\lambda \]

Then \( \lambda \gamma'' + \lambda \gamma = 0, \quad 0 < y < 1 \)
\( \gamma(0) = \gamma(1) = 0 \)
\( \gamma = \sin(\pi y) \)
\( \lambda = \pi^2 \)

Then \( \frac{X''}{X} = -\pi^2 \) so \( X = e^{-\pi^2 x} \) bounded as \( x \to \infty \).

This yields that
\[ u(x,y) = \sum_{n=1}^{\infty} b_n e^{-\pi^2 n x} \sin(\pi n y) \]

Now
\[ u(0,y) = 1 - y = \sum_{n=1}^{\infty} b_n \sin(\pi n y) \]
\[ b_n = 2 \int_0^1 (1 - y) \sin(\pi n y) \, dy \]

If we calculate
\[ b_n = \frac{2}{\pi n} \]
\( n = 1, 2, 3, \ldots \)

This yields
\[ u(x,y) = \sum_{n=1}^{\infty} \frac{2}{\pi n} e^{-\pi^2 n x} \sin(\pi n y) \]

We let \( z = e^{-\pi x + i\pi y} \) so
\[ u(x,y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \text{IM} \left( \frac{z^n}{n} \right) = \frac{2}{\pi} \text{IM} \left( \sum_{n=1}^{\infty} \frac{z^n}{n} \right) \]

Now
\[ z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots = \sum_{n=1}^{\infty} \frac{z^n}{n} = -\log(1 - z) \]

Thus
\[ u(x,y) = -\frac{2}{\pi} \text{IM} \left[ \log(1 - z) \right] = -\text{IM} \left[ \frac{2}{\pi} \log(1 - z - i\,\phi) \right] \quad (\phi = \tan^{-1} \left( \frac{\text{IM}(1 - z)}{\text{RE}(1 - z)} \right)) \]

\[ 1 - z = 1 - e^{-\pi x} \cos(\pi y) - i \, e^{-\pi x} \sin(\pi y) \]
\[ \text{RE}(1 - z) = 1 - e^{-\pi x} \cos(\pi y), \quad \text{IM}(1 - z) = -e^{-\pi x} \sin(\pi y). \]
\[ Q = \tan^{-1}\left[ \frac{e^{-\pi x} \sin(\pi y)}{1 - e^{-\pi x} \cos(\pi y)} \right] = -\tan^{-1}\left( \frac{\sin(\pi y)}{e^{\pi x} - \cos(\pi y)} \right). \]

This yields that
\[ U(x, y) = \frac{2}{\pi} \tan^{-1}\left( \frac{\sin(\pi y)}{e^{\pi x} - \cos(\pi y)} \right). \]

Notice that there is a discontinuity in the boundary data near \( x = 0, y = 0 \).

Then \( \sin(\pi y) \approx \pi y \)
\[ e^{\pi x} - \cos(\pi y) \approx 1 + \pi x - \left[ 1 - \left( \frac{\pi y}{2} \right)^2 \right] \approx \pi x. \]

This yields near \( x \approx 0 \) and \( y \approx 0 \) that
\[ U(x, y) \approx \frac{2}{\pi} \tan^{-1}\left( \frac{Y}{X} \right) = \frac{2}{\pi} \omega \]

when \( \omega = 0 \rightarrow U = 0 \)
\( \omega = \frac{\pi}{2} \rightarrow U = 1 \)

Now near the corner,
\[ U_{x} \approx \frac{2}{\pi} \frac{-Y/X^2}{1 + Y^2/X^2} \approx -\frac{2}{\pi} \frac{Y}{X^2 + Y^2} \]
which looks like \[ U_{x} \approx -\frac{2}{\pi} \left( \frac{\sin Q}{\Gamma} \right) \]

Similarly,
\[ U_{y} \approx \frac{2}{\pi} \left( \frac{\cos Q}{\Gamma} \right) \]

and so the derivatives blow up \( U(0, 0) \rightarrow (0, 0) \).
Suppose that we want to solve

\[ \Delta u = f \]

in the domain as shown below with \( u = 0 \) on all sides.

We want to see what is the behavior of the solution near the corners at points A and B.

We zoom in a neighborhood near A and introduce a local coordinate system near A.

