Be sure that this examination has 3 pages.

The University of British Columbia

Final Examinations - December 2015

Mathematics 400

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Closed book examination | Time: $2\frac{1}{2}$ hours

Special Instructions: Closed Book and Notes. Calculators are not allowed.

Marks

[15] **1.** Consider the following first order PDE for $u = u(x, y)$:

$$u_x + 2xu_y = (3y + x)u\,.$$

(i) Find the general solution to this PDE.

(ii) Find the specific solution to this PDE that satisfies the data

$$u = f(y)\,, \quad \text{on } x = 0 \text{ for } -1 \le y \le 1\,.$$

In which region of the (x, y) plane is this solution defined?

[20] **2.** Consider axially symmetric diffusion for $u(r, z, t)$ in a finite cylinder of radius $a > 0$ and height $H > 0$ with insulating boundary conditions modeled by

$$u_t = u_{rr} + \frac{1}{r}u_r + u_{zz}\,, \quad 0 \le r \le a\,, \quad 0 \le z \le H\,, \quad t \ge 0\,,$$
$$u_z = 0 \quad \text{on } z = 0 \text{ and } z = H\,; \qquad u_r = 0 \quad \text{on } r = a\,, \quad u \text{ bounded as } r \to 0\,,$$
$$u(r, z, 0) = f(r, z)\,.$$

Determine an eigenfunction expansion representation for the time-dependent solution $u(r, z, t)$, and also calculate the steady-state solution.

Continued on page 2

[20] **3.** Assume that $\beta > 0$ and $\alpha \geq 0$ are constants, and consider the following traffic flow model for the density $\rho(x,t)$ of cars given by

$$\rho_t + (2-\rho)\,\rho_x + \alpha\rho = 0\,, \quad -\infty < x < \infty\,, \quad t > 0\,,$$
$$\rho(x,0) = \frac{3\beta^2}{\beta^2 + x^2}\,.$$

(i) **First let $\alpha = 0$.** Determine a parametric form for the solution $\rho(x,t)$. Plot qualitatively the characteristics in the (x,t) plane, and sketch the solution $\rho(x,t)$ versus x at different times. Determine the time t_b as a function of β when the solution first becomes multi-valued.

(ii) **Now let $\alpha > 0$.** Find a value α_c, which depends on β, such that the solution does not become multi-valued for any $t > 0$ if and only if $\alpha > \alpha_c$.

(iii) **Let $\alpha \geq 0$.** Calculate explicitly the total number $N(t)$ of cars on the road, defined by $N(t) = \int_{-\infty}^{\infty} \rho(x,t)\,dx$.

[20] **4.** Consider the diffusion problem for $u(r,\theta,t)$ in a disk of radius a with an inflow/outflow flux boundary condition modeled by

$$u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\,, \quad 0 \leq r \leq a\,, \quad 0 \leq \theta \leq 2\pi\,, \quad t \geq 0\,,$$
$$u_r(a,\theta,t) = f(\theta)\,, \quad u \text{ bounded as } r \to 0\,, \quad u \text{ and } u_\theta \text{ are } 2\pi \text{ periodic in } \theta\,,$$
$$u(r,\theta,0) = g(r,\theta)\,.$$

(i) Write the problem that the **steady-state solution** $U(r,\theta)$ would satisfy. Prove that such a steady-state solution $U(r,\theta)$ does not exist when $\int_0^{2\pi} f(\theta)\,d\theta \neq 0$.

(ii) Assume that $\int_0^{2\pi} f(\theta)\,d\theta = 0$. Calculate an integral representation for the **steady state solution** $U(r,\theta)$ by summing an appropriate eigenfunction expansion.

(ii) Assume that $\int_0^{2\pi} f(\theta)\,d\theta \neq 0$. Determine an approximation to the time-dependent solution $u(r,\theta,t)$ that is valid for large time t.

