Introduce spherical coordinates

\[ x = r \sin \phi \cos \theta \]
\[ y = r \sin \phi \sin \theta \]
\[ z = r \cos \phi \]

The poles are at \( \phi = 0, \pi \)

The equator is at \( \theta = \pi/2 \)

\[ q = \tan^{-1}\left( \frac{\sqrt{x^2 + y^2}}{z} \right) \]
\[ \theta = \tan^{-1}\left( \frac{y}{x} \right) \]

In terms of these coordinates, \( \Delta u = 0 \) becomes

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin q} \frac{\partial}{\partial q} \left( \sin q \frac{\partial u}{\partial q} \right) + \frac{1}{r^2 \sin^3 q} \frac{\partial^2 u}{\partial \theta^2} = 0 \]

For simplicity, we will consider the axi-symmetric case where there is azimuthal symmetry so that \( \frac{\partial u}{\partial \phi} = 0 \).

Then

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin q} \frac{\partial}{\partial q} \left( \sin q \frac{\partial u}{\partial q} \right) = 0, \quad 0 < q < \pi \]
\[ 0 < r < a \]

We impose that \( u \) has no singularities at the pole \( q = 0 \) and \( q = \pi \).

We separate variables

\[ u(\rho, q) = R(\rho) \Phi(q) \]

Then

\[ -\frac{r^2}{r^1} \left( \frac{R'}{R} \right)' = \frac{1}{\sin q} \left( \frac{\Phi'}{\Phi} \right)' \quad \text{constant}. \]

We write the constant \( A = -\sqrt{\nu + 1} \).

Then

1. \( \left( \frac{\sin q}{\Phi} \right)' + \sqrt{\nu + 1} \sin q \Phi = 0, \quad 0 < q < \pi \)
2. \( \left( r^2 R' \right)' - \sqrt{\nu + 1} R = 0 \)
Now examine the equation for $\Phi$.

We let $\chi = \cos \varphi$ and replace $y \mapsto \Phi$.

Then

$$\sin \varphi \dddot{\Phi} + \cos \varphi \ddot{\Phi}' + \sqrt{(\nu + 1)} \sin \varphi \dddot{\Phi} = 0.$$

Now

$$\frac{d}{dq} \frac{d\Phi}{dx} = \frac{d}{dq} \frac{dy}{dx} = -y' \sin \varphi$$

$$\frac{d^2 \Phi}{dq^2} = -y' \cos \varphi + y'' \sin^3 \varphi$$

So

$$\ddot{\Phi} + \frac{\cos \varphi \ddot{\Phi}'}{\sin \varphi} + \sqrt{(\nu + 1)} \dddot{\Phi} = 0 \quad \text{become}$$

$$y'' \sin^3 \varphi - y' \cos \varphi + \frac{\cos \varphi (-y' \sin \varphi) + \sqrt{(\nu + 1)} y}{\sin \varphi} = 0.$$

This yields

$$y'' \sin^3 \varphi - 2 y' \cos \varphi + \sqrt{(\nu + 1)} y = 0.$$

But $\sin^3 \varphi = 1 - \chi^2$. Hence

$$\begin{cases} 
(1 - \chi^2) \dddot{y} - 2 \chi \dot{y}' + \sqrt{(\nu + 1)} y = 0 & -1 < \chi < 1 \\
\ddot{y} \text{ bounded at } \chi = \pm 1 
\end{cases}$$

Remarks

(i) Equation (3) is Legendre's equation.

(ii) $\chi = \pm 1$ are regular singular points.

(iii) We want to find the eigenvalue parameter $\sqrt{\nu}$ so that (3) has non-trivial solutions that are bounded at the pole $\chi = \pm 1$. Note that $\chi = \pm 1$ are $\varphi = 0, \pi$. 
To look for solution of (3) we put

$$\psi = \sum_{j=0}^{\infty} a_j x^j$$

and define a recursion relation for $a_j$. The upshot of this calculation is the following:

(i) If $\psi = 0$, $n = 0, 1, 2, \ldots$ then (3) has solution of the form

$$\psi(x) = A P_n(x) + B Q_n(x)$$

where $P_n(x)$ is a polynomial of degree $n$ in $x$.

$Q_n(x)$ has singularities at $x = \pm i$.

(ii) Therefore the eigenvalues and eigenfunctions of (3) are

$$\psi_n = P_n(x), \quad \psi = 0, \quad n = 0, 1, 2, \ldots$$

(iii) $P_n(x)$ are called Legendre polynomials and are defined by

$$P_n(x) = \sum_{r=0}^{n} \frac{(-1)^r (2n-2r)!}{2^r r! (n-r)!^2} x^{n-2r}, \quad m = \text{integer part of} \frac{n}{2}$$

In addition, $P_n(x)$ is normalized by

$$P_n(1) = 1.$$ There is another way of defining these polynomials. See the appendix A page A2 below.

