DERIVATION OF THE HEAT EQUATION

Let \( V \) be an arbitrarily small domain inside a given volume \( \Omega \)

rate of change = heat flowing + heat energy of heat energy across boundaries generated per unit time internally per unit time

heat energy = \( \int_V c \rho u \, dx \) with \( c \) = specific heat
\( \rho \) = density, \( u = u(x,t) \) temperature

heat energy generated internally per unit time = \( \int_V Q(x,t,u) \, dx \)

Let \( \phi \) = heat flux vector on \( \partial V \). (boundary of \( V \)).

The normal component on \( V \) is \( \phi \cdot \hat{n} \)

If \( \phi \cdot \hat{n} < 0 \) heat flux is inward to \( V \)

Then
\[
\frac{d}{dt} \int_V c \rho u \, dx = - \int_{\partial V} \phi \cdot \hat{n} \, ds + \int_V Q \, dx
\]

Now use the divergence theorem
\[
\int_V \nabla \cdot A \, dx = \int_{\partial V} A \cdot \hat{n} \, ds.
\]

Thus
\[
\frac{d}{dt} \int_V c \rho u \, dx + \int_V \nabla \cdot \phi \, dx - \int_V Q \, dx = 0.
\]

Now if \( c, \rho \) independent of \( t \) we have
\[
\int_V (c \rho u_t + \nabla \cdot \phi - Q) \, dx = 0.
\]
Since \( V \) is arbitrary domain we have
\[
C P U_t = -\nabla \cdot \phi + Q(x,t,u) .
\]

Finally by Fourier's law (of Fick's law of diffusion)
\[
\phi = -K \nabla u . \quad K = \text{thermal conductivity}.
\]

This leads to \( U_t = D \nabla^2 U + Q(x,t,u) \)
with \( Q(x,t,u) = Q/pc \), \( D = K/pc \) thermal diffusivity.

We then have the boundary condition that
\[
\phi \cdot \hat{n} = h(u - u_b) \quad h > 0 \quad h = \text{constant}.
\]

Then we write \( \phi = -K \nabla u \) so that Newton's law of cooling is
\[
\nabla u \cdot \hat{n} = \partial_n u = -b(u - u_b) \quad \text{on } \partial \Omega \quad b = h/K \text{ Biot number},
\]
with \( u_b \) given, is the outside temperature.

This gives the heat conduction problem
\[
U_t = D \nabla^2 U + Q(x,t,u) \quad \text{in } \Omega
\]
\[
\partial_n u = -b(u - u_b) \quad \text{on } \partial \Omega \quad \text{(boundary condition)}
\]
\[
U(x,0) = u_0(x) \quad \text{initial temperature}.
\]

Remark
(i) \( \nabla^2 U = U_{xx} + U_{yy} + U_{zz} \) when \( \mathcal{X} = (x,y,z) \).

Occasionally one writes \( \nabla^2 = \Delta \).

(ii) We can only solve linear problems where
\[
Q(x,t,u) = b(x,t)u + F(x,t);
\]
if \( b < 0 \) sink of heat proportional to \( u \).
\( F < 0 \) corresponds to sink of heat.
THE ONE-DIMENSIONAL PROBLEM WITH $Q(x, t, u) = F(x)$ IS SIMPLY

\[\begin{align*}
\frac{\partial u}{\partial t} &= D \frac{\partial^2 u}{\partial x^2} + F(x) & 0 < x < L \\
\frac{\partial u}{\partial x} &= h_2 \left(u - u_2\right) & \text{on } x = L \\
\frac{\partial u}{\partial x} &= h_1 \left(u - u_1\right) & \text{on } x = 0 \\
u(x, 0) &= u_0(x)
\end{align*}\]

THE STEADY-STATE SOLUTION CORRESPONDING TO $u_s(x) = \lim_{t \to \infty} u(x, t)$ SATISFIES

\[\begin{align*}
D u_s'' &= -F(x) & 0 < x < L \\
u_s' &= h_2 \left(u_s - u_2\right) & \text{on } x = L; \quad u_s' = h_1 \left(u_s - u_1\right), \; x = 0
\end{align*}\]

REMARK

(i) FOR RADIALY SYMMETRIC DIFFUSION IN A CIRCLE THEN $\nabla^2 u = u_{rr} + \frac{1}{r} u_r$

(ii) FOR RADIALY SYMMETRIC DIFFUSION IN A SPHERE THEN $\nabla^2 u = u_{rr} + \frac{2}{r} u_r$.

(iii) TO SOLVE $u_{rr} + (n-1) u_r = F_0$ ($F_0 = \text{constant}$), LET $u = A r^2$

TO OBTAIN $2A + 2(n-1)A = F_0$ SO THAT $A = F_0 / 2n$

(iv) IN A CIRCLE $\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$.

(v) SUPPOSE $u_t = [D u_x]_x$ THEN FOR THE STEADY-STATE SOLUTION WE REQUIRE

\[D_+ u_x \bigg|_{x_0^+} = D_- u_x \bigg|_{x_0^-} \quad \text{when } D = \begin{cases} D_+, & x > x_0 \\
D_-, & x < x_0 \end{cases}\]
INSULATING BOUNDARY CONDITIONS

Consider

\[ \begin{align*}
\frac{\partial u}{\partial t} &= D \frac{\partial^2 u}{\partial x^2}, & 0 < x < L, \quad t > 0 \\
\frac{\partial u}{\partial x} &= 0 \quad \text{on} \quad x = 0, \quad L; \quad u(x, 0) = f(x) \quad \text{at} \quad x = 0, \quad x = L
\end{align*} \]

The steady-state solution is

\[ u_s'' = 0, \quad 0 < x < L; \quad u_s'(0) = u_s'(L) = 0. \]

This gives \( u_s = A \), \( A \) = constant to be found.

We will show that \( A = \frac{1}{L} \int_0^L f(x) \, dx = \text{average of initial temperature}. \)

To show this we integrate

\[ \int_0^L \frac{\partial u}{\partial t} \, dx = \int_0^L D \frac{\partial^2 u}{\partial x^2} \, dx = D \left. \frac{\partial u}{\partial x} \right|_0^L = 0. \]

Hence

\[ \frac{d}{dt} \int_0^L u(x, t) \, dx = 0 \quad \implies \quad \int_0^L u(x, t) \, dx = C \quad \text{constant for all time. Evaluate at} \quad t = 0 \quad \text{to obtain} \quad \int_0^L u(x, 0) \, dx = \int_0^L f(x) \, dx. \]

Then let \( t \to \infty \) and use \( u(x, t) \to A \) as \( t \to \infty \)

so that

\[ \int_0^L A \, dx = AL = \int_0^L f(x) \, dx \quad \implies \quad A = \frac{1}{L} \int_0^L f(x) \, dx. \]

In the multi-dimensional case \( \frac{\partial u}{\partial t} = D \nabla^2 u \) in \( \Omega \)

with \( \frac{\partial u}{\partial n} = 0 \) on \( \partial \Omega \) and \( u(x, 0) = f(x) \)

you will show that

\[ \lim_{t \to \infty} u(x, t) = \frac{1}{\text{Vol}(\Omega)} \int_\Omega f(x) \, dx. \]
Alternatively, for (1*) in 1-dimension, we write

\[ u(x, t) = x(x) t(t) \quad \frac{T'}{D T} = \frac{X''}{X} = -\lambda. \]

Then

\[ \phi'' + \lambda \phi = 0 \quad \implies \lambda_0 = 0, \quad \phi_0 = 0 \]
\[ \phi'(0) = \phi'(L) = 0 \quad \lambda_n = \frac{n^2 \pi^2}{L^2}, \quad \phi_n(x) = \cos\left(\frac{n \pi x}{L}\right) \]

\( n = 1, 2, 3, \ldots \)

Thus

\[ u(x, t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-\lambda_n D t} \cos\left(\frac{n \pi x}{L}\right) \]

Then

\[ f(x) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n \pi x}{L}\right) \]

Integrating from \( x = 0, L \)

\[ \int_0^L f(x) \, dx = c_0 \cdot L \]

So

\[ c_0 = \frac{1}{L} \int_0^L f(x) \, dx. \]

As \( t \to \infty \) then

\[ \lim_{t \to \infty} u(x, t) = c_0 = \frac{1}{L} \int_0^L f(x) \, dx. \]
STURM–LIOUVILLE PROBLEMS