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[25] **5. Each of these five short-answer questions below is worth 5 points. Very little calculation is needed for any of these problems.**

(i) In the circular disk $0 < r < a$, $0 \le \theta \le 2\pi$ find the explicit solution to Laplace's equation for $u(r,\theta)$:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0\,, \quad 0 \le r \le a\,, \quad 0 \le \theta \le 2\pi\,,$$
$$u(a,\theta) = 2\cos^2(\theta) + 1\,, \quad u \text{ bounded as } r \to 0\,, \quad u \text{ and } u_\theta \ 2\pi \text{ periodic in } \theta\,.$$

(ii) Outside the sphere $r > a$, where θ is the polar angle with $0 \le \theta \le \pi$, find the explicit solution to Laplace's equation $\Delta u = 0$ with $u = 2\cos^2(\theta) + 1$ on $r = a$, $u \to 0$ as $r \to \infty$, and u is bounded at the poles $\theta = 0, \pi$. (Recall that the first three Legendre polynomials are $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = (3x^2 - 1)/2$.)

(iii) Consider the radially symmetric diffusion problem for $u(r,t)$ in the unit sphere $0 < r < 1$, modeled by

$$u_t = D\left(u_{rr} + \frac{2}{r}u_r\right)\,, \quad 0 \le r \le 1\,, \quad t > 0$$
$$u(1,t) = 0\,, \quad u \text{ bounded as } r \to 0\,; \qquad u(r,0) = f(r)\,,$$

where $D > 0$ is constant. Show that for long time, i.e. for $t \to +\infty$, that the solution can be approximated by $u(r,t) \approx Ae^{-D\pi^2 t}\sin(\pi r)/r$ for some $A > 0$ to be found.

(iv) Consider the damped wave-equation on a finite interval $0 < x < L$, and with time-periodic forcing modeled by

$$u_{tt} + au_t = c^2 u_{xx} + \sin(\omega t)\,, \quad 0 < x < L\,, \quad t > 0\,,$$
$$u(0,t) = 0\,, \quad u(L,t) = 0\,; \qquad u(x,0) = 0\,, \quad u_t(x,0) = 0\,.$$

Here $c > 0$, while $a > 0$ is small. For what frequencies $\omega > 0$ will we obtain a very large response for u when $a > 0$ is small? (Hint: these are the frequencies where resonance would occur if $a = 0$).

(v) Solve the signalling problem for the wave equation $u(x,t)$ where the signal is applied on a space-time curve as follows:

$$u_{tt} = c^2 u_{xx}\,, \quad c_0 t < x < \infty, \quad t > 0\,,$$
$$u(c_0 t, t) = \sin(\omega t)\,, \quad u(x,0) = u_t(x,0) = 0\,.$$

Here $0 < c_0 < c$.

[100] **Total Marks**

The End

MATH 400 FINAL 2015

PROBLEM 1 $\qquad u_x + 2x\,u_y = (3y + x)\,u$

(i) ON $\quad dy/dx = 2x \qquad du/dx = (3y+x)\,u$

SO $\quad y = x^2 + \lambda. \qquad$ WE PUT $\quad U(x', \lambda).$

$$u_x = U_{x'} + U_\lambda(-2x) \qquad u_y = U_\lambda$$

SO $\qquad U_{x'} + (-2x)\,U_\lambda + 2x\,U_\lambda = (3y+x)\,U$

OR $\qquad U_{x'} = [3(x'^2 + \lambda) + x']\,U$

$$U_{x'} = (3x'^2 + 3\lambda + x')\,U$$

OR $\qquad U = F(\lambda)\,e^{x'^3 + 3\lambda x' + x'^2/2}$

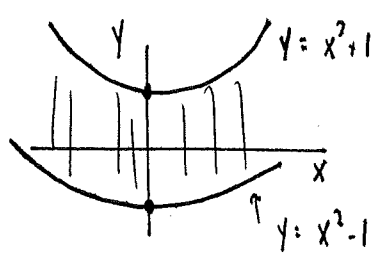
$$U = F(y - x^2)\,e^{x'^3 + 3(y - x'^2)x' + x'^2/2}$$