(iv) The first few such polynomials are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2} (3x^2 - 1), \quad P_3(x) = \frac{1}{2} (5x^3 - 3x),$$

$$P_4(x) = \frac{35}{8} x^4 - \frac{15}{4} x^2 + \frac{3}{8}.$$
LEGENDRE POLYNOMIALS: SERIES EXPANSION

Consider \((1 - x^2) y'' - 2xy' + \sqrt{\nu + 1} y = 0\)

We put in \(y = \sum_{m=0}^{\infty} a_m x^m\) to get

\[(1 - x^2) \sum_{m=1}^{\infty} a_m x^{m-2} - 2x \sum_{m=1}^{\infty} m a_m x^{m-1} + \nu \sqrt{\nu + 1} \sum_{m=0}^{\infty} a_m x^m = 0\]

This gives \(\sum_{m=1}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=1}^{\infty} m a_m x^{m-1} - \sum_{m=1}^{\infty} 2m a_m x^m + \nu \sqrt{\nu + 1} \sum_{m=0}^{\infty} a_m x^m = 0\).

\[X^0 \text{ term: } 2a_2 + \nu \sqrt{\nu + 1} a_0 = 0\]

\[X^1 \text{ term: } 6a_3 - 2a_4 + \nu \sqrt{\nu + 1} a_1 = 0\]

Coefficient of \(x^m\) for \(m \geq 2\)

\[(m+2)(m+1)a_{m+2} - m(m+1)a_m - 2m a_m + \nu \sqrt{\nu + 1} a_m = 0\]

Thus \(a_{m+2} = \frac{(m^2 + m - \nu \sqrt{\nu + 1})a_m}{(m+2)(m+1)}\), \(m \geq 2\).

But notice that (x) also work if \(m=0,1\) in comparing with (x).

To generate two linearly independent solutions:

(1) Let \(a_0 = 1, a_1 = 0\). Then \(a_3 = a_5 = \ldots = 0\). Also notice that

If \(\nu = \eta\) with \(\eta\) even then we get a polynomial of degree \(\eta\), i.e.

\(\text{eq: let } \eta = 2, \text{ then } a_0 = 1, a_2 = \frac{-2(3)}{2(1)} a_0 = -3x^2 a_0\) \(a_{n+2} = a_{n+4} = \ldots = 0\)

Thus \(y_1 = a_0 (1 - 3x^2)\). If \(\eta = 1\) we instead \(y_1 = \frac{(3x^2-1)}{2}\).

\(\text{eq: let } \eta = 4, \text{ then } a_0 = 1, a_2 = \frac{-20}{2} a_0, a_4 = -20 a_0, a_6 = \ldots = 0\)

\(a_4 = \frac{6 - 20}{30} a_0, a_2 = \frac{-7}{15} \left[-20 a_0\right] = \frac{70}{15} a_0\).

So \(y_1 = a_0 - 10a_0 x^2 + \frac{70}{15} x^4 a_0\).
(ii) If $q_0 = 0$, $q_1 = 1$, then $q_2 = q_4 = q_6 = \ldots = 0$.

If $Y = N$ and $N = 0, \ldots, N$ then we get a polynomial of degree $N$ since $a_{p+1} = a_{p+3} = \ldots = 0$.

Set $N = 1$, then $q_1 = 1$ and $q_3 = 0 \implies Y_2 = X$

$N = 3$, then $a_1 = 1$ and $a_3 = \frac{2 - 12}{6} q_1 = -\frac{5}{3}$

so $Y_2 = q_1 X - 5 a_1 X^{3/3}$. 

The $Q_n(x)$ are

$$Q_0(x) = \frac{1}{2} \log \left( \frac{1+x}{1-x} \right), \quad Q_1(x) = \frac{1}{2} x \log \left( \frac{1+x}{1-x} \right) - 1, \ldots$$

Now returning to Laplace's equation on page 1 we write the equation for $R(x)$,

$$r^2 R'' + 2r R' - n(n+1) R = 0$$

We put $R = r^B$ in Euler's equation $\implies B(B-1) + 2B - n(n+1) = 0$,

$$B^2 + B - n(n+1) = (B + n(n+1))(B - n) = 0$$

so $B = n, -n(n+1)$.

$$R_n(r) = A_n r^n + B_n r^{-n(n+1)}$$

Thus the general solution to Laplace's equation in terms of spherical coordinates, and assuming azimuthal symmetry, is

$$U(r, \phi) = \sum_{n=0}^{\infty} \left( A_n r^n + B_n r^{-n(n+1)} \right) P_n(\cos \phi)$$

If the domain is the sphere $0 < r < a$ with boundary condition $U(a, \phi) = F(\phi)$, then $B_n = 0$, $\forall n$ and

$$U(r, \phi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos \phi)$$

with

$$F(\phi) = \sum_{n=0}^{\infty} A_n a^n P_n(\cos \phi)$$

It remains to find the $A_n$. 

\[ \tag{4} \]
**IF THE DOMAIN IS CONCENTRIC SPHERE** \( 0 < r < R < b \) **THEN**

\[
U(\gamma, \varphi) = \sum_{n=0}^{\infty} \left( A_n \gamma^n + B_n \gamma^{-(n+1)} \right) P_n(\cos \varphi)
\]

**IF THE DOMAIN IS OUTSIDE THE SPHERE** \( R > a \) **WITH**

\[U \to 0 \quad \text{as} \quad \gamma \to \infty\] **THEN**

\[
U(\gamma, \varphi) = \sum_{n=0}^{\infty} B_n \gamma^{-(n+1)} P_n(\cos \varphi).
\]

---

**PROPERTIES OF LEGENDRE POLYNOMIALS**

We consider

\[
(1 - x^2) y'' - 2x y' + n (n+1) y = 0 \quad -1 < x < 1
\]

The solution is \( y_n = P_n(x) \). We can write this in Sturm-Liouville form as

\[
\left[ (1 - x^2) y' \right]' = -n (n+1) y, \quad -1 < x < 1
\]

Hence the weight function is \( w(x) = 1 \) and \( p(x) = 1 - x^2 \).