Near A we introduce polar coordinates \( \Gamma \) and \( \theta \) to get

\[ \Gamma_{\Gamma\Gamma} + \frac{1}{\Gamma} \Gamma_{\Gamma} + \frac{1}{\Gamma^2} \Gamma_{\theta\theta} = F(\Gamma, \theta), \quad \Gamma = 0 \to \text{point A} \]

We then let \( \Gamma = \varepsilon p \) where \( \varepsilon \) is small to localize the region near A. Then \( \Gamma_{\Gamma} = \frac{1}{\varepsilon} \Gamma_{p} \), \( \Gamma_{\Gamma\Gamma} = \frac{1}{\varepsilon^2} \Gamma_{pp} \)

and so

\[ \frac{1}{\varepsilon^2} (\Gamma_{pp} + \frac{1}{p} \Gamma_{p} + \frac{1}{p^2} \Gamma_{\theta\theta}) = F(\varepsilon p, \theta). \]

So if \( F \) is bounded as \( \varepsilon \to 0 \) then in a small neighborhood of point A we must solve

\[ \begin{align*}
\Gamma_{pp} + \frac{1}{p} \Gamma_{p} + \frac{1}{p^2} \Gamma_{\theta\theta} &= 0 \\
\Gamma = 0 \text{ on } \theta = 0 \\
\Gamma = 0 \text{ on } \theta = B
\end{align*} \]

We separate variables \( \Gamma = P(\rho) \Phi(\theta) \)

\[ \frac{\rho^2 (\rho'' + \frac{1}{\rho} \rho')}{\rho} = -\frac{\Phi''}{\Phi} = \lambda \]
This yields that
\[ \Phi' + A \Phi = 0 \quad \rightarrow \quad \Phi = \sin(\sqrt{A} \varphi) \quad \text{so} \quad \sqrt{A} B = \pi \]
\[ \Phi(0) = \Phi(B) = 0 \]

or \( A = \frac{\pi^2}{B^2} \) is smallest \( A \).

Then \( p^2 \Phi'' + p \Phi' - \frac{\pi^2}{B^2} \Phi = 0 \) so \( \Phi = p^{\pi/B} \) is bounded solution.

Therefore near the corner we have
\[ U(p, \varphi) \approx C p^{\pi/B} \sin\left(\frac{\pi \varphi}{B}\right) \quad \text{for some} \quad C. \]

We calculate the electric field
\[ \nabla U \approx \left( \frac{\partial U}{\partial p}, \frac{1}{p} \frac{\partial U}{\partial \varphi} \right) = C \left( \frac{\pi}{B} p^{\pi/B-1} \sin\left(\frac{\pi \varphi}{B}\right), \frac{\pi}{B} p^{\pi/B-1} \cos\left(\frac{\pi \varphi}{B}\right) \right) \]

This yields
\[ \nabla U \approx C \frac{\pi}{B} p^{\pi/B-1} \left( \sin\left(\frac{\pi \varphi}{B}\right), \cos\left(\frac{\pi \varphi}{B}\right) \right) \]

We consider a few cases:

- \( B = \pi/2 \)
\[ \rightarrow \nabla U = O(p) \rightarrow \text{does not blow up as} \quad p \to 0. \]

- \( B = \pi \)
\[ \rightarrow \nabla U = O(1) \rightarrow \text{no blow-up as} \quad p \to 0. \]

- \( B = 3\pi/2 \)
\[ \rightarrow \nabla U = O(p^{-1/2}) \rightarrow \text{blow up as} \quad p \to 0. \]

- \( B = 2\pi \)
\[ \rightarrow \nabla U = O(p^{-1/2}) \rightarrow \text{blow up as} \quad p \to 0. \]

This last example is a "lightning rod" that can store lots of charge at tip.
We want to prove that there is a unique solution to
\[ \Delta u = F \text{ in } \Omega \]
\[ \partial_\nu u + \kappa (u-g) = 0 \text{ on } \partial \Omega \] with \( \kappa > 0 \) and constant.

Here \( \Omega \) is an arbitrary bounded domain.

Suppose that \( u_1, u_2 \) are two solutions and let \( v = u_1 - u_2 \).

Then \( v \) satisfies
\[ \Delta v = 0 \text{ in } \Omega \]
\[ \partial_\nu v + \kappa v = 0 \text{ on } \partial \Omega, \kappa > 0. \]

We want to prove that \( v \equiv 0 \text{ in } \Omega \) so that \( u_1 \equiv u_2 \text{ in } \Omega \).