WE HAVE $\qquad U = F(y - x^2)\,e^{-2x^3 + 3yx + x^2/2}$

(ii) PUT $\quad u = f(y) \quad$ ON $-1 \le y \le 1 \quad$ FOR $\quad x = 0.$

$$f(y) = F(y)$$

SO $\qquad u(x,y) = f(y)\,e^{-2x^3 + 3yx + x^2/2}$



PROBLEM 2 $\quad u_t = u_{rr} + \frac{1}{r} u_r + u_{zz}, \quad 0 < r < a, \quad 0 < z < H, \quad t > 0$

$u_r = 0$ ON $r = a$, u BOUNDED AS $r \to 0$

$u_z = 0$ ON $z = 0, H$

$u(r, z, 0) = f(r, z).$

NOW WE SEPARATE VARIABLES TO OBTAIN

$$\frac{T'}{T} = \frac{R'' + 1/r\, R'}{R} + \frac{Z''}{Z} = -\lambda.$$

THEN $\quad Z'' + \mu Z = 0 \qquad Z_m(z) = \cos\left(\frac{m\pi z}{H}\right), \quad \mu_m = \frac{m^2\pi^2}{H^2}, \quad m = 0, 1, 2, \ldots$

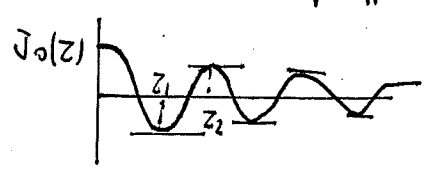
$Z'(0) = Z'(H) = 0$

THEN WE HAVE $\quad R'' + \frac{1}{r} R' + (\lambda - \mu) R = 0.$

$R'(a) = 0 \qquad R(0)$ NICE

let $\sigma = \lambda - \mu$. THEN $\quad R_n(r) = J_0(\sqrt{\sigma_n}\, r)$ $\quad$ ALSO NOTE $\sigma = 0$, $R_0 = 1$ IS EIGENPAIR

WITH $\quad J_0'(\sqrt{\sigma_n}\, a) = 0 \qquad \sqrt{\sigma_n}\, a = z_n$



SO $\quad \sigma_n = z_n^2 / a^2.$

THEN $\quad \lambda_{mn} = \sigma_n + \mu_m = \sigma_n + m^2\pi^2/H^2.$ $\quad$ DEFINE $\sigma_0 = 0$ SINCE $J_0(0) = 1$ will get FIRST eigenpair for $R(r)$.

THEN
$$u(r, z, t) = A_{00} + \sum_{\substack{m=0 \\ (m,n) \neq (0,0)}}^{\infty} \sum_{n=0}^{\infty} A_{mn} J_0(\sqrt{\sigma_n}\, r) \cos\left(\frac{m\pi z}{H}\right) e^{-\lambda_{mn} t}$$

NOW FOR $t > 0$
$$A_{00} = \frac{2\pi \int_0^a \int_0^H r f(r, z)\, dr\, dz}{\pi a^2 H}.$$

FOR $t >> 1$, $u(r, z, t) \approx A_{00}$.

THEN
$$A_{mn} = \frac{2 \int_0^a \int_0^H r J_0(\sqrt{\sigma_n}\, r) \cos(m\pi z/H)\, dr\, dz}{H \int_0^a r J_0^2(\sqrt{\sigma_n}\, r)\, dr}.$$

PROBLEM 3

$$\rho_t + (2-\rho)\rho_x + \alpha\rho = 0, \quad -\infty < x < \infty, \quad t > 0$$

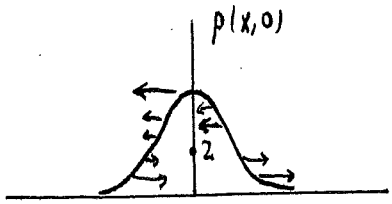
$$\rho(x,0) = \frac{3B^2}{B^2+x^2} := f(x)$$