Consequently, we have the orthogonality relation

\[
\int_{-1}^{1} P_n(x) P_m(x) \, dx = 0 \quad \text{for} \quad n \neq m.
\]

If we let \( x = \cos \varphi \) then this becomes

\[
\int_{0}^{\pi} P_n(\cos \varphi) P_m(\cos \varphi) \sin \varphi \, d\varphi = 0 \quad \text{for} \quad n \neq m.
\]

Below we will show that

\[
\int_{-1}^{1} \left| P_n(x) \right|^2 \, dx = \frac{2}{2n+1}
\]
(A) Thus we have the orthogonality relation
\[ \int_{-1}^{1} P_n(x) P_m(x) \, dx = \frac{2}{2n+1} \delta_{mn}, \quad \delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n. \end{cases} \]

Or equivalently
\[ \int_{0}^{\pi} P_n(\cos \varphi) P_m(\cos \varphi) \sin \varphi \, d\varphi = \frac{2}{2n+1} \delta_{mn}. \]

(B) Next, as for all Sturm-Liouville problems, we have a completeness property that says that for any continuous function \( f(x) \) with \(-1 < x < 1\) that
\[ f(x) = \sum_{n=0}^{\infty} Q_n P_n(x) \]
with mean square convergence.

To find the coefficients, we calculate
\[ \int_{-1}^{1} f(x) P_m(x) \, dx = \sum_{n=0}^{\infty} Q_n \int_{-1}^{1} P_n(x) P_m(x) \, dx = Q_m \frac{2}{2m+1} \]
so
\[ Q_m = \frac{(2m+1)}{2} \int_{-1}^{1} f(x) P_m(x) \, dx. \]

This gives the Fourier-Legendre series
\[ f(x) = \sum_{n=0}^{\infty} \left( \frac{(2n+1)}{2} \int_{-1}^{1} f(x) P_n(x) \, dx \right) P_n(x) \, dx. \]

(C) We have that the first few Legendre polynomials are
\[ P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2} (3x^2 - 1), \quad P_3(x) = \frac{1}{2} (5x^3 - 3x), \quad P_4(x) = \frac{35}{8} x^4 - \frac{15}{4} x^2 + \frac{3}{8}. \]

\[ P_0(\cos \varphi) = 1, \quad P_1(\cos \varphi) = \cos \varphi, \quad P_2(\cos \varphi) = \frac{1}{2} (3 \cos^2 \varphi - 1), \quad P_3(\cos \varphi) = \frac{1}{2} (5 \cos^3 \varphi - 3 \cos \varphi), \quad P_4(\cos \varphi) = \frac{35}{8} \cos^4 \varphi - \frac{15}{4} \cos^2 \varphi + \frac{3}{8}. \]
Generating Function

One of the most convenient ways to derive properties of Legendre polynomials is to introduce a generating function \( G(x,t) \). The function is defined in such a way that the coefficients of \( G(x,t) \) in the Taylor series about \( t = 0 \) are \( P_n(x) \).

1. \[ G(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n. \]

We will now show that

\[
(\star) \quad \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n, \quad \text{for } 0 < t < 1.
\]

To show this we write

\[
(\dagger) \quad (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} Z_n(x) t^n
\]

Using the binomial series on the LHS for \( t \) small it follows that \( Z_n(x) \) must be a polynomial of degree \( n \).

To get some intuition for this, we let \( t \) be small and calculate using

\[
(1+h)^{-1/2} \approx 1 - \frac{1}{2} h + \frac{3}{8} h^2 + \ldots
\]

with \( h = t^2 - 2xt \)

\[
\left[1 + (t^2 - 2xt)\right]^{-1/2} \approx 1 - \frac{1}{2} [t^2 - 2xt] + \frac{3}{8} [t^2 - 2xt]^2 + \ldots
\]

\[
\approx 1 + xt + t^2 \left( \frac{3x^2 - 1}{2} \right) + \ldots
\]

\[
\approx P_0(x) + t P_1(x) + t^2 P_2(x) + \ldots
\]

So at least up to \( n = 0, 1, 2 \), \((\star)\) is correct. Can we show \((\star)\) for all \( n \)?
To do so we show from (1) that $Z_n(x)$ satisfy Legendre's differential equation. Then since $Z_n(x)$ is a polynomial of degree $n$ and $Z_n(1) = 1$ for $n = 0, \ldots$ (Note: $(1 - 2x + t^2)^{-1/2} = (1 - t)^{-1} = \sum_{n=0}^{\infty} Z_n(1) t^n \rightarrow Z_0(1) t^1$)

It follows that $Z_n(x) = P_n(x)$.

