**Problem 1**

Find a formula analogous to Poisson's integral formula for the solution $U(r, \phi)$ to

$$
\Delta U = 0 \quad \text{in} \quad 0 \leq r < R, \quad 0 \leq \phi < 2\pi
$$

with $U(0, \phi) = F(\phi)$

$U$, $U_r$ bounded as $r \to 0$

$U$, $U_\phi$ are $2\pi$ periodic in $\phi$.

Show in two different ways that we need $\int_0^{2\pi} F(\phi) \, d\phi = 0$

for a solution to exist. Is the solution unique?

**Problem 2**

Consider Laplace's equation in an infinite strip satisfying

$$
U_{xx} + U_{yy} = 0 \quad \text{in} \quad 0 < x < \infty, \quad 0 < y < 1
$$

with $U(x, 0) = U(x, 1) = 0$, $U(0, y) = F(y)$

$U$ bounded as $x \to \infty$.

By summing an appropriate eigenfunction expansion

Show that the solution is

$$
U(x, y) = \int_0^1 F(s) K(x, y, s) \, ds
$$

where $K(x, y, s)$ is to be found.

**Hint:** You will need the identity $\sin A \sin B = \frac{1}{2} \left[ \cos(A - B) - \cos(A + B) \right]$ when dealing with the eigenfunction expansion.
Problem 3: Diffusion on a Sphere of Radius R with Azimuthal Symmetry

Consider the diffusion equation for $u(\phi, t)$ on a sphere of radius $R$ with initial temperature $u(\phi, 0) = f(\phi)$, modeled by

$$u_t = \frac{D}{R^2} \frac{1}{\sin \phi} \frac{d}{d\phi} (\sin \phi \cdot u_\phi)$$

where $0 < \phi < \pi$, $t > 0$.

And $D > 0$ is the diffusivity. The initial condition is

$$u(\phi, 0) = f(\phi)$$

$q$ is polar angle.

And $u, u_\phi$ bounded as $\phi \to 0, \pi$ (the two poles).

1. Find an infinite series separation of variables solution for $u(\phi, t)$.

2. Find an explicit solution if $f(\phi) = 2 \cos^2(\phi)$. For this choice calculate $\lim_{t \to \infty} u(\phi, t)$.

Problem 4: By starting from the generating function for Legendre polynomials, complete the derivation in the notes to show that $P_n(x)$ satisfies

(i) $(n+1) P_{n+1}(x) = x (2n+1) P_n(x) - n P_{n-1}(x)$ \hspace{1cm} (n \geq 1 \text{ integer})

(ii) $\int_{-1}^{1} \left[ P_n(x) \right]^2 \, dx = \frac{2}{2n+1}$, $n = 0, 1, 2, \ldots$

(iii) Use (i) above together with $P_0(x) = 1$, $P_1(x) = x$ to calculate $P_2(x)$, $P_3(x)$ and $P_4(x)$. 
**Problem 5 (Charged Balloon Problem)**

Assuming azimuthal symmetry the potential \( U \) is assumed to satisfy, inside and outside a sphere of radius \( R \), the PDE:

\[
\Delta U = \frac{1}{r} \frac{d}{dr} \left( r \frac{dU}{dr} \right) + \frac{1}{r^2 \sin \varphi} \frac{d}{d\varphi} \left( \sin \varphi \frac{dU}{d\varphi} \right) = 0 \quad \text{in} \quad 0 < \varphi < \pi, \quad 0 < r < R
\]

with \( U = \cos(3\varphi) \) on \( r = R \) (surface of balloon)

\( U \) bounded as \( r \to 0 \) and as \( r \to \infty \)

\( U \) bounded as \( \varphi \to 0 \) and as \( \varphi \to \pi \) (poles).

Impose that \( U \) is continuous across \( r = R \).

(i) Calculate \( U(r, \varphi) \) inside and outside the balloon.

(ii) From your solution in (i) calculate the surface charge density

\[
\sigma(\varphi) = \frac{1}{2\pi} \left( \left. \frac{dU}{dr} \right|_{r=R^+} - \left. \frac{dU}{dr} \right|_{r=R^-} \right)
\]

Remark: This problem models the electromagnetic potential of a thin balloon that has some potential applied to its surface.
PROBLEM 1

SEPARATION OF VARIABLES LEADS TO

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^\infty R^n \left( A_n \cos n\theta + B_n \sin n\theta \right)$$

NOW

$$u(r, \theta) = \frac{1}{\pi} \int_0^{2\pi} f(\phi) \left( \cos n\phi + \sin n\phi \right) d\phi$$

INTEGRATING BOTH SIDES, IT IS CLEAR THAT WE NEED

$$\int_0^{2\pi} f(\phi) d\phi = 0$$

NOW IF THIS IS SATISFIED, (X) IS CONSISTENT AND

$$n R^{n-1} A_n = \frac{1}{\pi} \int_0^{2\pi} f(\phi) \cos n\phi d\phi \quad n = 1, 2, \ldots$$

$$n R^{n-1} B_n = \frac{1}{\pi} \int_0^{2\pi} f(\phi) \sin n\phi d\phi \quad n = 1, 2, \ldots$$