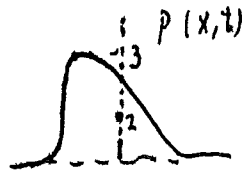
(i) LET $\alpha = 0$: $\quad \rho_t + c(\rho)\rho_x = 0 \qquad c(\rho) = 2-\rho. \quad c'(\rho) = -1 < 0$

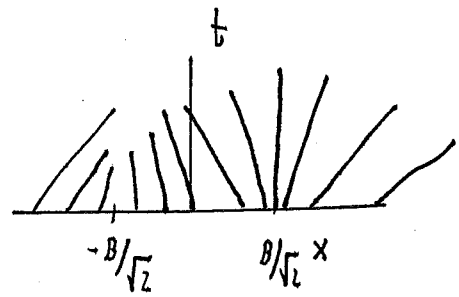
$$\rho(x,0) = f(x)$$

ON $\quad dx/dt = c(\rho), \quad x(0) = \xi \qquad$ SO $\quad x = c[f(\xi)]t + \xi$

$\qquad d\rho/dt = 0, \quad \rho(0) = f(\xi) \qquad \rho = f(\xi)$







IF $\rho(x,0) > 2 \rightarrow$ MOVES LEFT

IF $\rho(x,0) < 2 \rightarrow$ MOVES RIGHT

WHERE IS $\rho(x,0) = 2 = \dfrac{3B^2}{B^2+x^2}$ SO $2B^2 + 2x^2 = 3B^2 \qquad x^2 = B^2/2$

SHOCK WILL FORM IN THE BACK.

(ii) FIND BREAKING TIME:

$$1 = c'[f(\xi)]\, f'(\xi)\, \xi_x\, t + \xi_x$$

SO $\quad \xi_x = \dfrac{1}{1 + f'(\xi)\, c'(f(\xi))\, t} \qquad t = -1/f'(\xi)\, c'[f(\xi)]$

BUT $c'(f) = -1.$

BREAKING TIME OCCURS AT $t = t_B$

WHERE $t_b = \dfrac{1}{\max\limits_{\xi<0} |f'(\xi)|}.$

NOW $\quad f(x) = \dfrac{3B^2}{B^2+x^2}, \qquad f'(x) = \dfrac{-6B^2x}{(B^2+x^2)^2}$

$$f''(x) = -\frac{6B^2}{(B^2+x^2)^2} + \frac{24B^2x^2}{(B^2+x^2)^3} = -\frac{6B^2(B^2+x^2)+24B^2x^2}{(B^2+x^2)^3}$$

SET $\quad f''=0 \qquad 18B^2x^2 = +6B^4 \qquad$ so $\quad x = -B/\sqrt{3}$

NOW $\quad f'(B/\sqrt{3}) = +\dfrac{6B^2(B/\sqrt{3})}{B^4(1+1/3)^2} = +\dfrac{1}{B}\dfrac{(6/\sqrt{3})}{(16/9)} = \dfrac{54}{B\,16\sqrt{3}} = \dfrac{27}{B\,8\sqrt{3}}$

$$f'(B/\sqrt{3}) = \frac{27\sqrt{3}}{24B} = \frac{9\sqrt{3}}{8B}$$

THUS $\quad t_B = \dfrac{8B}{9\sqrt{3}}.$

(iii) NOW WE WRITE

ON $\quad dx/dt = c(p), \qquad x(0) = \xi$

$\quad dp/dt = -\alpha p, \qquad p(0) = f(\xi)$

SO $\quad p = f(\xi)e^{-\alpha t}$

$$\frac{dx}{dt} = c[f(\xi)e^{-\alpha t}] = 2 - f(\xi)e^{-\alpha t}$$

OR $\quad x = 2t + \dfrac{1}{\alpha}f(\xi)(e^{-\alpha t}-1) + \xi.$

NOW FIND WHERE ξ_x IS MAXIMUM, I.E. BLOW UP.

