Be sure that this examination has 3 pages.

The University of British Columbia
Final Examinations – December 2015

Mathematics 400

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Closed book examination

Time: 2\(\frac{1}{2}\) hours

Special Instructions: Closed Book and Notes. Calculators are not allowed.

Marks

[15] 1. Consider the following first order PDE for \(u = u(x, y)\):

\[ u_x + 2xu_y = (3y + x)u. \]

(i) Find the general solution to this PDE.

(ii) Find the specific solution to this PDE that satisfies the data

\[ u = f(y), \text{ on } x = 0 \text{ for } -1 \leq y \leq 1. \]

In which region of the \((x, y)\) plane is this solution defined?

[20] 2. Consider axially symmetric diffusion for \(u(r, z, t)\) in a finite cylinder of radius \(a > 0\) and height \(H > 0\) with insulating boundary conditions modeled by

\[
\begin{align*}
u_t &= u_{rr} + \frac{1}{r}u_r + u_{zz}, & 0 \leq r \leq a, & 0 \leq z \leq H, & t \geq 0, \\
u_z &= 0 & \text{on } z = 0 \text{ and } z = H; & u_r = 0 & \text{on } r = a, & u \text{ bounded as } r \to 0, \\
u(r, z, 0) &= f(r, z).
\end{align*}
\]

Determine an eigenfunction expansion representation for the time-dependent solution \(u(r, z, t)\), and also calculate the steady-state solution.
3. Assume that $\beta > 0$ and $\alpha \geq 0$ are constants, and consider the following traffic flow model for the density $\rho(x,t)$ of cars given by

$$\rho_t + (2 - \rho) \rho_x + \alpha \rho = 0, \quad -\infty < x < \infty, \quad t > 0,$$

$$\rho(x,0) = \frac{3\beta^2}{\beta^2 + x^2}.$$

(i) **First let $\alpha = 0.$** Determine a parametric form for the solution $\rho(x,t)$. Plot qualitatively the characteristics in the $(x,t)$ plane, and sketch the solution $\rho(x,t)$ versus $x$ at different times. Determine the time $t_b$ as a function of $\beta$ when the solution first becomes multi-valued.

(ii) **Now let $\alpha > 0.$** Find a value $\alpha_c$, which depends on $\beta$, such that the solution does not become multi-valued for any $t > 0$ if and only if $\alpha > \alpha_c$.

(iii) **Let $\alpha \geq 0.$** Calculate explicitly the total number $N(t)$ of cars on the road, defined by $N(t) = \int_{-\infty}^{\infty} \rho(x,t) \, dx$.

4. Consider the diffusion problem for $u(r, \theta, t)$ in a disk of radius $a$ with an inflow/outflow flux boundary condition modeled by

$$u_t = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta}, \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi, \quad t \geq 0,$$

$$u_r(a, \theta, t) = f(\theta), \quad u \text{ bounded as } r \to 0, \quad u \text{ and } u_{\theta} \text{ are } 2\pi \text{ periodic in } \theta,$$

$$u(r, \theta, 0) = g(r, \theta).$$

(i) **Write the problem that the steady-state solution $U(r, \theta)$ would satisfy.** Prove that such a steady-state solution $U(r, \theta)$ does not exist when $\int_0^{2\pi} f(\theta) \, d\theta \neq 0$.

(ii) **Assume that $\int_0^{2\pi} f(\theta) \, d\theta = 0.$** Calculate an integral representation for the steady state solution $U(r, \theta)$ by summing an appropriate eigenfunction expansion.

(ii) **Assume that $\int_0^{2\pi} f(\theta) \, d\theta \neq 0.$** Determine an approximation to the time-dependent solution $u(r, \theta, t)$ that is valid for large time $t$.  

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5. Each of these five short-answer questions below is worth 5 points. Very little calculation is needed for any of these problems.