To show that $Z_n(x)$ satisfy Legendre's equation we differentiate (1) with respect to $x$ to obtain

$$t \left(1 - 2x + t^2\right)^{-3/2} = \sum_{n=0}^{\infty} Z_n'(x) t^n$$

Differentiating again we get

$$3t^2 \left(1 - 2x + t^2\right)^{-5/2} = \sum_{n=0}^{\infty} Z_n''(x) t^n$$

Now differentiate (1) with respect to $t$ to get

$$\left(x - t\right) \left(1 - 2x + t^2\right)^{-3/2} = \sum_{n=0}^{\infty} Z_n(x) t^{n-1}$$

Now multiply (4) by $t$ and differentiate with respect to $t$

$$\frac{d}{dt} \left[ t^2 \left(x - t\right) \left(1 - 2x + t^2\right)^{-3/2}\right] = \sum_{n=0}^{\infty} n(n+1) Z_n(x) t^n$$

$$t^2 \left[\left(x - t\right) \left(1 - 2x + t^2\right)^{-3/2} - \left(x - t\right) \left(1 - 2x + t^2\right)^{-5/2}\right] + 2t \left(x - t\right) \left(1 - 2x + t^2\right)^{-3/2}$$

$$= \sum_{n=0}^{\infty} n(n+1) Z_n(x) t^n$$

Thus is simplified to

$$\left(1 - 2x + t^2\right)^{-3/2} \left[ 3t^2 \left(x - t\right)^2 \left(1 - 2x + t^2\right)^{-1} - 1 + 2t \left(x - t\right)\right] = \sum_{n=0}^{\infty} n(n+1) Z_n(x) t^n$$
NOW COMBINE (2), (3) AND (5)

\[
(1 - x^2) \sum_{n=0}^{\infty} Z_n''(x) t^n - 2x \sum_{n=0}^{\infty} Z_n'(x) t^n + \sum_{n=0}^{\infty} n(n+1) Z_n(x) t^n
\]

\[
= 3t^2 (1 - x^2) (1 - 2xt + t^2)^{-3/2} - 2xt (1 - 2xt + t^2)^{-3/2}
\]

\[
+ (1 - 2xt + t^2)^{-3/2} \left[ 3t^2 (x-t)^2 (1 - 2xt + t^2)^{-1} + 1 + 2t(x-t) \right]
\]

THE RHJ IS IDENTICALLY ZERO AFTER SOME ALGEBRA, AND SO

\[
\sum_{n=0}^{\infty} \left[ (1 - x^2) Z_n'' - 2x Z_n' + n(n+1) Z_n \right] t^n = 0
\]

HENCE

\[
\left( 1 - x^2 \right) Z_n'' - 2x Z_n' + n(n+1) Z_n = 0 \quad \Rightarrow \quad Z_0(x) = P_0(x)
\]

WITH \( Z_0(1) = 1 \)

**Remark:** In the appendix A we give a different, easier, proof of (1).

Now the generating function can be used for many results relating to \( P_0(x) \).

**Problem 1** Show from \( (1 - 2xt + t^2)^{-3/2} = \sum_{n=0}^{\infty} P_n(x) t^n \)

THAT

\[
(n+1) P_{n+1}(x) - (2n+1) x P_n(x) + n P_{n-1}(x) = 0 \quad (\star)
\]

WITH \( P_0(x) = 1, P_1(x) = x \) THIS RECURSION RELATION CAN BE READILY USED TO CALCULATE ALL THE \( P_n(x) \).

I.E. FOR \( P_2(x) \):

\[
2 P_2(x) = 3 x P_1(x) - P_0(x)
\]

\[
\Rightarrow \quad P_2(x) = \frac{3x^2}{2} - \frac{1}{2}
\]

TO SHOW (\star) DIFFERENTIATE THE GEN. FUNCTION WRT \( t \):

\[
(x-t) (1 - 2xt + t^2)^{-3/2} = \sum_{n=0}^{\infty} P_n(x) t^n t^{n-1}
\]
Multiply by \((1 - 2xt + t^2)\) to get

\[
(x - t) \left(1 - 2xt + t^2\right)^2 y_1 = (1 - 2xt + t^2) \sum_{n=0}^{\infty} P_n(x) n t^{n-1}
\]

\[
(x - t) \sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} P_n(x) n t^{n-1} - 2x \sum_{n=0}^{\infty} P_n(x) n t^n + \sum_{n=0}^{\infty} P_n(x) n t^{n+1}
\]

Shifting indices etc, this can be shown to yield \((x)\)

**Problem 2** Use the generating function to prove that

\[
\int_{-1}^{1} (P_n(x))^2 \, dx = \frac{2}{2n+1}
\]

**Proof** We outline the proof. The details are in HW.

\[
\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n \quad \text{now square both sides and combine}
\]

\[
\frac{1}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_n(x) P_m(x) t^n t^m
\]

\[
\int_{-1}^{1} \frac{dx}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} \left( \int_{-1}^{1} (P_n(x))^2 \, dx \right) t^n \quad \text{using} \int_{-1}^{1} P_n(x) P_m(x) \, dx = 0 \quad \text{if} \ n \neq m
\]

Now simply integrate the LHS and expand the result in a Taylor series in \(t\). This yields

\[
\int_{-1}^{1} (P_n(x))^2 \, dx = \frac{2}{2n+1}.
\]
Determine the Fourier-Legendre series of \( f(x) = x^2 \) on \(-1 < x < 1\).