$$1 = \left[\frac{1}{\alpha}f'(\xi)(e^{-\alpha t}-1)+1\right]\xi_x.$$

THUS THE BREAKING TIME t_B IF IT EXISTS IS

$$\frac{1}{\alpha}f'(\xi)(1-e^{-\alpha t}) = 1$$

$$\underbrace{1-e^{-\alpha t}}_{\text{ON } (0,1)} = \alpha/f'(\xi)$$

THUS IF $\alpha > \max_{\xi\le 0} f'(\xi)$

$\rightarrow$ NO BREAKING.

$\alpha > 8B/9\sqrt{3}.$

(iii) THE NUMBER OF CARS ON THE ROAD.

$$\rho_t + (q(\rho))_x + \alpha \rho = 0. \qquad q(\rho) = (2\rho - \rho^2/2) \text{ . LET } N = \int_{-\infty}^{\infty} \rho \, dx$$

SO $$N'(t) + \alpha N(t) = 0$$

$$N(0) = \int_{-\infty}^{\infty} \frac{3B^2}{B^2 + x^2} \, dx \qquad \text{let } x = BS$$

$$= 3B \int_{-\infty}^{\infty} \frac{1}{1+S^2} \, dS = 3B \tan^{-1} S \Big|_{-\infty}^{\infty} = 3B\pi.$$

SO $$N = (3B\pi) e^{-\alpha t}.$$

PROBLEM 4

$$u_t = \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi}\right) \qquad 0 \le r \le a, \quad 0 \le \varphi \le 2\pi$$

$$u_r(a, \varphi, t) = f(\varphi) \qquad u \text{ BOUNDED AS } r \to 0$$

$$u, u_\varphi \quad 2\pi \text{ periodic}$$

$$u(r, \varphi, 0) = g(r, \varphi).$$

(i) THE STEADY-STATE SATISFIES

$$\Delta U = 0 \quad \text{IN} \quad 0 \le r \le a, \quad 0 \le \varphi \le 2\pi$$

$$U_r = f(\varphi) \quad \text{ON} \quad r = a, \quad U \text{ BOUNDED AS } r \to 0$$

$$U, U_\varphi \text{ ARE } 2\pi \text{ periodic.}$$

NOW NO SOLUTION UNLESS $\int_0^{2\pi} f(\varphi)\, d\varphi = 0$, SINCE $\int_\Omega \nabla \cdot (\nabla U)\, d\underline{x} = \int_{\partial\Omega} \underbrace{\nabla U \cdot \hat{n}}_{\partial U/\partial r|_{r=a}}\, dS$

$$0 = a \int_0^{2\pi} f(\varphi)\, d\varphi$$

(ii) ASSUME $\int_0^{2\pi} f(\varphi)\, d\varphi = 0.$

THEN $$U(r, \varphi) = A_0 + \sum_{n=1}^{\infty} (r/a)^n \left[A_n \cos(n\varphi) + B_n \sin(n\varphi) \right].$$

WE HAVE $$U_r\big|_a = f(\varphi) = \sum_{n=1}^{\infty} \frac{n r^{n-1}}{a^n} \left[A_n \cos(n\varphi) + B_n \sin n\varphi \right]$$

SO $$f(\varphi) = \frac{1}{a} \sum_{n=1}^{\infty} \left(n A_n \cos(n\varphi) + n B_n \sin(n\varphi) \right)$$

THUS $$\frac{n A_n}{a} = \frac{1}{\pi} \int_0^{2\pi} f(w) \cos(nw)\, dw \qquad \frac{n B_n}{a} = \int_0^{2\pi} f(w) \sin(nw)\, dw.$$

