CONSIDER A FAMILY OF CURVES

\[ G(x, t, s) = 0 \]

THE ENVELOPE OF THE FAMILY OF CURVES (IF IT EXISTS) MUST CONTAIN
POINT ON THE FAMILY OF CURVES, AND THE FAMILY OF CURVES MUST
INTERSECT TANGENTIALLY WITH THE ENVELOPE:

\[ \text{curves of different} \]
\[ \text{members of} \]
\[ \text{family of curves} \]

SO ON THE POINT \( P_0, s = s_0 \) AND AT THE POINT \( P_1, s = s_1 \).

AT THE POINT OF CONTACT WITH THE FAMILY OF CURVES \( G(x, t, s) = 0 \)
WITH \( s = s(x, t) \).

THUS

\[ G(x, t, s(x, t)) = 0 \]

NOW LET \((x, t)\) AND \((x + dx, t + dt)\) BE TWO NEIGHBORING POINTS
ON THE ENVELOPE CORRESPONDING TO \( s \) AND \( s + ds \) RESPECTIVELY.

THUS

\[ ds = G_x \, dx + G_t \, dt + G_s \, ds = 0 \]  \( \text{(since} \ G(x, t, s(x, t)) = 0) \)

HOWEVER

\[ G_x \, dx + G_t \, dt = \left( G_x, G_t \right) \cdot (dx, dt) = 0 \]  \( \text{by the fact that} \)

THE FAMILY OF CURVES INTERSECT TANGENTIALLY. THIS IMPLIES \( G_s = 0 \).

THUS THE ENVELOPE OF A FAMILY OF CURVES (IF IT EXISTS)
IS THE SOLUTION OF

\[ G(x, t, s) = 0, \quad G_s(x, t, s) = 0. \]
Example

A concentrated sound pulse emitted at \( t = 0 \) in a quiescent medium propagates through space as an ever expanding sphere whose radius is \( ct \) at any time \( t \). Assume that the source is moving along \( x \)-axis at a speed \( U > 0 \). Then the family of wavefronts satisfy

\[
(x + Ut)^2 + y^2 + z^2 = c^2 t^2
\]

Remark

(i) If we consider the 2-D version of this and let \( t = 5 \), we can write \( G(x, t, 5) = 0 \) \( G(x, t, 5) = (x + 5t)^2 + y^2 - c^2 \).

This is precisely of the form of the theory.

We write

\[
(x + Ut)^2 + y^2 + z^2 - c^2 t^2 = G(x, y, z, t) = 0.
\]

The equation of the envelope if it exists is to solve \( G = 0 \) together with \( G_t = 0 \).

\[
G = 0 \quad \Rightarrow \quad (x + Ut)^2 + y^2 + z^2 = c^2 t^2
\]

\[
G_t = 0 \quad \Rightarrow \quad 2(x + Ut)U - 2c^2 t = 0 \quad \Rightarrow \quad U(x + Ut) = c^2 t.
\]

Now we combine to get

\[
x^2 + 2Ut x + U^2 t^2 + y^2 + z^2 = U(x + Ut)^2
\]

This gives:

\[
x^2 + UXt + y^2 + z^2 = 0
\]

We then solve for \( t \) in \( U(x + Ut) = c^2 t \) to get

\[
t = \frac{UX}{(U^2 - C^2)}
\]

This gives

\[
x^2 + UX \left( \frac{UX}{U^2 - C^2} \right) + y^2 + z^2 = 0.
\]

We solve to get

\[
y^2 + z^2 = \frac{X^2}{M^2 - 1} \quad M = \frac{U}{C} \quad \text{MACH NUMBER}
\]

\( M < 1 \) SUBSONIC

\( M > 1 \) SUPERSONIC
If $M > 1$ then the envelope of the family of curves is simply a cone (called a Mach cone) with semi-angle $\theta = \sin^{-1}(1/M)$. The picture in the 2-D case is:

\[ y^2 = \frac{x^2}{M^2 - 1} \quad (M > 1) \]

\[ (x + ut)^2 + y^2 = c^2 t^2 \]

Notice that when $M < 1$ subsonic, then $y^2 + z^2 = \frac{x^2}{M^2 - 1} < 0$.

Hence there is no envelope in the subsonic case.