THEREFORE,

$$u(r, \theta) = \frac{A_0}{2} + \frac{1}{\pi} \int_0^{2\pi} f(\phi) \sum_{n=1}^\infty \frac{R^n}{n R^{n-1}} \left( \cos n\phi \cos (n\theta - \phi) + \sin n\phi \sin (n\theta - \phi) \right) d\phi$$

NOW

$$u(r, \theta) = \frac{A_0}{2} + \frac{R}{\pi} \int_0^{2\pi} f(\phi) \left( \sum_{n=1}^\infty \frac{1}{n} \left( \frac{r}{R} \right)^n \cos (n\phi - \theta) \right) d\phi$$

LET

$$z = \frac{r}{R} e^{i(\phi - \theta)}$$

THEN

$$\sum_{n=1}^\infty \frac{1}{n} z^n = -\log(1 - z)$$

$$u(r, \theta) = \frac{A_0}{2} + \frac{R}{2\pi} \int_0^{2\pi} f(\phi) \left( -\log(1 - z) \right) d\phi$$

SO

$$u(r, \theta) = \frac{A_0}{2} + \frac{R}{2\pi} \int_0^{2\pi} f(\phi) \log \left[ 1 - 2 \frac{r}{R} \cos(\phi - \theta) + \left( \frac{r}{R} \right)^2 \right] d\phi$$

REMARK

(i) THE CONSTANT $A_0$ IS ARBITRARY. IF $f(\phi) = 0$ AND WE HAD A TIME-DEPENDENT HEAT EQUATION THEN $A_0$ WOULD BE THE STEADY-STATE = THE AVERAGE OF THE INITIAL CONDITION. WITHOUT SPECIFYING THE TIME-DEPENDENT PROBLEM WE HAVE THAT $A_0$ IS ARBITRARY.

(ii) TO SEE THAT $\int_0^{2\pi} f(\phi) d\phi = 0$ IS NEEDED IN A DIFFERENT WAY WE USE THE DIVERGENCE THEOREM

$$\int_D \nabla \cdot \nabla u \, dV = \int_{\partial D} \nabla u \cdot \hat{n} \, dS \rightarrow 0 = \int_0^{2\pi} du/d\phi \bigg|_{r=R} R \, d\phi \rightarrow \int_0^{2\pi} f(\phi) d\phi = 0$$

NO NET FLOW OF HEAT ACROSS SURFACE ALLOWED FOR A STEADY-
PROBLEM 2

\[ u_{xx} + u_{yy} = 0, \quad 0 \leq x \leq \infty, \quad 0 \leq y < 1 \]

\[ u(x, 0) = u(x, 1) = 0, \quad u(0, y) = f(y) \]

\[ u \text{ bounded as } x \to \infty \]

\[ y \]

\[ u = 0. \]

SOLUTION in the y-direction there are eigenfunctions. Let \( u = X \varphi \). Then,

\[ \frac{X''}{X} = -\frac{\varphi''}{\varphi} = \lambda. \]

\[ \begin{cases} \varphi'' + \lambda \varphi = 0 \\ \varphi(0) = \varphi(1) = 0 \end{cases} \]

\[ \Rightarrow \varphi_n(y) = \sin(n \pi y) \]

\[ \lambda_n = n^2 \pi^2, \quad n = 1, 2, ... \]

Now \( X'' - \lambda X = 0 \) so \( X = e^{\sqrt{\lambda} x} \) \( u \) bounded as \( x \to \infty \). Thus,

\[ u(x, y) = \sum_{n=1}^{\infty} c_n e^{-\sqrt{n \pi} x} \sin(n \pi y) \]

Now \( u(0, y) = f(y) = \sum_{n=1}^{\infty} c_n \sin(n \pi y) \) so \( c_n = \frac{1}{\int_0^1 f(s) \sin(n \pi s) \, ds}{\int_0^1 (\sin(n \pi s))^2 \, ds} \)

Thus \( c_n = 2 \int_0^1 f(s) \sin(n \pi s) \, ds \). Thus given,

\[ u(x, y) = 2 \int_0^1 f(s) \left( \sum_{n=1}^{\infty} \sin(n \pi s) \sin(n \pi y) e^{-\sqrt{n \pi} x} \right) \, ds \]

NOW use \( \sin A \sin B = \frac{1}{2} \left[ \cos(A - B) - \cos(A + B) \right] \). Then define

\[ J(k) = \sum_{n=1}^{\infty} \cos(n \pi k) e^{-\sqrt{n \pi} x} \]

From (*) we have

\[ u(x, y) = \int_0^1 f(s) W(x, y, s) \, ds \]

where \( W(x, y, s) \equiv J(n \pi (s - y)) \]

so we need only calculate \( J(k) \):

\[ J(k) = RF \left( \sum_{n=1}^{\infty} e^{-\sqrt{n \pi} (x - i y)} \right) \]