We write
\[
F(x) = \sum_{n=0}^{\infty} a_n P_n(x)
\]
so
\[
a_n = \frac{(2n+1)}{2} \int_{-1}^{1} F(x) P_n(x) \, dx \quad (\forall n)
\]

Since \( F(x) = x^2 \) then \( a_n = 0 \) for \( n \geq 3 \), this is because any polynomial of degree 2 can be written as \( x^2 = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) \) for some \( c_0, c_1, c_2 \). Hence
\[
\int_{-1}^{1} (c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x)) P_n(x) \, dx = 0
\]
\( \forall n \geq 3 \) by orthogonality
\[
\int_{-1}^{1} P_m(x) P_n(x) \, dx = 0 \quad \text{for } m \neq n.
\]

Thus we need only calculate \( a_0, a_1, a_2 \) in (4).

\underline{Method 1}
\[
P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)
\]
Hence
\[
2 P_2 = 3 x^2 - 1, \quad x^2 = \frac{2}{3} P_2(x) + \frac{1}{3} P_0(x).
\]
We claim \( a_2 = \frac{2}{3}, \quad a_1 = 0, \quad a_0 = \frac{1}{3} \).

\underline{Method 2}
We calculate directly that
\[
a_0 = \frac{1}{2} \int_{-1}^{1} x^1 P_0(x) \, dx = \frac{1}{2} \int_{-1}^{1} x^2 \, dx = \frac{1}{3}
\]
\[
a_1 = \frac{3}{2} \int_{-1}^{1} x^2 x \, dx = 0
\]
\[
a_2 = \frac{5}{2} \int_{-1}^{1} x^2 (\frac{3x^2}{2} - \frac{1}{2}) \, dx = \frac{5}{2} \left( \frac{3x^5}{10} - \frac{x^3}{6} \right) \bigg|_{-1}^{1} = \frac{2}{3}.
\]
**Problem 2** Solve Laplace's Equation outside a sphere,

\[ \Delta U = 0, \quad \Gamma > a, \quad 0 < \varphi < \pi \]

This represents the velocity potential of a fluid that is incompressible and inviscid.

With \( U \sim \Gamma \cos \varphi \) as \( \Gamma \to \infty \).

We write the general solution as

\[ U(\Gamma, \varphi) = \sum_{n=0}^{\infty} \left( A_n \Gamma^n + B_n \Gamma^{-1-n} \right) P_n(\cos \varphi) \]

Now \( A_n = 1, \; B_n = 0 \) for \( n = 0, 2, \ldots \). \( P_1(\cos \varphi) = \cos \varphi \).

Hence

\[ U(\Gamma, \varphi) = \left( \Gamma + \frac{B}{\Gamma^2} \right) \cos \varphi \]

Now \( U_\Gamma = 0 \) on \( \Gamma = a \)

\[ 1 + \left( \frac{-2B}{a^3} \right) = 0 \]

\( B = \frac{a^3}{2} \)

Thus, \( U(\Gamma, \varphi) = \left( \Gamma + \frac{a^3}{2\Gamma^2} \right) \cos \varphi \).

**Problem 3** Solve Laplace's equation in a sphere with azimuthal symmetry.

\[ \Delta U = 0 \text{ in } 0 < \Gamma < a, \quad 0 < \varphi < \pi \]

\( U(a, \varphi) = f(\varphi) \) on \( \Gamma = a \).

We write

\[ U(\Gamma, \varphi) = \sum_{n=0}^{\infty} A_n \Gamma^n P_n(\cos \varphi) \]

Now

\[ f(\varphi) = \sum_{n=0}^{\infty} \underbrace{A_n a^n P_n(\cos \varphi)}_{n=0} \]

Orthogonality gives

\[ A_n a^n \frac{2}{2n+1} = \int_{0}^{\pi} f(\varphi) P_n(\cos \varphi) \sin \varphi \, d\varphi \]

So

\[ A_n = \frac{(2n+1)}{2} \frac{1}{a^n} \int_{0}^{\pi} f(\varphi) P_n(\cos \varphi) \sin \varphi \, d\varphi \]
Suppose now that \( F(q) = T_0 \sin^4 q \) with \( T_0 \) constant. We will calculate the solution explicitly.

We write

\[
U(r, \varphi) = T_0 \sum_{n=0}^{\infty} A_n \left( \frac{r}{a} \right)^n P_n(\cos \varphi)
\]

Then

\[
\sin^4 \varphi = \sum_{n=0}^{\infty} A_n P_n(\cos \varphi)
\]

But if \( x = \cos \varphi \) then

\[
\sin^4 \varphi = (1 - x^2)^2 = x^4 - 2x^2 + 1
\]

so

\[
x^4 - 2x^2 + 1 = \sum_{n=0}^{\infty} A_n P_n(x)
\]

We must have \( A_0 = 0, \quad n \geq 5 \)

Now recall

\[
P_4(x) = \frac{35}{8} x^4 - \frac{15}{4} x^2 + \frac{3}{2} P_2(x) = \frac{3}{2} x^2 - \frac{1}{2}
\]

\[
P_0(x) = 1
\]