The Sturm–Liouville problem has the form

\[
\left( \rho(x) \phi'(x) \right)' - q(x) \phi + \lambda w(x) \phi = 0, \quad 0 < x < 1 \quad w(x) > 0, \quad \text{subject to} \quad \phi'(0) - h_1 \phi(0) = 0, \quad \phi'(1) + h_2 \phi(1) = 0.
\]

We write the eigenvalue problem as

\[
\begin{align*}
\left\{ \begin{array}{ll}
\phi''(x) - \lambda w(x) \phi(x) = 0, & \text{on } 0 < x < 1, \\
\phi'(0) - h_1 \phi(0) = 0, & \phi'(1) + h_2 \phi(1) = 0.
\end{array} \right.
\]

We will derive many properties of (x). To do so we first derive Lagrange's identity:

\[
(1) \quad \int_0^1 (v' u - u' v) \, dx = \left. -p(x) u' v \right|_0^1 + p(x) u v \bigg|_0^1.
\]

Proof: We write

\[
\int_0^1 v' u \, dx = \left. \int_0^1 [-v(p u')' + v q u] \, dx \right. = \left. -p u' v \right|_0^1 + \int_0^1 (p v') u + v q u \, dx \right.
\]

\[
= -p u' v \bigg|_0^1 + p v' u \bigg|_0^1 - \int_0^1 (p v') u - v q u \, dx.
\]

This yields that

\[
\int_0^1 v' u \, dx = -p u' v \bigg|_0^1 + p v' u \bigg|_0^1 + \int_0^1 u v' \, dx \quad \square
\]

Now suppose that \( u, v \) satisfy the boundary conditions in (x) so that \( u'(0) - h_1 u(0) = 0, \ u'(1) + h_2 u(1) = 0, \ v'(0) - h_1 v(0) = v'(1) + h_2 v(1) = 0 \). Then we can add and subtract in (1):

\[
\int_0^1 (v' u - u' v) \, dx = \left. p(x) u'(1) \left( v'(1) + h_2 v(1) \right) - p(x) v(1) \left( u'(1) + h_2 u(1) \right) \right.
\]

\[
+ p(0) v(0) \left( u'(0) - h_1 u(0) \right) - p(0) u(0) \left( v'(0) - h_1 v(0) \right) = 0.
\]
Therefore, we obtain
\[ \int_0^1 v(x) u(x) \, dx = \int_0^1 u(x) v(x) \, dx \quad \forall v, u \text{ satisfy the B.C.} \]

If we define \((a, b) = \int_0^1 a \, b \, dx\) then we can write \((v, u) = (u, v)\)

We now derive (or state) many of the key properties of the Sturm-Liouville problem.

Properties

(i) The eigenvalues have the properties

a) \(\lambda_j\) is real
b) There are an infinite no. of eigenvalues \(\lambda_j\) with \(\lambda_1 < \lambda_2 < \lambda_3 < \ldots\) and \(\lambda_j \rightarrow +\infty\) as \(j \rightarrow \infty\).

c) \(\lambda_j > 0\) when \(h_1 > 0\) and \(h_2 > 0\), and \(q_j(x) > 0\), and \(h_1, h_2 > 0\).

(ii) The eigenfunctions \(y = \phi_j(x)\) for \(j = 1, 2, 3, \ldots\) have the properties

a) \(\phi_j(x)\) are real and can be normalized \(\int_0^1 w(x) \phi_j^2 \, dx = 1\)

b) \(\int_0^1 \phi_j(x) \phi_k(x) w(x) \, dx = 0\) \(j \neq k\)

(iii) Expansion property

Any function \(f(x)\) with \(\int_0^1 (f(x))^2 \, dx < \infty\) can be expanded as \(f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)\)

where by orthogonality \(c_n = \frac{\int_0^1 f(x) \phi_n(x) w(x) \, dx}{\int_0^1 (\phi_n(x))^2 w(x) \, dx}\)
(i) Eigenvalues Are Real

Let \( \lambda \phi = \Delta W \phi \) with \( \phi'(0) - h_1 \phi(0) = 0 \) and \( \phi'(1) + h_2 \phi(1) = 0 \).

Take the conjugate \( \bar{\phi} = \bar{\Delta} W \bar{\phi} \). Now use Lagrange's identity

\[
\int_0^1 (\phi \bar{\phi} - \bar{\phi} \phi) \, dx = 0 \quad \Rightarrow \quad (\phi, \bar{\phi}) - (\bar{\phi}, \lambda \phi) = 0.
\]

This yields that

\[
0 = (\phi, \bar{\lambda} \phi) - (\bar{\phi}, \lambda \phi) = (\bar{\lambda} - \lambda) (\phi, \bar{\phi}) = (\bar{\lambda} - \lambda) \int_0^1 W \phi \bar{\phi} \, dx
\]

Hence

\[
(\bar{\lambda} - \lambda) \int_0^1 W |\phi|^2 \, dx = 0 \quad \Rightarrow \quad \lambda = \bar{\lambda} \quad \Rightarrow \quad \lambda \text{ is real.}
\]

(ii) Show \( \lambda_j > 0 \) when \( h_1 > 0 \) and \( h_2 > 0, q_1(x) > 0, h_1 h_2 > 0 \)

We write \( \lambda \phi = \Delta W \phi \)

Multiply by \( \phi \) and integrate

\[
\int_0^1 \phi \bar{\phi} \, dx = \lambda \int_0^1 W |\phi|^2 \, dx
\]

Now integrate by parts:

\[
- p(x) \phi'(x) \phi(x) \bigg|_0^1 + \int_0^1 (p \phi'^2 + q \phi^2) \, dx = \lambda \int_0^1 W |\phi|^2 \, dx
\]

Now \( \phi'(1) = - h_2 \phi(1) \) and \( \phi'(0) = h_1 \phi(0) \), which yields

\[
p(1) h_2 (\phi(1))^2 + p(0) h_1 (\phi(0))^2 + \int_0^1 (p \phi'^2 + q \phi^2) \, dx = \lambda \int_0^1 W |\phi|^2 \, dx.
\]

Now since \( q_1(x) > 0 \) (by assumption) and \( p(x) > 0, W(x) > 0 \)

on \( 0 < x < 1 \) for Sturm-Liouville then we have \( \lambda > 0 \).

Remark if \( q_1 = 0 \) for \( 0 < x < 1 \) and \( h_1 = h_2 = 0 \) then

we have \( (p(x) \phi')' + \Delta W(x) \phi = 0 \) with \( \phi'(0) = \phi'(1) = 0 \).

This has an eigenvalue \( \lambda = 0 \) and eigenfunction \( \phi = 1 \).
(ii) Eigen functions corresponding to different eigenvalues are orthogonal.

We write \[ \phi_j \cdot A_j \phi_j = \phi_k \cdot A_k \phi_k \] with \( \phi_j \) and \( \phi_k \) satisfying the boundary condition.

Then \( (\phi_k, \phi_j^*) = (\phi_j, \phi_k^*) \) by Lagrange's identity.

This yield \( \lambda_j (\phi_k, \phi_j^*) = \lambda_k (\phi_j, \phi_k^*) \).

Therefore \( (\lambda_j - \lambda_k) \int_0^1 w \phi_j \phi_k \, dx = 0 \).

If \( \lambda_j \neq \lambda_k \) then \( \int_0^1 w \phi_j \phi_k \, dx = 0 \) \( \rightarrow \) orthogonality.

(iii) It is difficult to prove that any \( f(x) \) with \( \int_0^1 (f(x))^2 \, dx < \infty \) can be expanded as \( f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \).

To determine the coefficients, multiply by \( w(x) \phi_m(x) \) and get \( \int_0^1 F(x) \phi_m(x) w(x) \, dx = \sum_{n=1}^{\infty} c_n \int_0^1 \phi_m(x) \phi_n(x) w(x) \, dx \).

By orthogonality \( C_m = \int_0^1 \phi_m(x)^2 w(x) \, dx \).

Which determine \( C_m \).
EXAMPLE 1: Find the normalized eigenfunctions for
\[
\phi'' + \lambda \phi = 0, \quad 0 < x < 1 \\
\phi(0) = 0, \quad \phi'(1) + \phi(1) = 0
\]

We know that \( \lambda > 0 \) since \( h_1, h_2 > 0 \) (property ii).