THIS GIVES $$U(r, \varphi) = A_0 + \sum_{n=1}^{\infty} \frac{a}{\pi n} \left(\frac{r}{a}\right)^n \left[\int_0^{2\pi} f(w) \left[\cos(nw)\cos(n\varphi) + \sin(nw)\sin n\varphi \right] dw \right.$$

OR $$U(r, \varphi) = A_0 + \frac{a}{\pi} \int_0^{2\pi} f(w) \left(\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{a}\right)^n \cos(n(\varphi - w)) \right) dw.$$

LET $$S = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{a}\right)^n \cos(n(\varphi - w)) = \mathrm{RE} \left(\sum_{n=1}^{\infty} z^n/n \right)$$

WITH $z = r/a\, e^{i(\varphi - w)}$. $\quad S = -\mathrm{RE}\left[\log(1 - z) \right].$

THUS $\quad S = -\,\mathrm{RE}\left[\log|1-z|\right] = -\frac{1}{2}\ln|1-z|^2.$

SO $\quad S = -\frac{1}{2}\ln\left[\left(1 - r/a\cos(\varphi-\omega)\right)^2 + r^2/a^2\sin^2(\varphi-\omega)\right]$

$$S = -\frac{1}{2}\ln\left[1 - 2r/a\cos(\varphi-\omega) + r^2/a^2\right].$$

THUS $\quad u(r,\varphi) = A_0 - \frac{a}{2\pi}\int_0^{2\pi} f(\varphi)\ln\left[1 - \frac{2r}{a}\cos(\varphi-\omega) + r^2/a^2\right]d\omega.$

NOW WE HAVE $\quad A_0 = \dfrac{\int_0^{2\pi}\int_0^a g(r,\varphi)\,r\,dr\,d\varphi}{\pi a^2}$ IS THE AVERAGE OF THE INITIAL DATA.

(iii) NOW SUPPOSE $\int_0^{2\pi} f(\varphi)\,d\varphi \neq 0$. THEN NO steady state solution.

WE INTEGRATE THE EQUATION OVER THE DISK

$$\int_\Omega u_t\,d\underline{x} = \int_{\partial\Omega}\nabla u\cdot\hat{n}\,ds.$$

SO $\quad \frac{d}{dt}\int_\Omega u\,d\underline{x} = \int_0^{2\pi} a\,f(\varphi)\,d\varphi.$

NOW IN A GENERALIZED EIGENFUNCTION EXPANSION

$$u(r,\varphi,t) = A_0(t) + \sum_{n=1}^{\infty}(r/a)^n\left[A_n(t)\cos n\varphi + B_n(t)\sin n\varphi\right]$$

WE OBTAIN THAT $\quad d/dt\; A_0\pi a^2 = \int_0^{2\pi} a\,f(\varphi)\,d\varphi.$

THUS $\quad A_0' = \frac{1}{\pi a^2}\int_0^{2\pi} a\,f(\varphi)\,d\varphi = \frac{1}{\pi a}\int_0^{2\pi} f(\varphi)\,d\varphi$

SO $\quad A_0 \simeq t/\pi a\int_0^{2\pi} f(\varphi)\,d\varphi$ for t large.

PROBLEM 5

(i) $\Delta u = 0$ IN $0 \le r \le a$, $0 \le \varphi \le 2\pi$

$u = 2\cos^2\varphi + 1$ ON $r = a$.

WE WRITE $u = \frac{2}{2}[1 + \cos 2\varphi] + 1 = 2 + \cos(2\varphi)$ ON $r = a$.

THE SOLUTION HAS THE FORM $u = A_0 + A_2 (r/a)^2 \cos(2\varphi)$

NOW ON $r = a$ $u = A_0 + A_2 \cos(2\varphi) = 2 + \cos(2\varphi)$ SO $A_0 = 2$, $A_2 = 1$

$$u = 2 + (r/a)^2 \cos(2\varphi).$$

(ii) NOW $\Delta u = 0$ WITH $a \le r < \infty$, $0 < \varphi < \pi$ IN 3-D.