We recall a vector identity
\[ \nabla \cdot [\mathbf{F} \cdot \mathbf{\phi}] = \mathbf{\phi} \cdot (\partial_\nu \mathbf{F} + \mathbf{F} \cdot \nabla \mathbf{\phi}) \]
so that
\[ \nabla \cdot [\nabla v] = \nabla \cdot \nabla v + \nabla \Delta v = |\nabla v|^2 + \nabla \Delta v. \]

Then
\[ 0 = \int \nabla \Delta v \, dx = \int \nabla \cdot [\nabla v] \, dx = \int \nabla \cdot [\nabla v] \, dx = \int |\nabla v|^2 \, dx \]

Now using divergence theorem,
\[ \int \nabla \cdot [\nabla v] \, \hat{n} \, ds = \int |\nabla v|^2 \, dx \]

But \( \nabla v \cdot \hat{n} = \partial_\nu v = -\kappa v \text{ on } \partial \Omega. \) Then,
\[ -\int \kappa v^2 \, ds = \int |\nabla v|^2 \, dx \rightarrow \int |\nabla v|^2 \, dx + \int \kappa v^2 \, ds = 0. \]

Since \( \kappa > 0 \) this implies that \( v \equiv 0 \text{ in } \Omega \) so \( u_1 \equiv u_2. \)

If \( \kappa = 0 \) (no flux) then \( v \) = constant in \( \Omega \) and so any two solutions differ by a constant.
UNBOUNDED REGIONS - UNIQUENESS THEOREMS

Consider \( \Omega \) to be a domain in \( \mathbb{R}^3 \) with
\[
\begin{align*}
\Delta u &= 0 \quad \text{outside } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

and let us impose \( u \) is bounded as \( \gamma = \|x\| \to \infty \).

Does this mean that \( u \equiv 0 \) at each \( x \) outside \( \Omega \)?

No! Let \( \Omega \) be a sphere of radius 1 so that
\[
\begin{align*}
u_{rr} + \frac{2}{r} u_r &= 0 \quad \text{with } \gamma > 1 \\
u &= 0 \quad \text{on } \gamma = 1 \\
u \text{ bounded as } \gamma \to \infty
\end{align*}
\]

Then this problem has an infinite number of solutions given by
\[
u = A \left( \frac{1}{\gamma} - 1/\gamma \right)
\]
for any \( A \).

In order to get the unique solution \( u = 0 \), we must impose the stronger condition that
\[
u = 0 \left( \frac{1}{\gamma} \right) \quad \text{as } \gamma \to \infty
\]

i.e., \( u \to 0 \) as \( \gamma \to \infty \) like \( u = C/\gamma \).

To see that this condition is sufficient to guarantee that \( u \equiv 0 \), we "solve" the following problem in the domain where \( \Omega \) is surrounded by a large sphere of radius \( R \).
Let \( S_R = \{ x \in R / \Omega \} \)
region outside \( \Omega \) but inside sphere.

Then recall
\[
\nabla \Delta u = \nabla \cdot (\nabla u) = \nabla \cdot [\nabla u \nabla] - |\nabla u|^2.
\]

Thus
\[
0 = \int_{S_R} u \Delta u \, d\mathbf{x} = \int_{S_R} \nabla \cdot [\nabla u \nabla] \, d\mathbf{x} - \int_{S_R} |\nabla u|^2 \, d\mathbf{x}.
\]

Now use divergence theorem
\[
\int_{\partial S_R} u \nabla u \cdot \mathbf{n}_1 \, d\mathbf{s} + \int_{\partial S_R} u \nabla u \cdot \mathbf{n}_2 \, d\mathbf{s} = \int_{S_R} |\nabla u|^2 \, d\mathbf{x}
\]

But \( u = 0 \) on \( \partial \Omega \) and \( \partial S_R \) is the boundary of a sphere of radius \( R \).

\[
\Rightarrow - \int_0^{2\pi} \int_0^\pi \frac{u}{\partial \mathbf{r}} \bigg|_{\mathbf{r} = R} \ R^2 \sin \phi \, d\phi \, d\theta = \int_{S_R} |\nabla u|^2 \, d\mathbf{x}
\]

Now if we can show that \( \text{LHS} \to 0 \) as \( R \to \infty \)

then \( |\nabla u|^2 \, d\mathbf{x} = 0 \) outside \( \Omega \)

\( \Rightarrow u = \text{constant} \) and since \( u = 0 \) on \( \partial \Omega \) \( \Rightarrow u \equiv 0 \).

So we need \( \frac{\partial u}{\partial \mathbf{r}} \big|_{\mathbf{r} = R} \leq \frac{1}{R^{2+\delta}} \) for any \( \delta > 0 \).