We obtain \( \phi = A \sin(\sqrt{\lambda} x) \). Now \( A \sqrt{\lambda} \cos(\sqrt{\lambda} x) + A \sin(\sqrt{\lambda} x) = 0 \). Thus \( \lambda \) satisfies \( \tan(\sqrt{\lambda}) = -\sqrt{\lambda} \). With \( \lambda > 0 \), plot \( \tan z = -z \).

There are an infinite \# of roots with \( z \approx \frac{2n+1}{2} \pi \) as \( n \to \infty \).

Next we write \( \phi_n(x) = A_n \sin(\sqrt{\lambda_n} x) \). Then to normalize we write

\[
\int_0^1 A_n^2 \sin^2(\sqrt{\lambda_n} x) \, dx = \frac{1}{\lambda_n} \int_0^1 (1 - \cos(2\sqrt{\lambda_n} x)) \, dx
\]

This yields

\[
\frac{A_n^2}{2} \left[ 1 - \frac{1}{2\lambda_n} \sin^2(2\sqrt{\lambda_n} x) \right] = 1
\]

or \( \frac{A_n^2}{2} \left[ 1 - \frac{\sin(\sqrt{\lambda_n}) \cos(\sqrt{\lambda_n})}{\sqrt{\lambda_n}} \right] = \frac{A_n^2}{2} \left[ 1 + \cos^2(\sqrt{\lambda_n}) \right] = 1 \Rightarrow A_n = \frac{\sqrt{2}}{\left[ 1 + \cos^2(\sqrt{\lambda_n}) \right]^{1/2}}
\]

(Here we used \( \sin(\sqrt{\lambda_n}) = -\frac{\lambda_n}{2} \cos(\sqrt{\lambda_n}) \) from the eigenvalue relation).

This yields

\[
\phi_n(x) = \frac{\sqrt{2}}{\left[ 1 + \cos^2(\sqrt{\lambda_n}) \right]^{1/2}} \sin(\sqrt{\lambda_n} x)
\]

Now if we expand \( f(x) = \sum C_n \phi_n(x) \). We have

\[
C_n = \int_0^1 x \phi_n(x) \, dx = A_n \int_0^1 x \sin(\sqrt{\lambda_n} x) \, dx = A_n \left[ \frac{-x}{\sqrt{\lambda_n}} \cos(\sqrt{\lambda_n} x) \right]_0^1 + \frac{1}{\lambda_n} \int_0^1 \cos(\sqrt{\lambda_n} x) \, dx
\]

This yields that \( C_n = A_n \left[ -\frac{1}{\sqrt{\lambda_n}} \cos(\sqrt{\lambda_n}) + \frac{1}{\lambda_n} \sin(\sqrt{\lambda_n}) \right] \)
Now replace $\cos (\sqrt{\alpha \theta}) = -\frac{1}{\sqrt{\alpha \theta}} \sin (\sqrt{\alpha \theta})$, which gives $C_n = \frac{2}{\alpha \theta} \sin (\sqrt{\alpha \theta}) A_n$.

The final result is $f(x) = x = \sum_{\theta = 1}^{\infty} \frac{2}{\alpha \theta} \sin (\sqrt{\alpha \theta}) A_n^2 \sin (\sqrt{\alpha \theta} x)$.

Finally, $f(x) = x = 4 \sum_{n=1}^{\infty} \frac{\sin (\sqrt{\alpha \theta})}{\alpha \theta (1 + \cos^2 (\sqrt{\alpha \theta}))} \sin (\sqrt{\alpha \theta} x)$.

Use this expansion to solve the heat conduction problem given by

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1 \\
u(0, t) &= 1, \quad u_x(1, t) + u(1, t) = 0 \\
u(x, 0) &= f(x)
\end{align*}
\]

We first calculate the steady-state solution $u_s(x)$ which satisfies

\[
u''_s = 0,
\]

\[
u_s(0) = 1, \quad \nu'_s(1) + \nu_s(1) = 0.
\]

We get $u_s(x) = Ax + B$ then $u_s(0) = 1$ gives $u_s(x) = Ax + 1$.

This yields $A + (A + 1) = 1$ or $A = -\frac{1}{2}$. \rightarrow $u_s(x) = -\frac{x}{2} + 1$.

Finally, we write $u = u_s + v$. This yields that

\[
\begin{align*}
\frac{\partial v}{\partial t} &= \alpha^2 \frac{\partial^2 v}{\partial x^2} \\
v(0, t) &= 0, \quad v_x(1, t) + v(1, t) = 0 \\
v(x, 0) &= f(x) - u_s(x)
\end{align*}
\]

We write $v = xt$ so that $\frac{\partial^2 v}{\partial x^2} = -\lambda x'' + \lambda x = 0$, \rightarrow $v = e^{-\alpha^2 \lambda t}$.
This yields the eigenvalue problem

\[ \phi'' + \lambda \phi = 0, \quad 0 < x < 1 \]
\[ \phi(0) = 0, \quad \phi'(1) + \phi(1) = 0 \]

The normalized eigenfunction are

\[ \phi_n(x) = \frac{\sqrt{2}}{\left[ 1 + \cos^2(\sqrt{\lambda_n}x) \right]^{\frac{1}{2}}} \sin(\sqrt{\lambda_n}x) \]

with \( \tan(\sqrt{\lambda_n}) = -\sqrt{\lambda_n} \). We expand

\[ \mathsf{V}(x, t) = \sum_{n=1}^{\infty} c_n e^{-d^2 \lambda_n t} \phi_n(x) \]

Now \( \mathsf{V}(x, 0) = \mathsf{f}(x) - \mathsf{U}_s(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \).

By orthogonality, \( c_n = \int_0^1 (\mathsf{f}(x) - \mathsf{U}_s(x)) \phi_n(x) \, dx \).

And then

\[ \mathsf{U}(x, t) = \mathsf{U}_s(x) + \sum_{n=1}^{\infty} c_n e^{-d^2 \lambda_n t} \phi_n(x) \]

For \( t \to \infty \), \( \mathsf{U}(x, t) \sim \mathsf{U}_s(x) + c_1 e^{-d^2 \lambda_1 t} \phi_1(x) + \cdots \).
**Example** Find the normalized eigenfunction for
\[
\begin{cases}
(x^2 \phi')' + \lambda \phi = 0, & 1 < x < 2 \\
\phi(1) = 0, & \phi(2) = 0
\end{cases}
\]

We expand out to obtain \(x^2 \phi'' + 2x \phi' + \lambda \phi = 0\).

We let \(\phi = x^\Gamma\) to obtain \(\Gamma(\Gamma - 1) + 2\Gamma + \lambda = 0\).

This yields \(\Gamma = \frac{-1 \pm \sqrt{1 - 4\lambda}}{2}\).

For an eigenvalue we need \(1 - 4\lambda < 0\) or \(\lambda > \frac{1}{4}\).

This yields \(\Gamma = \frac{-1}{2} \pm i \frac{\sqrt{4\lambda - 1}}{2}\).

Our solution is
\[
\phi(x) = C_1 x^{-\frac{1}{2}} \sin \left( \frac{\sqrt{4\lambda - 1}}{2} \log x \right) + C_2 x^{-\frac{1}{2}} \cos \left( \frac{\sqrt{4\lambda - 1}}{2} \log x \right)
\]

Now \(\phi(1) = 0\) gives \(C_2 = 0\).

\(\phi(2) = 0\) gives \(\sin \left( \frac{\sqrt{4\lambda - 1}}{2} \log 2 \right) = 0\) or \(\sqrt{4\lambda - 1} \log 2 = n\pi\), \(n = 1, 2, 3, \ldots\).

This yields \(\lambda_n = \frac{1}{4} + \frac{n^2 \pi^2}{(\log 2)^2}\), \(n = 1, 2, 3, \ldots\).

And \(\phi_n(x) = C x^{-\frac{1}{2}} \sin \left( \frac{n\pi \log x}{\log 2} \right)\).