(i) In the circular disk \(0 < r < a, 0 \leq \theta \leq 2\pi\) find the explicit solution to Laplace's equation for \(u(r, \theta)\):

\[
\frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = 0, \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi,
\]

\[
u(a, \theta) = 2 \cos^2(\theta) + 1, \quad u \text{ bounded as } r \to 0, \quad u \text{ and } u_{\theta} \text{ 2\pi periodic in } \theta.
\]

(ii) Outside the sphere \(r > a\), where \(\theta\) is the polar angle with \(0 \leq \theta \leq \pi\), find the explicit solution to Laplace's equation \(\Delta u = 0\) with \(u = 2 \cos^2(\theta) + 1\) on \(r = a\), \(u \to 0\) as \(r \to \infty\), and \(u\) is bounded at the poles \(\theta = 0, \pi\). (Recall that the first three Legendre polynomials are \(P_0(x) = 1, P_1(x) = x,\) and \(P_2(x) = (3x^2 - 1)/2.\)

(iii) Consider the radially symmetric diffusion problem for \(u(r, t)\) in the unit sphere \(0 < r < 1\), modeled by

\[
u_t = D \left( u_{rr} + \frac{2}{r} u_r \right), \quad 0 \leq r \leq 1, \quad t > 0
\]

\[
u(1, t) = 0, \quad u \text{ bounded as } r \to 0; \quad u(r, 0) = f(r),
\]

where \(D > 0\) is constant. Show that for long time, i.e. for \(t \to +\infty\), that the solution can be approximated by \(u(r, t) \approx Ae^{-D\pi^2t}\sin(\pi r)/r\) for some \(A > 0\) to be found.

(iv) Consider the damped wave-equation on a finite interval \(0 < x < L\), and with time-periodic forcing modeled by

\[
u_{tt} + au_t = c^2 u_{xx} + \sin(\omega t), \quad 0 < x < L, \quad t > 0,
\]

\[
u(0, t) = 0, \quad u(L, t) = 0; \quad u(x, 0) = 0, \quad u_t(x, 0) = 0.
\]

Here \(c > 0\), while \(a > 0\) is small. For what frequencies \(\omega > 0\) will we obtain a very large response for \(u\) when \(a > 0\) is small? (Hint: these are the frequencies where resonance would occur if \(a = 0\).

(v) Solve the signalling problem for the wave equation \(u(x, t)\) where the signal is applied on a space-time curve as follows:

\[
u_{tt} = c^2 u_{xx}, \quad c_0 t < x < \infty, \quad t > 0,
\]

\[
u(c_0 t, t) = \sin(\omega t), \quad u(x, 0) = u_t(x, 0) = 0.
\]

Here \(0 < c_0 < c\).
Problem 1

\[ u_x + 2x u_y = (3y + x^2) u \]

(i) on \[ \frac{dy}{dx} = 2x \] \[ \frac{du}{dx} = (3y + x^2) u \]

so \[ y = x^2 + \lambda \]

we put \[ U(x', \lambda) \].

\[ u_x = U_x + U_\lambda (\cdot 2x) \quad u_y = U_\lambda \]

so \[ U_x + (-2x) U_\lambda + 2x U_\lambda = (3y + x^2) U \]

or \[ U_x' = \left[ 3(x^2 + \lambda) + x' \right] U \]

\[ U_x' = (3x^2 + 3\lambda + x') U \]

or \[ U = f(\lambda) e^{x^2 + 3\lambda x' + x'^2/2} \]

\[ U = f(y - x^2) e^{x^2 + 3(y - x^2)x' + x'^2/2} \]

we have \[ U = f(y - x^2) e^{-2x^3 + 3yx^2 + x^2} \]

(ii) put \[ u = f(y) \] on \(-1 < y < 1\) for \( x = 0 \).

\[ f(y) = f(y) \]

so \[ U(x, y) = f(y) e^{-2x^3 + 3yx^2 + x^2} \]
\textbf{Problem 2}

\[ U_t = U_{rr} + \frac{1}{r} U_r + U_{zz}, \quad 0 < r < a, \quad 0 < z < H, \quad t > 0 \]

\[ U_r = 0 \text{ on } r = a, \quad U \text{ bounded as } r \to 0 \]

\[ U_z = 0 \text{ on } z = 0, 1 \]

\[ U(r, z, 0) = f(r, z) \]

\textbf{Now we separate variables to obtain}

\[ \frac{T'}{T} = \frac{R'' + \frac{1}{r} R'}{R} + \frac{Z''}{Z} = -\lambda. \]