This is clearly satisfied if \( u \equiv \frac{c}{\mathbf{r}} \) as \( \mathbf{r} \to \infty \)

for then \( \frac{\partial u}{\partial \mathbf{r}} \big|_{\mathbf{r} = R} = -\frac{c^2}{R^3} \) with \( \delta = 1 \).
Therefore in \( \mathbb{R}^3 \) the following problem has a unique solution:

\[
\begin{align*}
    \Delta u &= 0 \quad \text{outside } \Omega \\
    u &= 1 \quad \text{on } \partial \Omega \\
    u &\approx c' / r, \quad r = |x| \to \infty
\end{align*}
\]

The constant \( c \) is called the "capacitance" of the "body" \( \Omega \). It measures how much charge can be stored on the surface of \( \Omega \) (hint: use the divergence theorem).

I Ill-posed problem

Consider the following problem where we give both \( u \) and the flux \( u_y \) at \( y = 0 \) for

\[
\begin{align*}
    u_{xx} + u_{yy} &= 0, \quad -\infty < x < \infty, \quad y > 0 \\
    u(x, 0) &= 0, \quad u_y(x, 0) = \frac{1}{K^2} \sin (Kx)
\end{align*}
\]

If \( K \gg 1 \) is large then \( |u_y| \leq \frac{1}{K^2} \to 0 \) but \( u_y \) highly oscillatory so that

\[
\begin{align*}
    u_y(x, 0) &\to 0(1/K) \\
    \quad &\rightarrow 0(1/K^2) \\
\end{align*}
\]

The exact solution is

\[
\begin{align*}
    u &= f(y) \sin (Kx) \quad \to \quad f'' - K^2 f = 0
\end{align*}
\]

with \( f(0) = 0 \) and \( f'(0) = 1/K^2 \). The solution is

\[
\begin{align*}
    f(y) &= \frac{1}{K^3} \sinh (Ky) \\
    u(x, y) &= \frac{1}{K^3} \sin (Kx) \sinh (Ky)
\end{align*}
\]

and so

\[
\begin{align*}
    |u| &\leq \frac{c}{K^3}
\end{align*}
\]

Notice that for each fixed \( y > 0 \) we get that \( |u| \leq \frac{c}{K^3} \), which has unbounded growth for high frequency \( K \to \infty \).

The solution does not have continuous dependence on data \( \to \text{ill-posed} \).
WE CONSIDER THE FOLLOWING PDE:

\[ u_t = \nabla \cdot (p(x) \nabla u) - q(x) u - f \text{ in } \Omega \]
\[ \partial_n u + k(u - g) = 0 \text{ on } \partial \Omega \]
\[ u(x, 0) = u_0(x). \]

WE ASSUME THAT \( p(x) > 0, \ q(x) > 0 \) AND \( k > 0 \) FOR \( x \in \Omega \).

NOW SHOW THERE IS A UNIQUE SOLUTION.

LET \( u_1, u_2 \) BE TWO SOLUTIONS AND DEFINE \( \psi = u_1 - u_2 \).

WE WANT TO SHOW THAT \( \psi \equiv 0 \) IN \( \Omega \) AND FOR \( t > 0 \).

BY SUBTRACTION, WE OBTAIN THAT

\[ \psi_t = \nabla \cdot (p(x) \nabla \psi) - q \psi \text{ in } \Omega \]
\[ \partial_n \psi + k \psi = 0 \text{ on } \partial \Omega \]
\[ \psi = 0 \text{ at } t = 0. \]

WE WANT TO SHOW \( \psi \equiv 0 \). WE MULTIPLY BY \( \psi \) AND INTEGRATE OVER \( \Omega \) TO OBTAIN:

\( \psi \psi_t = \psi \nabla \cdot (p(x) \nabla \psi) - q \psi^2. \)

THEN,

\[ \frac{1}{2} \frac{d}{dt} \int \Omega \psi^2 \, dx = \int \Omega \nabla \cdot (p(x) \nabla \psi) \, dx - \int \Omega q \psi^2 \, dx. \]

NOW

\( \nabla \cdot (p \psi \nabla \psi) = \psi \nabla \cdot (p \nabla \psi) + p \nabla \psi \cdot \nabla \psi \)

THIS GIVES

\[ \frac{d}{dt} \left( \frac{1}{2} \int \Omega \psi^2 \, dx \right) = \int \Omega \nabla \cdot (p \psi \nabla \psi) - p |\nabla \psi|^2 \, dx - \int \Omega q \psi^2 \, dx. \]
We use the divergence theorem next to obtain,

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \nabla^2 \, dx = \int_{\partial \Omega} \mathbf{v} \cdot \mathbf{n} \, ds - \int_{\Omega} \left( p |\nabla \mathbf{v}|^2 + g \nabla \mathbf{v}^2 \right) \, dx \]

\( \tag{x} \)

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \nabla^2 \, dx = -\int_{\partial \Omega} p \kappa \nabla \mathbf{v} \, ds - \int_{\Omega} \left( p |\nabla \mathbf{v}|^2 + g \nabla \mathbf{v}^2 \right) \, dx. \]

Now define \( E(t) = \frac{1}{2} \int_{\Omega} \nabla^2 (x,t) \, dx \).