Then we normalize with \(\int_1^2 \phi_n^2(x) dx = 1\). This yields that
\[
C^2 \int_1^2 \frac{1}{x \sin^2 \left( \frac{n\pi \log x}{\log 2} \right)} dx = 1.
\]

Let \(y = \log x/\log 2\) so that \(dy = \frac{1}{x \log 2} dx\)

\[
\rightarrow \quad C^2 \log 2 \int_0^1 \sin^2 (n\pi y) dy = 1 \rightarrow C^2 (\log 2)^2/2 = 1.
\]
This yields that \( C = (2 / \log 2)^{1/2} \).

Finally, we obtain that
\[
\phi_n(x) = \left(\frac{2}{\log 2}\right)^{1/2} x^{-1/2} \sin\left(\frac{n\pi \log x}{\log 2}\right), \quad n = 1, 2, 3, \ldots
\]

\[
\lambda_n = \frac{1}{4} + \frac{n^2 \pi^2}{(\log 2)^2}.
\]

Now if we expand \( f(x) \) in terms of this series we obtain
\[
f(x) = \sum_{n=1}^{\infty} C_n \phi_n(x), \quad C_n = \int_1^2 f(x) \phi_n(x) \, dx.
\]

**Example** Solve the heat conduction problem

\[
\begin{aligned}
&\quad \begin{cases}
\frac{\partial u}{\partial t} = D \left( x^2 \frac{\partial u}{\partial x} \right)_x - u \quad 1 < x < 2, \quad t > 0 \\
\quad u(1, t) = u(2, t) = 0, \quad u(x, 0) = f(x)
\end{cases} \\
&\text{separating variables we obtain} \quad u(x, t) = X(x) T(t)
\end{aligned}
\]

Then \( X T' = DT \left( x^2 X' \right)' - X T \rightarrow \frac{1}{D} \left( \frac{T'}{T} + 1 \right) = \frac{(x^2 X')'}{X} = -\lambda \).

This leads to the eigenvalue problem
\[
\begin{cases}
(x^2 \phi')' + \lambda \phi = 0 \quad 1 < x < 2 \\
\phi(1) = \phi(2) = 0
\end{cases}
\]

\[
\phi_n(x) = \left(\frac{2}{\log 2}\right)^{1/2} x^{-1/2} \sin\left(\frac{n\pi \log x}{\log 2}\right), \quad n = 1, 2, \ldots
\]

\[
\lambda_n = \frac{1}{4} + \frac{n^2 \pi^2}{(\log 2)^2}, \quad k = 1, 2, \ldots
\]

We then obtain \( T_k' = -(1 + D\lambda_k) T \)

This yields that \( T_k(t) = e^{-t} e^{-D\lambda_k t} \)

\[
u(x, t) = \sum_{n=1}^{\infty} C_n e^{-D\lambda_n t} \phi_n(x), \quad \text{with} \quad \nu(x, 0) = \sum_{n=1}^{\infty} C_n \phi_n(x)
\]

Thus \( f(x) = \sum_{n=1}^{\infty} C_n \phi_n(x) \) so that \( C_n = \int_1^2 f(x) \phi_n(x) \, dx \).
(i) \[ \phi'' + x \phi' + \lambda \phi = 0 \]
\[ \phi(0) = \phi(1) = 0 \]

Multiply by \( e^{x^2/2} \) so that
\[ (e^{x^2/2} \phi)' + \lambda e^{x^2/2} \phi = 0 \]
\[ \phi(0) = \phi(1) = 0 \]

The weight function is \( W = e^{x^2/2} \) so that
\[ \int_0^1 \phi_n \phi_m e^{x^2/2} \, dx = 0 \]
if \( n \neq m \).

(ii) \[ \phi'' + \frac{2}{x} \phi' + \lambda \phi = 0 \]
\[ \phi(1) = \phi(2) = 0 \]

In Sturm-Liouville form
\[ (x^2 \phi')' - \lambda x^2 \phi = 0 \]

The weight function is \( W = x^2 \) and
\[ \int_0^1 \phi_n \phi_m x^2 \, dx = 0 \], \( n \neq m \)

We can solve for \( \phi \) by writing
\[ \phi(x) = \frac{v(x)}{x} \]

so that
\[ v'' + \lambda v = 0 \]
and
\[ v = A \cos(\sqrt{\lambda} x) + B \sin(\sqrt{\lambda} x) \]

This yields
\[ \phi = \frac{1}{x} \left[ A \cos(\sqrt{\lambda} x) + B \sin(\sqrt{\lambda} x) \right] \]

It is more convenient to write
\[ \phi = \frac{1}{x} \left[ A \cos(\sqrt{\lambda} (x-1)) + B \sin(\sqrt{\lambda} (x-1)) \right] \]

Now \( \phi(1) = 0 \) \( \rightarrow A = 0 \).
\( \phi(2) = 0 \) \( \rightarrow \sin(\sqrt{\lambda}) = 0 \)

Hence \( \sqrt{\lambda} = n \pi \) or \( \lambda = n^2 \pi^2 \)

And
\[ \phi_n(x) = \frac{1}{x} \sin(\pi n (x-1)) \]
When $\phi(x)$ vanishes at one of the endpoints, or when either endpoint is $0$ we have a singular Sturm-Liouville problem.

(iii) $\phi'' + \frac{1}{x} \phi' + A \phi = 0$, $0 < x < 1$; $\phi(0)$ finite, $\phi(1) = 0$.

This is Bessel's equation with $\phi = A J_0(\sqrt{A} x) + B Y_0(\sqrt{A} x)$ and in Sturm-Liouville form $(x \phi')' + A x \phi = 0$ so that $\int_0^1 x \phi_n \phi_m \, dx = 0$ for $n \neq m$.

(iv) $\phi'' - 2x \phi' + A \phi = 0$ for $-\infty < x < \infty$; Hermite's equation in Sturm-Liouville form $(e^{-x^2} \phi')' + (A - x^2) \phi = 0$ so that $\int_\infty^\infty e^{-x^2} \phi_n \phi_m \, dx = 0$ for $n \neq m$.

Using Frobenius series there are polynomial solutions to this equation when $A = 2 \eta$, for $\eta = 0, 1, 2, \ldots$.

Then $\phi_n(x) = H_\eta(x)$ Hermite polynomials $A = 2 \eta$, $\eta = 0, 1, 2, \ldots$.

$H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, etc.

(v) $\phi'' - \frac{2x}{1-x^2} \phi' + \frac{A}{1-x^2} \phi = 0$ in $-1 < x < 1$, $\phi(\pm 1)$ finite.

This is Legendre's equation and in SL form we get $[(1-x^2) \phi']' + A \phi = 0$ so $\int_{-1}^1 \phi_n(x) \phi_m(x) \, dx = 0$ for $n \neq m$.

The solutions are $\phi = A P_n(x) + B Q_n(x)$ when $A = \eta(n+1)$. Now $P_n(x)$ are Legendre polynomials of degree $n$. 
Remarks in general for eigenvalue problem with non-separated boundary condition such as

\[
\begin{align*}
\phi'' + \lambda \phi &= 0, \quad 0 < x < 1 \\
\phi(0) &= 0 \\
\phi'(1) &= \eta \phi'(0) 
\end{align*}
\]

Here the condition at \( x = 1 \) depends on that at \( x = 0 \rightarrow \) non-separated BC.

We can expect the possibility of complex eigenvalues. This is because the proof that eigenvalue is real fails since Lagrange's identity in equation (1) on page 6 does not give

\[ \int_0^1 (\psi^T \psi - v^2) \, dx = 0. \]

An analysis of (v) showing an infinite \( \# \) of complex eigenvalue when \( \eta > 1 \) is done in the homework.

However, there is one type of non-separated boundary condition which occurs often and leads to real eigenvalues. Consider the case of periodic boundary conditions:

\[
\phi'' + \lambda \phi = 0, \quad 0 < x < L; \quad \phi(0) = \phi(L), \quad \phi'(0) = \phi'(L).
\]

For this problem we calculate for \( \lambda > 0 \) that

\[ \phi(x) = A \cos(\sqrt{\lambda} x) + B \sin(\sqrt{\lambda} x). \]

We put

\[
\begin{align*}
\phi(0) &= \phi(L) \quad \rightarrow \quad A = A \cos(\mu) + B \sin(\mu) \quad \mu = \sqrt{\lambda} L \\
\phi'(0) &= \phi'(L) \quad \rightarrow \quad B = -A \sin(\mu) + B \cos(\mu)
\end{align*}
\]

This gives

\[
\begin{pmatrix}
1 - \cos(\mu) & -\sin(\mu) \\
\sin(\mu) & 1 - \cos(\mu)
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix} = 0
\]

which has a non-trivial solution \( \det A \neq 0 \) when \( \det A = 0 \).