\textbf{Then}

\[ Z'' + \mu Z = 0 \quad Z_m(z) = \cos \left( \frac{m\pi z}{H} \right), \quad \mu_m = \frac{m^2 \pi^2}{H^2}, \quad m = 0, 1, 2, \ldots \]

\[ Z'(0) = Z'(H) = 0 \]

\textbf{Then we have}

\[ R'' + \frac{1}{r} R' + (\lambda - \mu) R = 0. \]

\[ R'(0) = 0 \quad R(a) \text{ nice} \]

\textbf{Let } \sigma = \lambda - \mu. \textbf{ Then}

\[ R_n(r) = J_0 \left( \sqrt{\sigma_n} r \right) \quad \text{also note } \sigma = 0 \]

\textbf{with}

\[ J_0' \left( \sqrt{\sigma_n} \right) = 0 \quad \sqrt{\sigma_n} \alpha = z_n \]

\[ J_0(z_n) \]

\textbf{Then}

\[ \sigma = \sigma_n + \mu_m = \sigma_n + m^2 \pi^2 / H^2 \]

\textbf{Define } \sigma = 0 \text{ since } J_0(0) \text{ will get first eigenpair for } R(r). \]

\textbf{Then}

\[ U(r, z, t) = A_{00} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} J_0 \left( \sqrt{\sigma_n} r \right) \cos \left( \frac{m\pi z}{H} \right) e^{-\sigma_n t} \]

\textbf{Now for } t > 0

\[ A_{00} = \frac{2\pi}{\alpha^3 H} \int_0^a \int_0^H r f(r, z) \, dr \, dz \]

\textbf{For } t > 1 \text{,}

\[ U(r, z, t) \approx A_{00} \]

\textbf{Then}

\[ A_{mn} = \frac{2}{\alpha^3 H} \int_0^a \int_0^H r J_0 \left( \sqrt{\sigma_n} r \right) \cos \left( \frac{m\pi z}{H} \right) \, dr \, dz \]
Problem 3

\[ \rho_t + (2 - \rho) \rho_x + \alpha \rho = 0, \quad -\infty < x < \infty, \quad t > 0 \]

\[ \rho(x, 0) = \frac{3B^2}{B^2 + x^2} \quad \rho(x, 0) \]

(i) Let \( \alpha = 0 : \)

\[ \rho_t + c(p) \rho_x = 0 \]

\[ c(p) \equiv 2 - \rho, \quad c'(p) \equiv -1 < 0 \]

\[ p(x, 0) = f(x) \]

on \[ \frac{dx}{dt} = c(p), \quad x(0) = \{ \]

so \[ x = c \int f(s) \, ds + \{ \]

\[ \frac{dp}{dt} = 0, \quad p(0) = \tilde{f}(x) \]

\[ p = F(s) \]

If \( \rho(x, 0) > 2 \to \text{M O V E L E F T} \)

If \( \rho(x, 0) < 2 \to \text{M O V E R I G H T} \)

Where \( u = \rho(x, 0) : 2 = \frac{3B^2}{B^2 + x^2} \)

so \[ 2 \beta^2 + 2 x^2 = 3B^2 \]

\[ x^2 = B^2/2 \]

Shock will form in the back.

(ii) Find breaking time:

\[ l = c' \int f(s) \, ds \int F(s) \, dx \]

\[ s_x = \frac{1}{1 + F(s) c'(f(s)) t} \]

Breaking time occurs at \( t = t_B \)

Where \( t = \frac{1}{b \max \lfloor f(s) \rfloor} \)

\[ s \to 0 \]

\[ t = -\frac{1}{f(s) c'(f(s))} \]

But \( c'(f) = -1. \)
\[
F(x) = \frac{3B^3}{B^2 + X^3}, \quad \frac{f(x)}{x} = -\frac{6B^3x}{(B^2 + X^2)^2}
\]
\[
f''(x) = -\frac{6B^3}{(B^2 + X^2)^2} + \frac{24B^3x^2}{(B^2 + X^2)^3} = -\frac{6B^3(B^2 + X^2)}{(B^2 + X^2)^3} + 24 \frac{B^2x^2}{(B^2 + X^2)^3}
\]

Set
\[
f'' = 0 \quad 18B^3x^2 = 6B^4 \quad \text{so} \quad x = -\frac{B}{\sqrt[3]{3}}
\]