Then \( E(t) \) is continuous, \( E(0) = 0 \) since \( \nabla (x,0) = 0 \) for \( x \in \Omega \), and \( E(t) \geq 0 \) since \( \nabla^2 \geq 0 \) in \( \Omega \).

But since \( p \geq 0, g \geq 0 \) and \( \kappa > 0 \), \( \text{(x)} \) yields \( \frac{dE}{dt} \leq 0 \).

Therefore, by calculus \( E(t) \equiv 0 \) for all \( t \).

This implies that \( \nabla = 0 \) in \( \Omega \) and for \( t > 0 \),

which yields \( u_1 = u_2 \).

Remark

(1) The condition for the existence of a solution

\[ \Delta u = F \quad \text{in} \quad \Omega \]

\[ \partial_{\Omega} u = g \quad \text{on} \quad \partial \Omega \]

is that \( \int_{\Omega} F \, dx = \int_{\partial \Omega} g \, ds \).

Proof: Use divergence theorem \( \int_{\Omega} \nabla \cdot (\nabla u) \, dx = \int_{\partial \Omega} \nabla u \cdot \mathbf{n} \, ds \).

Hence \( \int_{\Omega} F \, dx = \int_{\partial \Omega} \partial_{\Omega} u \, ds = \int_{\partial \Omega} g \, ds \) is needed.
Nonlinear problems can have more than one solution. For instance, consider the nonlinear problem in a disk:

\[ \Delta u + Be^u = 0, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi \]

\[ u = 0 \text{ on } \Gamma = 1, \quad u \text{ bounded as } \Gamma \to 0. \]

We look for radially symmetric solutions \( u = u(r) \) satisfying

\[ u_{rr} + \frac{1}{r} u_r + Be^u = 0, \quad 0 \leq r \leq 1 \]

\[ u = 0 \text{ on } r = 1, \quad u \text{ bounded as } r \to 0. \]

The solutions have the form

\[
\begin{align*}
\alpha > 0 \\
\gamma = 2 \log \left( \frac{1 + \alpha}{1 + \alpha \gamma^2} \right)
\end{align*}
\]

(We verify this below.) Notice \( u(0) = 2 \gamma \log (1 + \alpha) \) so that \( \alpha \) is a measure of the temperature at the center. Notice that \( u(0) \) is the maximum temperature.

If we plot \( \alpha \) versus \( B \), then by calculus we obtain \( \alpha \) vs. \( B \) graph.

Observe \( dB/\alpha = 0 \) at \( \alpha = 1 \).

Then \( B(\alpha) = \frac{8}{(1 + \alpha)^2} = 2 = B_c \)

- For \( 0 < B < B_c = 2 \), there are two radially symmetric solutions.
- No radially symmetric solution for \( B > B_c = 2 \).
WE WRITE
\[ U = -2 \log(1 + \alpha \Gamma^2) + 2 \log(1 + \alpha) \].

THEN
\[ U_\Gamma = -4 \alpha \Gamma (1 + \alpha \Gamma^2)^{-1} \]
\[ U_{\Gamma \Gamma} = -4 \alpha (1 + \alpha \Gamma^2)^{-1} + 8 \alpha^2 \Gamma^2 (1 + \alpha \Gamma^2)^{-2} \]

NOW
\[ U_{\Gamma \Gamma} + \frac{1}{\Gamma} U_\Gamma = 8 \alpha^2 \Gamma^2 (1 + \alpha \Gamma^2)^{-2} - 8 \alpha (1 + \alpha \Gamma^2)^{-1} = 8 \alpha (1 + \alpha \Gamma^2)^{-2} \left[ \alpha \Gamma^2 - (1 + \alpha \Gamma^2) \right] \]
\[ \therefore U_{\Gamma \Gamma} + \frac{1}{\Gamma} U_\Gamma = -8 \alpha (1 + \alpha \Gamma^2)^{-2} \]

NOW
\[ e^U = \frac{(1 + \alpha)^2}{(1 + \alpha \Gamma^2)^2} \]
\[ \therefore U_{\Gamma \Gamma} + \frac{1}{\Gamma} U_\Gamma + 8 e^U \]
\[ = -8 \alpha \]
\[ + \frac{B(1 + \alpha)^2}{(1 + \alpha \Gamma^2)^2} = 0 \]

THUS \[ 8 \alpha = 8 (1 + \alpha)^2 \] OR \[ B = 8 \alpha / (1 + \alpha)^2 \].