Thus the eigenvalues correspond to \( \det A = 0 \rightarrow (1 - \cos(\mu))^2 + \sin^2(\mu) = 2 - 2 \cos(\mu) = 0 \).

Thus, \( \cos(\mu) = 1 \rightarrow \mu = \sqrt{\lambda} L = 2\pi n, \quad n = 0, 1, 2, \ldots \) The eigenvalues are

\[ \lambda_n = (2\pi n)^2 \quad \text{for} \quad n = 1, 2, \ldots \text{ and } \phi_n = A \cos\left(\frac{2\pi n x}{L}\right) + B \sin\left(\frac{2\pi n x}{L}\right). \]
EXAMPLE (EULER'S EQUATION)

Solve Laplace's equation in a wedge with cut-out

\[ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi \varphi} = 0 \quad \text{in} \quad 1 \leq r \leq 2, \quad 0 \leq \varphi \leq \alpha \]

\[ u(1, \varphi) = u(2, \varphi) = 0 \]

\[ u(r, 0) = f(r), \quad u(r, \alpha) = 0 \]

Solution: We look for an eigenfunction expansion in radial direction.

Separate variables:

\[ u = R(r) \Phi(\varphi) \]

Then

\[ \left( r \frac{d}{dr} + \frac{1}{r} \right) \Phi + \frac{1}{r^2} R \Phi'' = 0. \]

So

\[ \frac{r^2 R'' + r R'}{R} = -\frac{\Phi''}{\Phi} = -\lambda. \]

\[ \Phi'' - \lambda \Phi = 0 \]

\[ \Phi(0) = 0. \]

So \( \frac{1}{2} r^2 R'' + r R' + \lambda R = 0 \), \( R(1) = R(2) = 0 \)

This gives in 5L form:

\[ \left\{ \begin{array}{l}
2 \left( r \frac{d}{dr} + \frac{1}{r} \right) R = (r R')' + \frac{\lambda}{r} R = 0. \end{array} \right. \]

Thus

\[ \left\{ \begin{array}{l}
(r R')' + \frac{\lambda}{r} R = 0 \quad \text{in} \quad 1 \leq r \leq 2 \\
R(1) = R(2) = 0 \end{array} \right. \]

From (v) we identify the weight function \( \Lambda = 1/r \).

Also from (v) we claim \( \Lambda > 0 \).

Proof: Multiply by \( R' \): \( R (r R')' + \lambda \frac{R^2}{r} = 0 \)

And integrate

\[ \int_1^2 R (r R')' dr = -\lambda \int_1^2 \frac{R^2}{r} dr. \]

Now integrate by parts

\[ R (r R')|_1^2 - \int_1^2 r (R')^2 dr = -\lambda \int_1^2 \frac{R^2}{r} dr. \]
\[ R(1) = R(2) = 0. \text{ Hence, boundary terms vanish.} \text{ And} \]
\[ \int_1^2 r(R')^2 \, dr + \lambda \int_1^2 \frac{R^2}{r} \, dr \to \lambda \geq 0. \]

\[ \Rightarrow \lambda \geq 0 \]

But \( \lambda = 0 \) is not an eigenvalue since if we set \( \lambda = 0 \) we get
\[ (\int rR')' = 0 \to rR' = c \]
so \( R = c \log r + b. \)

But \( R(1) = R(2) = 0 \to c = b = 0 \Rightarrow R \equiv 0 \text{ not an eigenfunction.} \)

Thus we need only consider \( \lambda > 0 \) in Euler's equation (4). We write
\[ R = r^B \text{ in (4). We get} \]
\[ B(B-1) + B + \lambda = 0 \to B^2 = -\lambda. \]

But \( \lambda > 0 \), so \( B = \pm \sqrt{\lambda}. \)

Without loss of generality, take + sign. \( R = r^B = r^{\sqrt{\lambda}} \).

\[ R = e^{i \sqrt{\lambda} \log r} \cos \left( \sqrt{\lambda} \log r \right) + i \sin \left( \sqrt{\lambda} \log r \right) \]

is a complex-valued solution. Since \[ \text{Im} R, \text{Re} \text{Re} R \] are both

separately solutions, the general solution is
\[ R(r) = c_1 \cos \left( \sqrt{\lambda} \log r \right) + c_2 \sin \left( \sqrt{\lambda} \log r \right). \]

But \( R(1) = 0 \to c_1 = 0 \)
\( R(2) = 0 \to c_2 \sin \left( \sqrt{\lambda} \log 2 \right) = 0 \to \sqrt{\lambda} \log 2 = n\pi \)

so
\[ \lambda_n = \frac{n^2 \pi^2}{(\log 2)^2}, \quad n = 1, 2, 3, \ldots \]

\[ R_n(r) = \sin \left( \frac{n\pi \log r}{\log 2} \right) \quad n = 1, 2, \ldots \text{ and} \int_1^2 \frac{1}{r} R_n R_m \, dr = 0 \quad n \neq m. \]
Now recall \( \Phi'' - \lambda \Phi = 0 \).
\[ \Phi(0) = 0 \]

So
\[ \Phi_n(\varphi) = A_n \sinh [A_n \varphi] + B_n \cosh [A_n \varphi]. \]

But \( \Phi_n(0) = 0 \rightarrow B_n = 0. \)

Thus give by superposition
\[ (1) \quad u(\rho, \varphi) = \sum_{n=1}^{\infty} A_n \sinh [A_n \varphi] R_n(\rho) \]

Now a final step set \( u(\rho, \alpha) = F(\rho). \)

Then
\[ F(\rho) = \sum_{n=1}^{\infty} A_n \sinh [\lambda_n \alpha] R_n(\rho). \]

Multiply by the weight \( w = \frac{1}{\rho} \) and \( R_m(\rho) \) and integrate.
\[ \int_{1}^{2} \frac{1}{\rho} F(\rho) R_m(\rho) = \sum_{n=1}^{\infty} A_n \sinh [\lambda_n \alpha] \int_{1}^{2} \frac{1}{\rho} R_n R_m \, d\rho. \]

Using orthogonality
\[ A_n \sinh [\lambda_n \alpha] = \frac{\int_{1}^{2} \frac{1}{\rho} F(\rho) R_m(\rho)}{\int_{1}^{2} \frac{1}{\rho} (R_m(\rho))^2 \, d\rho} = \frac{2}{\log 2} \int_{1}^{2} \frac{F(\rho) R_m(\rho) \, d\rho}{\int_{1}^{2} \frac{1}{\rho} \log 2 \, d\rho} \]

which determines \( A_n \) in solution (1).

Note:
\[ \int_{1}^{2} \frac{1}{\rho} R_m^2(\rho) \, d\rho = \int_{1}^{2} \frac{1}{\rho} \sin^2 \left( \frac{m \log \rho}{\log 2} \right) \, d\rho \quad \text{let} \quad t = \log \rho \quad dt = \frac{1}{\rho} \, dr \]
\[ = \int_{\log 2}^{\log 4} \sin^2 \left( \frac{m \log t}{\log 2} \right) \, dt = \frac{1}{2} \log 2. \]
There are two sources of inhomogeneous terms:

(i) In the equation, i.e.

\[ U_t = U_{rr} + \frac{2}{r} U_r + G(r, t), \quad 0 \leq r \leq a, \quad t > 0 \]

where \( G(r, t) \) measures internal heating in a sphere of radius \( a \).

Another example is

\[ U_{tt} = C^2 U_{xx} + F(x, t), \quad 0 < x < L, \quad t > 0 \]

\[ U = 0 \text{ at } x = 0, L \]

\[ U(x, 0) = F(x) \]

where \( F(x, t) \) measures external force on a string whose linearized deflection in vertical direction is \( U \).