So

\[
x^4 = \frac{8}{35} P_4(x) - \frac{8}{35} \left( -\frac{15}{4} x^2 + \frac{3}{2} \right) = \frac{8}{35} P_4(x) + \frac{6}{7} x^2 - \frac{3}{35}
\]

\[
x^4 - 2x^2 = \frac{8}{35} P_4(x) - \frac{8}{7} x^2 - \frac{3}{35}
\]

\[
= \frac{8}{35} P_4(x) - \frac{8}{7} \left( \frac{2}{3} P_2(x) + \frac{1}{3} \right) \frac{3}{35}
\]

\[
x^4 - 2x^2 = \frac{8}{35} P_4(x) - \frac{16}{71} P_1(x) - \frac{8}{21} P_2(x) - \frac{3}{35}
\]

\[
x^4 - 2x^2 + 1 = \frac{8}{35} P_4(x) - \frac{16}{71} P_1(x) + \frac{8}{15}
\]

Hence

\[
A_0 = \frac{8}{15}, \quad A_2 = -\frac{16}{21}, \quad A_4 = \frac{8}{35}, \quad A_6 = 0 \quad \text{for} \quad n \neq 0, 2, 4
\]

\[
U(r, \varphi) = T_0 \sum_{n=0}^{\infty} A_n \left( \frac{r}{a} \right)^n P_n(\cos \varphi)
\]
Remark. There is another way to find coefficients for
\[ x^4 - 2x^2 + 1 = \sum_{n=0}^{\infty} A_n P_n(x). \]

Now the LHS is even in \( x \) and is degree 4. Hence since \( P_1, P_3 \) are odd
we need
\[ x^4 - 2x^2 + 1 = A_0 P_0(x) + A_2 P_2(x) + A_4 P_4(x) \]

Equating coefficients of \( x^4, x^2 \) and 1 give
\[ \begin{align*}
  x^4: & \quad 1 = \frac{85}{8} A_4 \quad \implies \quad A_4 = \frac{8}{35} \\
  x^2: & \quad -2 = -\frac{15}{4} A_4 + \frac{3}{2} A_2 \quad \implies \quad -2 = -\frac{15}{4} \left( \frac{8}{35} \right) + \frac{3}{2} A_2 \quad \implies \quad A_2 = -2 + \frac{6}{7} = -\frac{8}{7} \\
  1: & \quad 1 = A_0 - \frac{3}{2} A_2 + 3 A_4/8 \quad \implies \quad A_0 = -16/21
\end{align*} \]

Then
\[ 1 = A_0 + \frac{9}{21} + \frac{3}{35} = A_0 + \frac{1}{7} \left( \frac{8}{3} + \frac{3}{5} \right) = A_0 + \frac{1}{7} \left( \frac{49}{15} \right) = A_0 + \frac{7}{15} \]

So \( A_0 = 1 - \frac{7}{15} = \frac{8}{15} \).

We conclude that
\[ x^4 - 2x^2 + 1 = \frac{8}{15} \cdot P_0(x) + \frac{16}{21} \cdot P_2(x) + \frac{8}{35} \cdot P_4(x). \]

Problem. Prove that
\[ \int_{-1}^{1} x^N P_m(x) \, dx = 0 \quad \text{if} \quad m > n. \quad m, n \text{ integers} \geq 0. \]

Proof. We can write
\[ x^N = \sum_{j=0}^{N} A_j P_j(x) \quad \text{since} \quad x^N \text{ is a polynomial of degree} \ n. \quad \text{Then coefficients} \ A_j \ \text{can be found. Substitute and use}
\]

Orthogonality
\[ \int_{-1}^{1} x^N P_m(x) \, dx = \int_{-1}^{1} \sum_{j=0}^{N} A_j P_j(x) \cdot P_m(x) \, dx = \sum_{j=0}^{N} A_j \int_{-1}^{1} P_j \cdot P_m \, dx = 0. \]

Thus for any polynomial \( Q(x) \) of degree \( n \) we have
\[ \int_{-1}^{1} Q(x) P_m(x) \, dx = 0 \quad \text{if} \quad m > n. \quad \text{(similar proof).} \]
\[ A_n = \frac{(2n+1)}{2} \int_{-1}^{1} (x^4 - 2x^2 + 1) P_n(x) \, dx \]

We would observe that \( A_1, A_3 = 0 \) since \( P_1, P_3 \) are odd functions, while \((x^4 - 2x^2 + 1)\) is even. But calculating these integrals for \( A_0, A_2, A_4 \) is tedious by hand. The other method is better.

**Problem 4**

Show that the electrostatic potential for a point charge at \( x_0 = (0, 0, a) \) on the z-axis can be expanded in terms of Legendre polynomials \( P_n \):

\[
\frac{1}{|x - x_0|} = \frac{1}{a} \sum_{n=0}^{\infty} P_n(\cos \varphi) \left( \frac{r}{a} \right)^n \quad \text{with} \quad r = |x|, \quad \text{when} \quad 0 \leq \varphi < \pi.
\]

**Derivation**

We write \( |x - x_0|^2 = x^2 + y^2 + (z - a)^2 = x^2 + y^2 + z^2 - 2az + a^2 = r^2 - 2az \)

Then put \( z = r \cos \varphi \) with \( \varphi = \) latitude.

Hence \( \frac{1}{|x - x_0|} = \frac{1}{a} \sqrt{r^2 - 2ar \cos \varphi + a^2} \)

So \( \frac{1}{|x - x_0|} = \frac{1}{a} \frac{1}{\sqrt{r^2/a^2 - 2 \cos \varphi + 1}} \)

Now recall the generating function \( \frac{1}{\sqrt{1 - 2tx + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n \).