STABILITY AND STURM-LIOUVILLE THEORY

CONSIDER
\[ U_t = U_{\Gamma \Gamma} + \frac{1}{\Gamma} U_\Gamma + 8 e^U \] IN \( 0 < \Gamma < 1, \ t > 0 \)
\[ U(1, t) = 0, \ U \text{ BOUNDED AS } \Gamma \to 0. \]

LET \( U_S(\Gamma) \) BE STEady-STATE SOLUTION GIVEN BY
\[ U_S(\Gamma) = 2 \log \left( \frac{1 + \alpha}{1 + \alpha \Gamma^2} \right), \ B = \frac{8 \alpha}{(1 + \alpha)^2} \].

IS THIS SOLUTION STABLE? IF WE START WITH INITIAL CONDITION NEAR \( U_S(\Gamma) \) DO WE REMAIN CLOSE AS \( t \to \infty \)?

TO STUDY THIS WE LINEARIZE THE PDE AROUND \( U_S(\Gamma) \). WE WRITE
\[ U(\Gamma, t) = U_S(\Gamma) + \delta e^{\frac{\lambda t}{\Gamma}} \phi(\Gamma) \]
WITH \( \delta \ll 1 \) (\( \delta \) SMALL).
This yields that
\[ A \exp \left( \frac{1}{\Gamma} \right) \phi = u_s'' + \frac{1}{\Gamma} u_s' + \exp \left( \frac{1}{\Gamma} \phi' \right) + B e^{u_s + \exp \left( \frac{1}{\Gamma} \phi \right)} \]
\[ = u_s'' + \frac{1}{\Gamma} u_s' + \exp \left( \frac{1}{\Gamma} \phi' \right) + B e^{u_s (1 + \exp \left( \frac{1}{\Gamma} \phi \right))} \]

Upon using \( e^h \approx 1 + h + \ldots \) as \( h \to 0 \).

Then since \( u_s'' + \frac{1}{\Gamma} u_s' + B e^{u_s} = 0 \) we obtain that
\[ A \phi = \phi'' + \frac{1}{\Gamma} \phi' + B e^{u_s} \phi \]

Now \( B e^{u_s} = \frac{8d}{(1 + d)^2} e^{2 \log \left( \frac{1 + d}{1 + d r^2} \right)} = \frac{8d}{(1 + d)^2 \left(1 + d r^2\right)^2} = \frac{8d}{(1 + d r^2)^2} \)

The eigenvalue problem becomes a SL problem
\[ \phi'' + \frac{1}{\Gamma} \phi' + q(\Gamma) \phi = \lambda \phi \]
\[ \phi(1) = 0, \ \phi(0) \text{ bounded} \]

with \( q(\Gamma) = \frac{8d}{(1 + d r^2)^2} \) an infinite number of eigenvalues \( \lambda_1, \lambda_2, \ldots \)

If one can show that any eigenvalue \( \lambda \) satisfies
\[ \lambda < 0, \ \forall \lambda_1, 2, \ldots \] then we have stability of \( u_s(\Gamma) \).

It turns out to be true only for \( 0 < \lambda < 1 \) (lower branch).

If \( \exists \lambda_1 > 0 \), then \( u_s(\Gamma) \) is unstable. This occurs on the high temperature branch where \( d > 1 \).
GIBBS PHENOMENA

WE CONSIDER THE FOURIER SINE SERIES OF THE FUNCTION

\[ f(x) = 1, \quad 0 < x < \pi \]

THE PERIODIC ODD-EXTENSION IS

\[ \frac{-\pi}{2} < x < \frac{\pi}{2} \]

NOW WE WANT TO ESTABLISH CONVERGENCE BEHAVIOR NEAR \( x = 0 \). WE WRITE

\[ f(x) = \sum_{n=1}^{\infty} b_n \sin(n x) \]

NOW

\[ b_n = \frac{2}{\pi} \int_{0}^{\pi} (1) \sin(n x) \, dx = \frac{2}{\pi n} \left[ \sin(n x) \right]_{0}^{\pi} = \begin{cases} \frac{4}{\pi n}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \]

THIS YIELD

\[ f(x) = \sum_{n=1}^{\infty} \frac{4}{n \pi} \sin(n x) \]

THIS CAN BE WRITTEN AS

\[ f(x) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin((2m+1)x)}{(2m+1)} \]