For a bird jumping on a wire we can model

\[ F(x, t) = \delta(x-x_0) \sin(\omega t), \text{ where } \delta(x-x_0) \text{ is Dirac delta function.} \]

For these problems where inhomogeneous terms appear in the equation we will show later that

\[ U(r, t) = \sum_{n=1}^{\infty} b_n(t) \phi_n(r) \]

where \( \phi_n(r) \) are eigenfunctions of SL problem and

where \( b_n(t) \) will satisfy ODE's to be derived.
**Inhomogeneous Term in the Boundary Conditions**

Consider

\[ U_t = U_{xx}, \quad 0 < x < L, \quad t > 0 \]

\[ U(0, t) = T_0, \quad U_x(L, t) = -h \left( U(L, t) - T_1 \right) \]

\[ U(x, 0) = U_0(x) \]

where \( T_0, T_1 \) (possibly dependent on time \( t \)) are the inhomogeneous terms. We can't do separation of variables immediately since if we put \( U(x, t) = \phi(x) \, T(t) \) \( \Rightarrow \) \( U(0, t) = T_0 \) gives \( T_0 = \phi(0) \, T(t) \) with \( \phi(0) = 0 \) which is a contradiction. We must make a transformation to get homogeneous boundary conditions.

Put

\[ U(x, t) = U_S + V(x, t) \]

where \( V \) is to have homogeneous BC and where

\[ U_{SS} = 0, \quad U_S = T_0 \text{ at } x = 0, \quad U_{Sx} = -h \left( U_S - T_1 \right) \text{ at } x = L. \]

The solution is \( U_S = A x + B \) where \( T_0 = B \) and \( A = -h \left( \frac{AL + T_0 - T_1}{1 + hL} \right) \)

Thus

\[ U_S = \frac{-h(\bar{T}_0 - \bar{T}_1)}{1 + hL} x + T_0. \]

Now obtain equation for \( V \):

\[ \partial_t U_S + V_t = U_{Sxx} + V_{xx} \text{ but } U_{Sxx} = 0 \text{ so } \]

\[
\begin{cases}
V_t = V_{xx} + g(x, t), & 0 < x < L, \quad t > 0 \\
V(0, t) = 0, & V_x(L, t) = -h \, V(L, t) \\
V(x, 0) = U(x, 0) - U_S \bigg|_{t=0}
\end{cases}
\]

The transformed problem (+) can be solved by separation of variables by looking for a solution in the form

\[ V(x, t) = \sum_{n=1}^{\infty} B_n(t) \varphi_n(x) \]

where \( \varphi_n(x) \) are eigenfunctions of \( SL \).
KEY POINT: IF THERE ARE INHOMOGENEOUS TERMS IN THE
BOUNDARY CONDITIONS YOU MUST FIRST MAKE A TRANSFORMATION
TO GET A PROBLEM WITH HOMOGENEOUS BOUNDARY CONDITIONS,
AT THE EXPENSE OF INSERTING AN INHOMOGENEOUS TERM INTO THE EQUATION.

REMARK 1: IN PREVIOUS EXAMPLE IF $T_0, T_1$ ARE CONSTANT INDEPENDENT
OF TIME, THEN $u_s(x) = \frac{-\lambda (T_0 - T_1)}{L} x + T_0$ IS THE STEADY-STATE SOLUTION
AND $v(x,t)$ SATISFIES

\[
\begin{align*}
\frac{\partial v}{\partial t} &= v_x x \\
(++) &
\begin{align*}
v(0,t) &= 0, \\
v_x(L,t) &= -h v(L,t) \\
v(x,0) &= u(x,0) - u_s(x)
\end{align*}
\]

AS A SECOND EXAMPLE CONSIDER

\[
\begin{align*}
u_t &= u_{rr} + \frac{2}{r} u_r, \quad 0 \leq r \leq a, \quad t > 0 \\
u(0,t) &= \sin(\omega t), \quad u(r,0) \text{ NICE} \\
u(r,0) &= f(r)
\end{align*}
\]

TO ELIMINATE INHOMOGENEOUS TERM IN BC LET $u(x,t) = u_s + v(r,t)$ WHERE

\[
\begin{align*}
u_{rr} + \frac{2}{r} u_r &= 0, \quad 0 \leq r \leq a, \\
u_s &= \sin(\omega t) \text{ AT } r = a, \quad u_s \text{ NICE } a, \quad r > 0.
\end{align*}
\]

THE SOLUTION IS $u_s = A + B/r$ WITH $B = 0$ AND $A = \sin(\omega t)$. IF WE PUT

\[
\begin{align*}
&u = u_s + v \\
&\text{WE GET}
\end{align*}
\]

\[
\begin{align*}
&\frac{\partial v}{\partial t} = v_{rr} + \frac{2}{r} v_r - \omega \cos(\omega t) \\
&(++) \\
&v(0,t) = 0, \quad v(0,t) \text{ NICE} \\
&v(r,0) = f(r) - u_s |_{r=a} = f(r).
\end{align*}
\]
EXAMPLE CONSIDER HEAT FLOW IN A SPHERE WITH PERIODIC TEMPERATURE VARIATIONS ON ITS BOUNDARY MODELED BY

\[ u_t = D \left( u_{rr} + \frac{2}{r} u_r \right) \text{ in } 0 < r < a, \ t \geq 0 \]

WITH (BC) \[ u(0, t) = \sin(wt) \; ; \; u, u_r \text{ bounded at } r \to 0 \]

AND (IC) \[ u(r, 0) : 0. \]

WE FIRST MUST GET HOMOGENEOUS BC. DEFINE \( \overline{u} \) BY

\[ \overline{u}_{rr} + \frac{2}{r} \overline{u}_r = 0 \; ; \; \overline{u} = \sin(wt) \text{ on } r = a, \; \overline{u}, \overline{u}' \text{ bounded at } r \to 0. \]

WE GET \( \overline{u} = d_0 + d_1 r \). THEN \( \overline{u} \) NICE \( r \to 0 \Rightarrow d_1 = 0 \) AND CONDITION AT \( r = a \) GIVES \( d_0 = \sin(wt) \). WE GET

\[ \overline{u} = \sin(wt). \]

WE WRITE \( u(r, t) = \overline{u} + v(r, t) \) AND SUBSTITUTE:

\[ \overline{u}_t + v_t = D \left( \overline{u}_{rr} + \frac{2}{r} \overline{u}_r \right) + D \left( v_{rr} + \frac{2}{r} v_r \right) \]

\[ v(a, t) = \sin(wt) - \overline{u} \big|_{r=a} \]

\[ v(r, 0) = u(r, 0) - \overline{u} \big|_{t=0} \]

WE GET BY USING \( \overline{u} = \sin(wt) \) THAT

\[ v_t = D \left( v_{rr} + \frac{2}{r} v_r \right) - w \cos(wt) \text{ in } 0 \leq r \leq a, \ t \geq 0 \]

(BC) \[ v(a, t) = 0, \; v, v_r \text{ bounded at } r \to 0 \]

(IC) \[ v(r, 0) = 0. \]

THE UNDERLYING IS PROBLEM IS

\[ \Phi'' + \frac{2}{r} \Phi' + \frac{\lambda}{r^2} \Phi = 0 \]

\[ \Phi(a) = 0; \; \Phi, \Phi' \text{ bounded at } r \to 0. \]

\[ \Phi_0 = \sin \left( \frac{\pi n \pi r}{a} \right)/r \]

\[ \lambda_n = \frac{n^2 \pi^2}{a^2}, \ n = 1, 2, \ldots \]

\[ \int_0^a r^2 \Phi_0 \Phi_m \, dr = 0 \text{ if } n \neq m \]
Now expand

$$V(\Gamma, t) = \sum_{n=1}^{\infty} b_n(t) \Phi_n(\Gamma).$$

Substitute to get

$$\sum_{n=1}^{\infty} b_n(t) \Phi_n(\Gamma) = -\frac{2}{\Gamma} \sum_{n=1}^{\infty} b_n(\tau_n') - w \cos(\omega t).$$

Using eigenvalue relation

$$\Phi_n'' + \frac{2}{\Gamma} \Phi_n' = -\lambda_n \Phi_n$$

yield

$$\sum_{n=1}^{\infty} \left( b_n(t) + D \lambda_n b_n \right) \Phi_n(\Gamma) = -w \cos(\omega t).$$

Now expand

$$1 = \sum_{n=0}^{\infty} f_n(\Phi_n(\Gamma))$$

so

$$f_n = \int_{0}^{\alpha} \frac{\Gamma}{\sin \left( \frac{\pi \gamma}{\alpha} \right)} \, d\Gamma = \int_{0}^{\alpha} \sin \left( \frac{\pi \gamma}{\alpha} \right) \, d\Gamma$$