Hence, we let \( t = \frac{r}{a}, \quad x = \cos \varphi \).

This gives \( \frac{1}{|x - x_0|} = \frac{1}{a} \sum_{n=0}^{\infty} \left( \frac{r}{a} \right)^n P_n(\cos \varphi) \)

which converges when \( 0 < r < a \).
Next we consider diffusion on the surface of a sphere modeled by

\[
U_t = \frac{1}{a^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial U}{\partial \theta} \right), \quad 0 < \theta < \pi, \quad t > 0
\]

\[
U(\theta, 0) = f(\theta)
\]

Here \(a\) is the radius of the sphere and we have assumed azimuthal symmetry.

We separate variables to obtain

\[
\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) = -\gamma(\gamma + 1) \Phi
\]

Hence

\[
(\sin \theta \frac{\partial \Phi}{\partial \theta})' + \sqrt{\gamma + 1} \sin \theta \Phi = 0, \quad 0 < \theta < \pi
\]

\[
T' = -\sqrt{\gamma + 1} \frac{T}{a^2} \quad \Phi \text{ bounded at } \theta = 0, \pi
\]

Then

\[
\Phi_n(\theta) = P_n(\cos \theta) \quad n = 0, 1, 2, \ldots
\]

\[
T = e^{\frac{-\gamma(\gamma + 1) t}{a^2}}
\]

Hence we obtain the separation of variables solution

\[
U(\theta, t) = \sum_{n=0}^{\infty} A_n P_n(\cos \theta) e^{\frac{-\gamma(\gamma + 1) t}{a^2}}
\]

If

\[
U(\theta, 0) = f(\theta) = \sum_{n=0}^{\infty} A_n P_n(\cos \theta)
\]

Then

\[
A_n = \frac{(2n + 1)}{2} \int_0^{\pi} f(\theta) P_n(\cos \theta) \sin \theta \, d\theta, \quad n = 0, 1, \ldots
\]
NOTICE THAT
\[
A_0 = \frac{1}{2} \int_0^\pi f(q) \sin q \, dq = \lim_{t \to \infty} u(q, t).
\]

NOW SUPPOSE \( f(q) = 2 \cos^2 q - 1 \).

WE WRITE
\[
2x^2 - 1 = \sum_{n=0}^\infty A_n p_n(x)
\]

HENCE \( A_0 = 0 \) \( \forall \, n \geq 3 \). WE HAVE
\[
p_2(x) = \frac{1}{2} (3x^2 - 1).
\]

HENCE
\[
2x^2 = \frac{4}{3} \left( p_2(x) + \frac{1}{2} \right) = \frac{4}{3} p_1(x) + \frac{2}{3}
\]

SO
\[
2x^2 - 1 = \frac{4}{3} p_2(x) - \frac{1}{3} = \frac{4}{3} p_2(x) - p_0(x).
\]

THIS YIELD THAT \( u(q, t) \) IS GIVEN EXPLICITLY BY
\[
u(q, t) = -\frac{1}{3} + \frac{4}{3} P_2(\cos q) \exp \left( -\frac{6}{q^2} t \right).
\]

REMARK IF WE CHANGED THIS PROBLEM TO
\[
u_t = \frac{1}{\alpha^2 \sin q} \left( \sin q \, \nu \right)'(q)
\] \( \phi_c < q < \pi \)

\( \phi_c > 0 \)

THEN
\[
\frac{1}{\sin q} \left( \sin q \, \phi' \right)' + \sqrt{\gamma + 1} \phi = 0, \quad \phi_c < q < \pi.
\]

THE SOLUTION THAT HAS NO SINGULARITY AT \( q = \pi \)

IS THE LEGENDRE FUNCTION \( P_\gamma (\cos q) \). (\( \gamma \) NOT NECESSARILY AN INTEGER)

THE CONDITION \( \phi(\phi_c) = 0 \) WOULD GIVE THE

EIGENVALUE RELATION \( P_\gamma (\cos \phi_c) = 0 \)

WHICH IS AN IMPLICIT EQUATION FOR \( \gamma = \gamma_0, \ldots \).
Appendix A

Derivation of generating function

Let \( x_0 \) be some fixed point in \( \mathbb{R}^3 \). Then \( \frac{1}{|x - x_0|} \) is a harmonic function in \( \mathbb{R}^3 \) so that \( \Delta \left( \frac{1}{|x - x_0|} \right) = 0 \). This follows since if we let

\[
p = |x - x_0|\]

then

\[
\Delta \left( \frac{1}{p} \right) \equiv \left( \frac{1}{p} \right)^{n+2} \left( \frac{n}{p} \right)^{n+1} = 0.
\]

Now without loss of generality let \( x_0 \) be aligned with positive \( z \)-axis. We take \( x_0 \) to be unit vector. We get the picture shown.

\[
|X - X_0|^2 = (X - X_0)^T (X - X_0)
\]

\[
= X^T X - 2X^T X_0 + X_0^T X_0
\]

\[
= r^2 - 2r \cos \varphi + 1
\]

(Note: \( X^T X = 1 \), \( r^2 = 1 \), \( X_0^T X_0 = 1 \), \( \cos \varphi = 1 \).