Now
\[
F'\left(\frac{B}{\sqrt[3]{3}}\right) = \frac{6B^3 \left(\frac{B}{\sqrt[3]{3}}\right)}{B^4 \left(1 + \frac{1}{3}\right)^2} = \frac{1}{B} \left(\frac{6}{\sqrt[3]{3}}\right) = \frac{54}{B \cdot 16 \sqrt[3]{3}} = \frac{27}{8 \sqrt[3]{3}}
\]
\[
F'\left(\frac{B}{\sqrt[3]{3}}\right) = \frac{27}{24B} = \frac{9\sqrt[3]{3}}{8B}
\]

Thus,
\[
t_B = \frac{8B}{9\sqrt[3]{3}}
\]

(iii) Now we write
\[
dx/dt = C(p), \quad x(0) = \xi
\]
\[
dp/dt = -\alpha p, \quad p(0) = F(\xi)
\]

So
\[
p = f(\xi) e^{-\alpha t}
\]
\[
\frac{dx}{dt} = \left[ f(\xi) e^{-\alpha t} \right] = 2 - f(\xi) e^{-\alpha t}
\]
\[
x = 2t + \frac{1}{\alpha} \cdot f(\xi) \left( e^{-\alpha t} - 1 \right) + \xi.
\]

Now find where \( x \) is maximum, i.e. blow up,
\[
l = \left[ \frac{1}{\alpha} \cdot f'(\xi) \left( e^{-\alpha t} - 1 \right) + 1 \right] \xi.
\]

Thus the breaking time \( t_B \) if it exists is
\[
\frac{1}{\alpha} \cdot f'(\xi) \left( 1 - e^{-\alpha t} \right) = 1
\]
\[
1 - e^{-\alpha t} = \frac{\alpha}{f'(\xi)} \rightarrow \text{no breaking.}
\]
\[
\alpha > \frac{6B}{9\sqrt[3]{3}}.
\]
The number of cars on the road.

\[ P_t + (b(p)) \chi + \alpha p = 0. \]

Let \( N = \int_0^\infty \rho \, d\chi \) and

\[ N(0) = \int_0^\infty \frac{3B^2}{B^2 + \chi^2} \, d\chi \]

Let \( x = Bs \)

\[ = 3B \int_0^\infty \frac{1}{1 + s^2} \, ds = 3B \tan^{-1} s \bigg|_0^\infty = 3B \pi. \]

So \( N = (3B \pi) e^{-\alpha t}. \)
Problem 4

\[ U_t = \left( \frac{1}{r} U_r + \frac{1}{r^2} U_r + \frac{1}{r} U_\theta \right) \quad 0 \leq r < a, \quad 0 \leq \theta < 2\pi \]

\[ U_r \left( a, \theta, t \right) = F(\phi) \quad U \text{ bounded at } r \to 0 \]

\[ U_r, U_\theta \text{ periodic in } 2\pi \]

\[ U(r, \theta, 0) = g(r, \theta) . \]

(i) The steady state satisfies

\[ \Delta U = 0 \quad \text{in } 0 \leq r < a, \quad 0 \leq \theta < 2\pi \]

\[ U_r \left( a, \theta \right) = F(\phi) \quad \text{on } r = a, \quad U \text{ bounded at } r \to 0 \]

\[ U_r, U_\theta \text{ are } 2\pi \text{ periodic.} \]

Now no solution unless

\[ \int_0^{2\pi} F(\phi) \, d\phi = 0, \quad \text{since} \quad \int_0^{2\pi} \nabla \cdot \nabla U \, d\theta = \int_0^{2\pi} \frac{\partial U}{\partial r} \left|_{r=a} \right. \, d\theta \]

0 = \int_0^{2\pi} F(\phi) \, d\phi.