$u = 1 + 2\cos^2(\varphi)$ ON $r = a$.

THE SOLUTION IS $u = \sum_{n=0}^{\infty} (a/r)^{n+1} P_n(\cos\varphi) A_n$

WE HAVE ONLY MODE $n = 0$ AND $n = 2$ SO

$$u = A_0 (a/r) + (a/r)^3 P_2(\cos\varphi) A_2. \quad (*)$$

NOW ON $r = a$: $1 + 2\cos^2(\varphi) = A_0 + A_2 P_2(\cos\varphi)$

LET $x = \cos\varphi$. $1 + 2x^2 = A_0 + A_2 P_2(x)$

$$1 + 2x^2 = A_0 + A_2\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$$

SO $2 = 3A_2/2$ $\quad 1 = A_0 - A_2/2$

$A_2 = 4/3$ $\quad 1 = A_0 - 2/3$ $\quad A_0 = 5/3$

SO $$u = \frac{5}{3}\left(\frac{a}{r}\right) + \left(\frac{a}{r}\right)^3 \frac{4}{3} P_2(\cos\varphi).$$

(iii) WE HAVE $u_t = D(u_{rr} + 2/r\, u_r)$

SO $u = RT$ gives $$\frac{T'}{DT} = \frac{R'' + 2/r\, R'}{R} = -\lambda.$$

THUS $\quad R'' + \frac{2}{r} R' + \lambda R = 0$

$R(0)$ NICE $\quad R(1) = 0.$

LET $\quad R = \Phi/r \quad \rightarrow \quad \Phi'' + \lambda \Phi = 0 \qquad \Phi = \sin(\sqrt{\lambda}\, r)$

SO $\quad \sqrt{\lambda_n} = n\pi \quad$ OR $\quad \lambda_n = n^2\pi^2.$

THUS $\quad T = e^{-Dn^2\pi^2 t}$

WE HAVE $\quad u(r,t) = \sum_{n=1}^{\infty} A_n e^{-Dn^2\pi^2 t} \sin\left(\frac{n\pi r}{r}\right)$

THUS $\quad u(r,t) \sim A_1 e^{-D\pi^2 t} \sin(\pi r)/r$

WITH $\quad A_1 = \dfrac{\int_0^1 r^2 f(r) \sin^2(\pi r)/r \, dr}{\int_0^1 \sin^2(\pi r)\, dr}$

THUS $\quad A_1 = 2\int_0^1 r f(r) \sin(\pi r)\, dr.$

(iv) EIGENFUNCTIONS ARE $\quad \Phi_n = \sin(n\pi x/L).$

PUT $\quad u = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi x}{L}\right)$

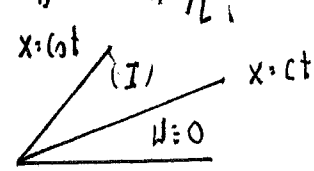
SO $\quad T_n'' + a T_n' = -c^2 \frac{n^2\pi^2}{L^2} T_n + \sin(\omega t)$

DEFINE $\quad \omega_n = c\, n\pi/L$

$$T_n'' + a T_n' + \omega_n^2 T_n = \sin(\omega t)$$

If $a > 0$ SMALL then <u>NEAR</u> RESONANCE AND A LARGE RESPONSE WHEN $\omega = \omega_n = c\, n\pi/L$.

(v) $\quad u_{tt} = c^2 u_{xx}$ IN $c_0 t < x < \infty, \; t > 0$



$u(c_0 t, t) = \sin(\omega t)$

$u(x,0) = u_t(x,0) = 0$

IN (I) $\quad u = f(x - ct)$, SO $f((c_0 - c)t) = \sin(\omega t)$

THUS $\quad f(z) = \sin\left(\frac{\omega z}{c_0 - c}\right) \rightarrow u = \sin\left(\frac{\omega(x - ct)}{c_0 - c}\right).$