Thus

\[
\frac{1}{|X - X_0|} = \frac{1}{(r^2 - 2r \cos \varphi + 1)^{1/2}}
\]

is a solution to Laplace's equation in 3-D with azimuthal symmetry, (independent of angle \( \phi \)).

Hence for \( \varphi \neq 0, r \neq 0 \), we must have for some \( A_n \), to find \( A_n \).

\[
\frac{1}{(r^2 - 2r \cos \varphi + 1)^{1/2}} = \sum_{\eta = 0}^{\infty} A_n \eta^n P_n(1, \cos \varphi)
\]

For some constant \( A_n \) (independent of \( \varphi \) and \( r \)), to find \( A_n \).

Set \( \varphi = 0 \) and take \( r < 1 \). Then

\[
(r^2 - 2r \cos \varphi + 1)^{-1/2} = (r^2 - 2r + 1)^{-1/2} = (1 - r^{-1})^{-1}.
\]

Hence

\[
\frac{1}{1 - r^{-1}} = \sum_{\eta = 0}^{\infty} A_n P_n(1) \eta^n.
\]

But \( P_n(1) = \frac{\delta_n}{(1 - r^{-1})} \to \frac{\delta_n}{1 - r} \), \( \delta_n \rightarrow 1 \), \( A_n \to 1 \).
Thus we have
\[
\frac{1}{(\Gamma - 2\Gamma \cos \phi + \Gamma)^{1/2}} = \sum_{n=0}^{\phi} \Gamma^n P_n (\cos \phi).
\]

Let \( x = \cos \phi, \ t = \Gamma \), with \( 0 < t < 1 \).

Hence
\[
\frac{1}{(t^2 - 2tx + 1)^{1/2}} = \sum_{n=0}^{\phi} t^n P_n (x).
\]

\[\square\]

Another way to define the Legendre polynomials

We claim that
\[
P_n (x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[ (x^2 - 1)^n \right]. \quad \text{(Rodrigues's formula)}.
\]

Clearly \( P_n (x) \) is a polynomial of degree \( n \) and by writing \( (x^2 - 1) = (x-1)(x+1) \)

we have
\[
P_n (x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[ (x-1)^n (x+1)^n \right].
\]

Upon differentiating \( n \) times and evaluating at \( x = 1 \) we get one non-zero term given by
\[
\left. \frac{d^n}{dx^n} \left[ (x-1)^n (x+1)^n \right] \right|_{x=1} = n! 2^n. \quad \text{Thus } P_n (1) = 1.
\]

Now we must show that \( P_n (x) \) satisfies
\[
(x) \quad (1-x^2) V'' - 2x V' + n(n+1) V = 0.
\]

Define \( h(x) = (1-x^2)^n \). We need only show that \( V = \frac{d^n}{dx^n} (1-x^2)^n \) satisfies (x) and then we are done.

We calculate
\[
h'(x) = -2nx (1-x^2)^{n-1}
\]

so
\[
(1-x^2) h' + 2n x (1-x^2)^n = 0 \quad \rightarrow \quad (1-x^2) h' + 2n x \cdot h = 0. \quad (\text{a})
\]
Now let \( A, B \) be any two functions of \( x \). By repeated application of the chain rule,

\[
(AB)'' = A''B + 2A'B' + AB''
\]

\[
(AB)''' = A'''B + 3A''B' + 3A'B'' + AB'''
\]

\[
(AB)^{(m)} = A^{(m)}B + mA^{(m-1)}B' + \binom{m}{2}A^{(m-2)}B'' + \ldots + mA^{(1)}B^{(m-1)} + AB^{(m)}.
\]

Now differentiate (+) \( n+1 \) times.

\[
[(1-x^2)h']^{(n+1)} + 2n[(Xh)'^{(n+1)}] = 0.
\]

Set \( A = h', B = (1-x^2), M = n+1 \) in first term.

\( A = h, B = x, M = n+1 \) in second term.

So

\[
(1-x^2)h^{(n+2)} + (n+1)h^{(n+1)}(-2x) + \left( \binom{n+1}{2} \right) h^{(n)}(-2)
\]

\[
+ 2n \left[ x h^{(n+1)} + (n+1)h^{(n)} \right] = 0.
\]

Thus

\[
(1-x^2)h^{(n+2)} + h^{(n+1)} \left[ -2X(n+1) + 2nx \right] + h^{(n)} \left[ 2n(n+1) - 2 \binom{n+1}{2} \right] = 0.
\]

But

\[
\binom{n+1}{2} = \frac{(n+1)!}{(n-1)!2!} = \frac{n(n+1)}{2}.
\]

This gives

\[
2n(n+1) - 2 \frac{n(n+1)}{2} = n(n+1).
\]

We conclude that

\[
h^{(n)} = \frac{d^n}{dx^n} \left[ (1-x^2)^n \right]
\]

\( (1-x^2)(h^{(n)})'' - 2x(h^{(n)})' + n(n+1)h^{(n)} = 0 \).

Thus \( h^{(n)} \) satisfies (1) on previous page (Legendre's differential equation).

\[
p_0(x) = \frac{1}{2^n} \frac{d^n}{dx^n} \left[ (x^2-1)^n \right].
\]