(ii) Assume

\[ \int_0^{2\pi} F(\phi) \, d\phi = 0. \]

Then

\[ U(r, \theta) = A_0 + \sum_{n=1}^{\infty} \left( \frac{r}{a} \right)^n \left[ A_n \cos(n\theta) + B_n \sin(n\theta) \right]. \]

We have

\[ U_r \big|_a = \lim_{\theta \to \frac{\pi}{2}} \frac{\partial U}{\partial \theta} = \sum_{n=1}^{\infty} \frac{a}{n} \left( A_n \cos(n\theta) + B_n \sin(n\theta) \right). \]

Thus

\[ F(\phi) = \frac{1}{a} \sum_{n=1}^{\infty} \left( \frac{n}{a} \right)^n \left[ A_n \cos(n\phi) + B_n \sin(n\phi) \right]. \]

The

\[ nA_n = \frac{1}{\pi} \int_0^{2\pi} F(\omega) \cos(n\omega) \, d\omega \quad nB_n = \frac{1}{\pi} \int_0^{2\pi} F(\omega) \sin(n\omega) \, d\omega. \]

This gives

\[ U(r, \theta) = A_0 + \sum_{n=1}^{\infty} \frac{a}{n \pi} \left( \frac{r}{a} \right)^n \left[ \int_0^{2\pi} F(\omega) \cos(n(\theta - \omega)) \, d\omega \right]. \]

Or

\[ U(r, \theta) = A_0 + \left( \frac{a}{\pi} \right)^n \left[ \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{r}{a} \right)^n \cos(n(\theta - \omega)) \right] d\omega. \]

Let

\[ S = \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{r}{a} \right)^n \cos(n(\theta - \omega)) = \text{RE} \left( \sum_{n=1}^{\infty} \frac{z^n}{n} \right) \]

with \[ z = r/a \exp(i(\theta - \omega)). \]

\[ S = -\text{RE} \left[ \log(1-z) \right]. \]
Thus \[ S = -\Re \left[ \log \left| 1 - z^2 \right| \right] = -\frac{1}{2} \log \left| 1 - z^2 \right|^2. \]

So \[ S = -\frac{1}{2} \int \left[ (1 - r/a \cos(q - \omega))^2 + \frac{r^3}{a^2} \sin^2(q - \omega) \right] \]

Thus \[ S = -\frac{1}{2} \int \left[ 1 - 2r/a \cos(q - \omega) + \frac{r^2}{a^2} \right]. \]

Thus \[ U(r, \varphi) = A_0 - \frac{a}{2\pi} \int_0^{2\pi} F(\psi) \int_0^a \frac{g(r, \varphi) \tau \, dr \, d\varphi}{\pi a^2} \] \text{ is the average of the initial data.}

(iii) Now suppose \[ \int_0^{2\pi} F(\psi) \, d\varphi \neq 0. \] Then no steady state solution.

We integrate the equation over the disk \[ \int_0^{2\pi} \frac{\partial U}{\partial t} \, d\varphi = \int_0^{2\pi} \nabla U \cdot \hat{n} \, ds. \]

So \[ \frac{d}{dt} \int_0^{2\pi} U \, d\varphi = \int_0^{2\pi} a \, f(\psi) \, d\varphi. \]

Now in a generalized eigenfunction expansion \[ U(r, \varphi, t) = A_0 \, |t| + \sum_{n=1}^\infty \left( \frac{r}{a} \right)^n \left[ A_n(t) \cos(n \varphi) + B_n(t) \sin(n \varphi) \right] \]

We obtain that \[ \frac{d}{dt} A_0 \frac{r}{a^2} = \int_0^{2\pi} a \, f(\psi) \, d\varphi. \]

Thus \[ A_0 \frac{r}{a} = \frac{1}{\pi a^2} \int_0^{2\pi} a \, f(\psi) \, d\varphi = \frac{1}{\pi a} \int_0^{2\pi} f(\psi) \, d\varphi \]

So \[ A_0 \approx \frac{1}{\pi a} \int_0^{2\pi} f(\psi) \, d\varphi \text{ for } t \text{ large.} \]
Problem 5

(i) \[ \Delta u = 0 \quad \text{in} \quad 0 \leq r \leq a, \quad 0 \leq \phi \leq 2\pi \]
\[ u = \frac{2}{r} \quad \text{on} \quad r = a. \]

We write
\[ u = \frac{2}{r} \left[ 1 + \cos(2\phi) \right] \quad \text{on} \quad r = a. \]

The solution has the form
\[ u = A_0 + A_2 \left( \frac{\alpha}{a} \right)^2 \cos(2\phi). \]

Now on \( r = a \),
\[ u = A_0 + A_2 \cos(2\phi) = 2 + \cos(2\phi) \quad \text{so} \quad A_0 = 2, \quad A_2 = 1 \]
\[ u = 2 + \left( \frac{\alpha}{a} \right)^2 \cos(2\phi). \]