WE WRITE THE PARTIAL SUM AS

\[ S_{N+1}(x) = \frac{4}{\pi} \sum_{k=0}^{N} \frac{\sin((2k+1)x)}{2k+1} = \frac{4}{\pi} \, IM \left( \sum_{k=0}^{N} \frac{e^{i(2k+1)x}}{2k+1} \right) \]

THEN

\[ S_{N+1}'(x) = \frac{4}{\pi} \, IM \left( i e^{ix} \sum_{k=0}^{N} \frac{e^{i(2k+1)x}}{2k+1} \right) = \frac{4}{\pi} \, IM \left( i e^{ix} \sum_{k=0}^{N} e^{i(2k)k} \right) \]
Let \( Z = e^{2ix} \)

Then
\[
\sum_{k=0}^{N} Z^k = \frac{1-Z^{N+1}}{1-Z} = \frac{1-e^{2i(N+1)x}}{1-e^{2ix}}
\]

Then
\[
S_{N+1}'(x) = \frac{4}{\pi} \text{IM} \left( i e^{iX} \left( \frac{1-e^{2i(N+1)x}}{1-e^{2ix}} \right) \right)
\]

so
\[
S_{N+1}'(x) = \frac{4}{\pi} \text{IM} \left[ \frac{i (1-e^{2i(N+1)x})}{e^{-ix}-e^{ix}} \right]
\]

so
\[
S_{N+1}'(x) = \frac{2}{\pi} \text{IM} \left[ \frac{e^{2i(N+1)x} - 1}{\sin x} \right]
\]

This yields that
\[
S_{N+1}(x) = \frac{2}{\pi} \frac{\sin \left[ 2(N+1)x \right]}{\sin x}
\]

This can be integrated to get
\[
S_{N+1}(x) = \frac{2}{\pi} \int_{0}^{x} \frac{\sin \left[ 2(N+1)\xi \right]}{\sin \xi} \, d\xi \quad \text{since} \quad S_{N+1}(0) = 0.
\]

Now let \( t = 2(N+1)\xi \). Then
\[
S_{N+1} = \frac{2}{\pi} \int_{0}^{2(N+1)x} \frac{\sin t}{\sin \left[ t/(2(N+1)) \right]} \frac{1}{2(N+1)} \, dt
\]

Now for \( N \gg 1 \) we use \( \sin \left[ \frac{t}{2(N+1)} \right] \sim \frac{t}{2(N+1)} \).

Then, we obtain for \( N \gg 1 \)
\[
S_{N+1}(x) \approx \frac{2}{\pi} \int_{0}^{2(N+1)x} \frac{\sin t}{t} \, dt
\]
Now notice that
\[ S_{N+1}(x) = \frac{4}{\pi} \left( \frac{N+1}{2N+1} \right) \sin\left( \frac{2(N+1)x}{2N+1} \right) \approx \frac{2}{\pi} \sin\left( \frac{2(N+1)x}{2N+1} \right) \]

We set \( \sin\left( \frac{2(N+1)x}{2N+1} \right) = 0 \) to obtain \( 2(N+1)x = \pi \)

or \( x_{\text{max}} = \frac{\pi}{2(N+1)} \).

This yields \( S_{N+1}\left( \frac{\pi}{2(N+1)} \right) \approx \frac{2}{\pi} \int_0^{\pi/2} \frac{\sin t}{t} \, dt = \text{finite and } > 0 \) as \( N \to \infty \).

This is the finite overshoot associated with the Gibbs phenomenon. It is easy to see that \( x_{\text{max}} = \frac{\pi}{2(N+1)} \) is a maximum of \( S_{N+1}(x) \) and has alternating local minimum and maximum of \( S_{N+1}(x) \) at \( x = \frac{\pi k}{2(N+1)} \), \( k = 1, 2, 3, \ldots \).

\[ S_{N+1}(x) \]

\[ \text{Finite overshoot} \]

\[ x_{\text{max}} = \frac{\pi}{2(N+1)} \]

Gibbs phenomena since \( f(x) = \sum_{n=1}^{\infty} q_n \sin(n \pi x) \)

with \( q_n = o\left( \frac{1}{n} \right) \) as \( n \to \infty \).

Remark (i) the same qualitative overshoot behavior occurs for Bessel expansion. For instance,
\[ F(\gamma) = \sum_{K=1}^{\infty} C_K J_0(\sqrt{A_K} \gamma) \quad \text{with} \quad J_0(\sqrt{A_K}) = 0 \]

if \( F(\gamma) = 1 \), then \( C_K = \frac{1}{\Gamma} \int_0^{1/\gamma} J_0(\sqrt{A_K} \gamma) \, d\gamma \rightarrow 0 \quad \text{as} \quad K \to \infty \).
Recall that \( \hat{f}(f(x)) = \hat{F}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx \) is the Fourier transform. Then \( F(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \, dk \) is \( \hat{F}^{-1}(\hat{F}(k)) \).