We then write (1) as

$$\sum_{n=1}^{\infty} \left( b_n(t) + D \lambda_n b_n \right) \Phi_n = \sum_{n=1}^{\infty} f_n(-w \cos(\omega t)) \Phi_n$$

We conclude that by orthogonality, and the fact that $V(\Gamma, 0) = 0 \rightarrow b_n(0) = 0$ that

$$b_n(t) + D \lambda_n b_n = C_n \cos(\omega t), \quad n=1,2,3,\ldots$$

where

$$C_n = -w f_n \quad \text{and} \quad f_n \text{ defined by } (2).$$

After solving for $b_n(t)$ and recalling

$$u(\Gamma, t) = \sin(\omega t) + \sum_{n=1}^{\infty} b_n(t) \Phi_n(\Gamma)$$

we have $u(\Gamma, t)$. Let's derive an expression for $u(0, t)$, i.e., the temperature at center of sphere, valid for large $t$. Since

$$\lim_{\Gamma \to 0} \frac{\Phi_n(\Gamma)}{\Gamma} = \lim_{\Gamma \to 0} \frac{\sin \left( \frac{\pi \gamma}{\alpha} \right)}{\Gamma} = \frac{\gamma}{\alpha}$$

we conclude that

$$u(0, t) = \sin(\omega t) + \sum_{n=1}^{\infty} \left( \frac{\gamma}{\alpha} \right) b_n(t)$$

$$u(0, t) = \sin(\omega t) + \sum_{n=1}^{\infty} \left( \frac{\gamma}{\alpha} \right) b_n(t)$$

(6).
Now we calculate $C_n$ and then find $b_n(t)$.

From (12) we have

$$F_n = \frac{2}{a} \int_0^\pi r \sin \left( \frac{n \pi r}{a} \right) dr = \frac{2}{a} \left[ -\frac{r}{n \pi} \cos \left( \frac{n \pi r}{a} \right) \right]_0^\pi + \frac{a}{n \pi} \int_0^\pi \cos \left( \frac{n \pi r}{a} \right) dr$$

so

$$F_n = \frac{2 (-a^2)}{a \pi} (-1)^n = \frac{2a}{n \pi} (-1)^n$$

Thus

$$C_n = \frac{2a \omega (-1)^n}{n \pi}$$

to be used in (4).

Now calculate the solution $b_n$:

Method

$$\left( e^{D \Delta_n t} b_n \right) = C_n e^{D \Delta_n t} \cos(\omega t)$$

so

$$e^{D \Delta_n t} b_n = C_n \left[ t \ e^{D \Delta_n t} \cos(\omega t) \right]$$

$$\text{sat in } b_n(0) = 0.$$

Now

$$\int_0^t e^{D \Delta_n t} \cos(\omega t) dt = \text{RE} \left[ \int_0^t e^{D \Delta_n i \omega t} \right] = \text{RE} \left[ \frac{1}{D \Delta_n + i \omega} \right]$$

$$= \text{RE} \left[ \frac{(D \Delta_n - i \omega)}{(D \Delta_n)^2 + \omega^2} \right] \left[ e^{D \Delta_n (\cos(\omega t) + i \sin(\omega t))} - 1 \right]$$

$$= \frac{1}{(D \Delta_n)^2 + \omega^2} \left[ D \Delta_n \left( e^{D \Delta_n \cos(\omega t)} - 1 \right) + \omega e^{D \Delta_n \sin(\omega t)} \right]$$

Thus

$$e^{D \Delta_n t} b_n = \frac{C_n}{(D \Delta_n)^2 + \omega^2} \left[ D \Delta_n \left( e^{D \Delta_n \cos(\omega t)} - 1 \right) + \omega e^{D \Delta_n \sin(\omega t)} \right]$$

so

$$b_n(t) = \frac{C_n}{(D \Delta_n)^2 + \omega^2} \left[ D \Delta_n \cos(\omega t) + \omega \sin(\omega t) \right] - D \Delta_n e^{-\Delta_n D t}$$

with

$$C_n = \frac{2a \omega (-1)^n}{n \pi}$$

Now in (6) we need $\frac{n \pi}{a} b_n$ so that

$$\frac{n \pi}{a} b_n = \frac{2 \omega (-1)^n}{(D \Delta_n)^2 + \omega^2} \left[ D \Delta_n \cos(\omega t) + \omega \sin(\omega t) - D \Delta_n e^{-\Delta_n D t} \right]$$

(7)
We now substitute into (6) to get, upon letting $t \to \infty$ to will exponential in (7):

$$u(0, t) \cong \sin(\omega t) + \sum_{n=1}^{\infty} \frac{2w^{-1} \eta^n}{(\eta_n^2 + \omega)^2} \left( \frac{D \Delta_n}{\omega} \cos(\omega t) + w \sin(\omega t) \right), \text{ for } t \gg 1.$$

Now write $D \Delta_n \cos(\omega t) + w \sin(\omega t) = R \sin(\omega t - \phi_n)$

so $R \cos \phi_n = \omega$,

$R \sin \phi_n = -D \Delta_n$,

$\tan \phi_n = -D \Delta_n / \omega$.

Then we have compactly that

$$u(0, t) \cong \sin(\omega t) + \sum_{n=1}^{\infty} \frac{2w^{-1} \eta^n}{\sqrt{(D \Delta_n)^2 + \omega^2}} \sin(\omega t - \phi_n), \text{ for } t \gg 1.$$

Now use $D \Delta_n = D \Delta_n \pi^2 / a^2 = \chi n^2$ where $\chi = \frac{D \pi^2}{a^2 \omega}$.

Our explicit formula becomes

$$u(0, t) \cong \sin(\omega t) + \sum_{n=1}^{\infty} \frac{2(-1)^n \eta^n}{\sqrt{\chi n^2 + 1}} \sin(\omega t - \phi_n), \text{ for } t \gg 1$$

where $\tan \phi_n = -\chi n^2$ and $\chi = \frac{D \pi^2}{a^2 \omega}$.

Notice that $\chi$ is dimensionless since $[D] = \text{length}^2 / \text{time}$, $[\eta] = \text{length}^{-2}$, and $[\omega] = 1 / \text{time}$. If $\chi$ is large, i.e. fast diffusion, then the infinite sum term is small and so $u(0, t) \approx u(t)$ close to $u(0, t)$ as we would expect.
Example Find the solution to the forced wave equation

\[ u_{tt} = c^2 u_{xx} + F(x,t), \quad 0 < x < L, \ t > 0 \]

\[ u(0,t) = u(L,t) = 0 \]

\[ u(x,0) = g(x), \ u_t(x,0) = 0 \]

Where \( F(x,t) = \delta(x-x_0) \sin \left( \frac{n \pi ct}{L} \right) \) for a positive integer.

This corresponds to plucking the string locally at some \( x = x_0 \) with \( 0 < x_0 < L \).

Remark Recall that \( \int_0^1 \delta(x-x_0) g(x) \, dx = g(x_0) \) when \( 0 < x_0 < 1 \).

Also \( \delta(x-x_0) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x-x_0)^2}{2\sigma^2}} \) in generalized function sense:

\[ \delta(x-x_0) = \begin{cases} 0 & \text{if } x \neq x_0 \\ \infty & \text{if } x = x_0 \end{cases} \quad \text{for } 0 < x_0 < b. \]

For the homogeneous problem the eigenfunctions are \( \sin \left( \frac{n \pi x}{L} \right) \) and the normalized eigenfunctions are

\[ \phi_n(x) = \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{n \pi x}{L} \right) \]

with \( \int_0^L (\phi_n(x))^2 \, dx = 1 \).