(ii) Now \( \Delta u = 0 \) with \( 0 \leq r < a, \quad 0 \leq \phi < 2\pi \) in 3-D.
\[ u = \frac{2}{r} \quad \text{on} \quad r = a. \]

The solution is
\[ u = \sum_{n=0}^{\infty} \left( \frac{a}{r} \right)^{n+1} P_n \left( \cos(\phi) \right) A_n. \]

We have only mode \( n = 0 \) and \( n = 2 \) so
\[ u = A_0 \left( \frac{a}{r} \right) + \left( \frac{a}{r} \right)^3 P_2 \left( \cos(\phi) \right) A_2. \]

Now on \( r = a \),
\[ 1 + 2 \cos^2(\phi) = A_0 + A_2 P_2 \left( \cos(\phi) \right) \]

Let \( x = \cos(\phi) \).
\[ 1 + 2x^2 = A_0 + A_2 P_2 \left( x \right) \]
\[ 1 + 2x^2 = A_0 + A_2 \left( \frac{3}{2} x^2 - \frac{1}{2} \right) \]

so
\[ 2 = 3A_2/2 \quad \text{and} \quad 1 = A_0 - A_2/2 \]

\[ A_2 = \frac{4}{3} \quad \quad 1 = A_0 - 2/3 \quad A_0 = 5/3 \]

so
\[ u = \left( \frac{a}{r} \right)^{5/3} \left( \frac{a}{r} \right)^{3/3} P_2 \left( \cos(\phi) \right). \]

(iii) We have
\[ u_t = \frac{\partial}{\partial r} \left( u r_\gamma + \frac{2}{r} u_\gamma \right) \]

so \( u = R T \) give,
\[ \frac{R'}{R} = -\Lambda. \]
Thus \( R'' + \frac{2}{r} R' + \Delta R = 0 \)

\[ R(0) \text{ nice} \quad R(1) = 0. \]

Let \( R = \tilde{\Phi}/r \rightarrow \tilde{\Phi}'' + \lambda \tilde{\Phi} = 0 \quad \tilde{\Phi} = \sin \left( \sqrt{\lambda} \frac{r}{c} \right) \)

so \( \sqrt{\lambda} = \frac{n \pi}{c} \quad \text{or} \quad \lambda = \frac{n^2 \pi^2}{c^2} \).

Thus \( T = e^{-Dn^2 \pi^2 t} \)

we have \( U(r, t) = \sum_{n=1}^{\infty} A_n e^{-Dn^2 \pi^2 t} \sin \left( \frac{n \pi r}{c} \right) \)

Thus \( U(r, t) \sim A_1 e^{-Dn^2 \pi^2 t} \sin \left( \frac{n \pi r}{c} \right) \)

with \( A_1 = \frac{2}{\int_0^1 r^2 f(r) \sin \left( \frac{n \pi r}{c} \right) dr} \)

Thus \( A_1 = 2 \int_0^1 r f(r) \sin \left( \frac{n \pi r}{c} \right) dr \).

**Eigenfunctions** are \( \tilde{\Phi}_n = \sin \left( \frac{n \pi x}{L} \right) \).

Put \( U = \sum_{n=1}^{\infty} T_n(t) \sin \left( \frac{n \pi x}{L} \right) \)

so \( T''_n + \alpha T'_n = -\frac{c^2 n^2 \pi^2}{L^2} T_n + \sin(\omega t) \)

**Defining** \( W_n = C n \pi / L \)

\( T''_n + \alpha T'_n + W_n^2 T_n = \sin(\omega t) \)

If \( \alpha \approx 0 \) small then near resonance and a large response when \( W = W_0 = C n \pi / L \)

(V) \( U_{tt} = c^2 U_{xx} \quad \text{in} \quad \cot < x < \omega, \quad t > 0 \)

\( U(\cot, t) = \sin(\omega t) \)

\( U(x, 0) = U_t(x, 0) = 0 \)

Thus \( \tilde{\Phi}(z) = \sin \left( \frac{\omega z}{c_0 - c} \right) \rightarrow U = \sin \left( \frac{\omega (x-c) t}{c_0 c} \right) \).