The idea of a "dispersion relation" for a PDE which is linear and has constant coefficients on \(-\infty < x < \infty\) is to look for a solution of the form
\[
\phi(x, t) = e^{ikx + \sigma t}
\]

The class of PDE's are, for constants \(c_0, b_0, b_1, b_2, b_3\)
\[
\phi_{tt} + c_0 \phi_t = b_0 \phi_x + b_1 \phi_{xx} + b_2 \phi_{xxx} + b_3 \phi_{xxxx} + \ldots
\]

The relation \(\sigma = \sigma(k)\) is called the dispersion relation. It may have more than one branch when there are two or more time derivatives.

Remark that if \(\sigma(k)\) has the form as shown, then

\[
\sigma(k) \quad \begin{cases} 
\sigma(k) \quad \begin{cases} 
0 < k < k_c & \text{long wavelength}
\end{cases} & \text{there disturbance decay in time since } \sigma < 0 \\
k > k_c & \text{short wavelength}
\end{cases}
\]

ILL-POSED

IF \(\sigma > 0\) FOR LARGE \(k\), then the problem is ill-posed in the sense that an arbitrary small wavelength initial perturbation can grow without bound as \(k \to 0\) at each fixed \(t\).
ALTERNATIVELY IF $\sigma$ IS IMAGINARY THEN THIS SIGNIFIES WAVE PROPAGATION.

(i) $u_t = Du_{xx}$ \quad \text{LET} \quad u = e^{ixx + \sigma t}$

then $\sigma = -DK^2$

\begin{align*}
\text{FOR } D > 0 & \quad \text{USUAL HEAT EQUATION, DECAY OF INITIAL WAVE WITH FASTER DECAY FOR SHORTER SPATIAL WAVELENGTH} \\
\text{FOR } D < 0 & \quad \text{BACKWARD HEAT EQUATION $\rightarrow$ ILL-POSED.}
\end{align*}

(ii) $u_t = Du_{xxxx}$ \quad \text{LET} \quad u = e^{ixx + \sigma t}$

then $\sigma = DK^4$

\begin{align*}
\text{FOR } D > 0 & \quad \text{ILL-POSED} \\
\text{FOR } D < 0 & \quad \text{BEHAVE LIKE HEAT EQUATION.}
\end{align*}

CONSIDER THE $D < 0$ CASE AND SET $D = -1$ FOR SIMPLICITY.

THEN $u(x, t) = e^{ixx - K^4 t}$ \quad \text{IS A SOLUTION.}

A MORE GENERAL SOLUTION IS

$$u(x, t) = \int_{-\infty}^{\infty} A(k) e^{ikx - K^4 t} \, dk.$$ 

TO SATISFY $u(x, 0) = f(x) = \int_{-\infty}^{\infty} A(k) e^{ikx} \, dk$

THEN $A(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) e^{-iks} \, ds$.

HENCE $u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \left( \int_{-\infty}^{\infty} e^{i(kx - s) - K^4 t} \, dk \right) ds$. 
Let $u_t = - Du_{xx} - u_{xxxx}$, and substitute $u = e^{ikx + \sigma t}$.

We substitute to obtain $\sigma = Dk^2 - k^4$.

- For $0 < k < 1/\sqrt{D}$, long spatial wavelengths lead to $\sigma > 0$ and so they grow in time.
- For $k > 1/\sqrt{D}$, short wavelengths lead to $\sigma < 0$ and so they decay in time.

Consider the linearized KDV equation

$$u_t + u_x + u_{xxx} = 0, \quad -\infty < x < \infty, \quad t > 0$$

Then with $u(x, t) = e^{ikx + \sigma t} \rightarrow \sigma + ik + (ik)^3 = 0$.

This yields that $\sigma = -ik + ik^3$.

Or $u = e^{ikx - ikt + ik^3 t} = e^{ik(x - t + k^2 t)}$.

This is dispersion; the "speed" of individual parts of the wave depend on their local wavelength.

Now the superimposed solution is

$$u(x, t) = \int_{-\infty}^{\infty} A(k) e^{ikx - ikt + ik^3 t} \, dk.$$ 

We let $t = 0$ and $u(x, 0) = f(x) = \int_{-\infty}^{\infty} A(k) e^{ikx} \, dk$.

So that $A(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) e^{-iks} \, ds$.

This yields that $u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \left( \int_{-\infty}^{\infty} e^{iks - ikt + ik^3 t} \, dk \right) ds$. 