We look for a solution of the form

\[ u(x,t) = \sum_{n=1}^\infty b_n(t) \phi_n(x), \]

Then

\[ u(x,0) = g(x) = \sum_{n=1}^\infty b_n(0) \phi_n(x) \quad \rightarrow \quad b_n(0) = \int_0^L g(x) \phi_n(x) \, dx \]

\[ u_t(x,0) = 0 = \sum_{n=1}^\infty b_n'(0) \phi_n(x) \quad \rightarrow \quad b_n'(0) = 0 \]

We then substitute (x) into the PDE to obtain
\[ \sum_{\eta=1}^{\infty} \left( b_{\eta}^2 + w_{\eta}^2 b_{\eta} \right) \phi_{\eta}(x) = \sum_{\eta=1}^{\infty} d_\eta(t) \phi_{\eta}(x), \]

WITH \( w_\eta = C \eta \pi / L \) AND \( d_\eta(t) = \int_0^L \phi_{\eta}(x) F(x,t) \, dx = \int_0^L \phi_{\eta}(x) F(x,t) \, dx. \)

NOW WE CALCULATE
\[ d_\eta(t) = \int_0^L \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{\eta \pi x}{L} \right) \delta(x-x_0) \sin \left( \frac{N \pi c t}{L} \right) \, dx. \]

THIS YIELDS \( d_\eta(t) = \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{\eta \pi x_0}{L} \right) \sin \left( \frac{N \pi c t}{L} \right), \) \( w_\eta = N \pi c / L. \)

BY ORTHOGONALITY \( f(x) = \sum_{\eta=1}^{\infty} b_{\eta}(0) \phi_{\eta}(x). \)

THIS YIELDS THAT \( b_{\eta}(0) = \int_0^L f(x) \phi_{\eta}(x) \, dx. \)

THE PROBLEM FOR \( b_{\eta}(t) \) BECOMES:
\[
\begin{cases}
 b_{\eta}'' + w_\eta^2 b_{\eta} = \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{\eta \pi x_0}{L} \right) \sin \left( \frac{N \pi c t}{L} \right), & w_\eta = N \pi c / L \\
 b_{\eta}(0) = \int_0^L f(x) \phi_{\eta}(x) \, dx, & b_{\eta}'(0) = 0
\end{cases}
\]

CASE 1 \( (\eta \neq 1) \) NON-RESONANT CASE

THE PARTICULAR SOLUTION HAS THE FORM
\[ b_1(t) = A_1 \sin(\omega_1 t) \] SO THAT \( A_1 (w_1^2 - w_1^2) = \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{\eta \pi x_0}{L} \right). \)

THIS YIELDS \( A_1 = \left( \frac{2}{L} \right)^{1/2} \frac{1}{w_1^2 - w_1^2} \sin \left( \frac{\eta \pi x_0}{L} \right). \)

THE GENERAL SOLUTION IS
\[ b_{\eta}(t) = A_\eta \sin(\omega_\eta t) + B_\eta \cos(\omega_\eta t) + A_1 \sin(\omega_1 t) \]

BUT \( b_{\eta}'(0) = 0 \) \( \rightarrow \) \( A_\eta \omega_\eta + A_1 \omega_1 = 0 \) \( \rightarrow \) \( A_\eta = -\frac{\omega_1}{\omega_\eta} A_1. \)

\[ b_{\eta}(0) = B_\eta. \]
This yields that for \( \Omega \neq N \),
\[
b_N(t) = A_N \left[ \sin \left( \omega_N t \right) - \frac{\omega_N}{\omega_D} \sin \left( \omega_D t \right) \right] + b_N(0) \cos \left( \omega_N t \right).
\]

**CASE 2 \((\Omega = N)\)** RESONANT CASE.

We write the particular solution as \( b_{NP} = 1M(b_N) \).

This yields
\[
\ddot{b}_N + \Omega^2 \dot{b}_N = \left( \frac{2}{L} \right) \frac{1}{2} \sin \left( \frac{N\pi x_0}{L} \right) e^{-i\Omega t}.
\]

We substitute \( \ddot{b}_N = b_N(t)e^{-i\Omega t} \)

to obtain that
\[
b_N = -\frac{i}{2\omega_N} \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{N\pi x_0}{L} \right)
\]

This yields that
\[
b_{NP}(t) = 1M(\ddot{b}_N) = -\frac{1}{2\omega_N} \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{N\pi x_0}{L} \right)(-\cos(\Omega t))
\]

Finally, we write
\[
b_N(t) = \alpha_N \sin(\omega_N t) + B_N \cos(\omega_N t) - \left( \frac{\omega_N}{2\omega_D} \right) \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{N\pi x_0}{L} \right) \cos(\omega_D t)
\]

Thus \( b_N(0) = 0 \) \( \rightarrow \)
\[
\alpha_N \omega_N - \frac{1}{2\omega_N} \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{N\pi x_0}{L} \right) = 0
\]
\[
\alpha_N = \frac{1}{2\omega_N^2} \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{N\pi x_0}{L} \right)
\]

This yields that
\[
b_N(t) = b_N(0) \cos(\omega_N t) + \frac{1}{2\omega_N^2} \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{N\pi x_0}{L} \right) \sin(\omega_N t)
\]
\[
- \left( \frac{\omega_N}{2\omega_D} \right) \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{N\pi x_0}{L} \right) \cos(\omega_D t)
\]

The final solution is
\[
U(x,t) = b_N(t) \sin \left( \frac{N\pi x}{L} \right) + \sum_{\Omega = 1, \Omega \neq N}^{0} b_N(0) \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{N\pi x}{L} \right)
\]
**Inhomogeneous Problems: Eigenfunction Expansions**

Suppose that we have a heat equation on $0 < x < 1$, $t > 0$

$$w(x) u_t = (p(x) u_x)_x - q(x) u + g(x,t)$$

$p(x) > 0$ on $0 < x < 1$

$w(x) > 0$ on $0 < x < 1$

$u_x(0,t) = h_1$, $u(0,t)$

$h_1 > 0$, $h_2 > 0$

$u_x(1,t) = -h_2$, $u(1,t)$

$u(x,0) = f(x)$

We first consider the eigenvalue problem for the homogeneous problem

$$\left( p(x) \phi' \right)' - q(x) \phi + \lambda w(x) \phi = 0, \quad 0 < x < 1$$

$$\phi'(0) - h_1 \phi(0) = 0, \quad \phi'(1) + h_2 \phi(1) = 0$$

The normalized eigenfunctions are $\phi_n(x)$ with eigenvalues $\lambda_n$ for $n = 1, 2, 3, \ldots$. The normalization condition is

$$\int_0^1 w(x) (\phi_n(x))^2 \, dx = 1.$$ 

We look for a solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \phi_n(x)$$

where $b_n(t)$ for $n = 1, 2, 3, \ldots$ are to be found.

We substitute into the PDE to obtain

$$\sum_{n=1}^{\infty} w(x) b_n'(t) \phi_n(x) = \sum_{n=1}^{\infty} b_n(t) \phi_n' + \frac{g(x,t)}{w(x)} w(x)$$

where $\phi_n' = \left( p \phi_n' \right)' - q \phi_n = -\lambda_n w \phi_n$. Now expand

$$\frac{g(x,t)}{w(x)} = \sum_{n=1}^{\infty} C_n \phi_n(x) \quad C_n = \int_0^1 g(x,t) \phi_n(x) \, dx.$$
This yields that
\[ \sum_{n=1}^{\infty} w(x) b_n' \phi_n = \sum_{n=1}^{\infty} -\lambda_n b_n w \phi_n + \sum_{n=1}^{\infty} c_n \phi_n w \]

Equivalently, we obtain that
\[ \sum_{n=1}^{\infty} \left( b_n' + \lambda_n b_n - c_n \right) \phi_n w = 0 \]

By orthogonality of the eigenfunctions \( \phi_n \), \( n = 1, 2, \ldots \)
we obtain that
\[ (\ast) \quad b_n' + \lambda_n b_n = c_n . \]

Now \( u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n(0) \phi_n(x) \).

This yields that
\[ b_n(0) = \int_0^1 f(x) \phi_n(x) w(x) \, dx \]

By orthogonality and \( \int_0^1 w \phi_n^2 \, dx = 1 \).

We solve (\ast) by writing
\[ (b_n e^{\lambda_n t})' = c_n(0) t e^{\lambda_n t} \]

We obtain
\[ b_n e^{\lambda_n t} = b_n(0) + \int_0^t e^{\lambda_n \tau} c_n(\tau) \, d\tau . \]

This yields that
\[ b_n(t) = b_n(0) e^{-\lambda_n t} + e^{-\lambda_n t} \int_0^t e^{\lambda_n \tau} c_n(\tau) \, d\tau . \]

The eigenfunction function expansion solution is
\[ u(x,t) = \sum_{n=1}^{\infty} b_n(t) \phi_n(x) \]

with \( b_n(t) \) as given above.