Let \( z = e^{-\pi (x - i y)} \)

Then \( J(k) = RF \left( \sum_{n=1}^{\infty} z^n \right) \) but \( 1 + z + z^2 + \ldots = \frac{1}{1-z} \) if \( |z| < 1 \).
\[ Z + Z^2 + \cdots = \frac{1}{1 - Z} - 1 = \frac{Z}{1 - Z} \quad \text{if} \quad |Z| < 1 \quad \text{which implies} \quad X > 0. \]

Thus, \[ J (x) = \Re \left( \frac{Z}{1 - Z} \right) = \Re \left( \frac{Z(1 - \bar{Z})}{1 - Z \bar{Z}} \right) = \frac{1}{1 - Z \bar{Z}} \left( \Re (Z) - |Z|^2 \right) \]

Now \[ |1 - Z|^2 = (1 - e^{-\pi x} \cos (\bar{\pi} x))^2 + (e^{-\pi x} \sin (\bar{\pi} x))^2 = 1 + e^{-2\pi x} - 2 e^{-\pi x} \cos (\bar{\pi} x). \]

So \[ |Z|^2 = e^{-2\pi x} \quad \text{and} \quad \Re Z = e^{-\pi x} \cos (\bar{\pi} x). \]

Thus \[ J (x) = \frac{1}{1 + e^{-2\pi x} - 2 e^{-\pi x} \cos (\bar{\pi} x)} \left( e^{-\pi x} \cos (\bar{\pi} x) - e^{-2\pi x} \right). \]

Multiply top and bottom by \( e^{\pi x} \) and we have \( 2 \cos (\bar{\pi} x) = e^{\pi x} + e^{-\pi x}. \)

Thus \[ J (x) = \cos (\bar{\pi} x) - e^{-\pi x} \]

\[ 2 \cos (\bar{\pi} x) - 2 \cos (\bar{\pi} x). \]

Thus \[ J (x) = \cos (\bar{\pi} x) - e^{-\pi x} \]

\[ 2 \left[ \cos (\bar{\pi} x) - \cos (\bar{\pi} x) \right] \]

And \[ U(x, y) = \int_0^1 \frac{1}{1 - s} \, K(x, y, s) \, ds \]

with \[ K = J \left[ \bar{n} (s - y) \right] - J \left[ \bar{n} (1 + y) \right]. \]
**Solution 3**

We separate variables \( u(\varphi, t) = T(t) \Phi(\varphi) \) in the PDE to get

\[
\frac{R^2 T'}{D T} = \frac{1}{\sin \varphi} \left( \sin \varphi \Phi' \right)' = -\Lambda, \quad \Phi(0) = \Phi(\bar{\varphi}) = 0, \quad 0 \leq \varphi \leq \bar{\varphi},
\]

Thus,

\[
\left( \sin \varphi \Phi' \right)' + \Lambda \sin \varphi \Phi = 0,
\]

Therefore, \( \Phi, \Phi' \) bounded at the pole \( \varphi = 0, \bar{\varphi} \).

If we put \( x = \cos \varphi \) and \( \varphi(x) = \Phi(\varphi) \) then we get

**Legendre's Equation**

\[
\left[ (1-x^2) v' \right]' + \lambda v = 0 \text{ in } -1 < x < 1.
\]

The bounded solution to this problem occurs for \( \lambda = n(n+1) \), \( n = 0, 1, 2, \ldots \) and \( V_n(x) = P_n(x) \).

Are the **Legendre Polynomials**. Thus,

\[
T_n = -\frac{D}{R^2} n(n+1) T_0 \rightarrow T_n = e^{-\frac{Dn(n+1)}{R^2} t}
\]

(1) By separation of variables,

\[
u(\varphi, t) = \sum_{n=0}^{\infty} a_n P_n(\cos \varphi) e^{-\frac{Dn(n+1)}{R^2} t}
\]

where \( F(\varphi) = \sum_{n=0}^{\infty} a_n P_n(\cos \varphi) \)

Thus,

\[
a_n = \frac{\int_0^\bar{\varphi} F(\varphi) P_n(\cos \varphi) \sin \varphi \, d\varphi}{\int_0^\bar{\varphi} \left[ P_n(\cos \varphi) \right]^2 \sin \varphi \, d\varphi}
\]
To calculate the denominator let \( x = \cos \phi \) \( dx = -\sin \phi \frac{d\phi}{2} \)

And \( \phi \to 0 \) mean \( x \to 1 \) to obtain

\[
\int_0^\infty \left[ \sum_{n=0}^\infty (\cos \phi)^n \right]^2 \sin \phi \frac{d\phi}{2} = -\int_1^1 \left( \frac{P_n(x)}{2} \right)^2 dx = \int_1^1 \left( \frac{P_0(1)}{2} \right)^2 dx = \frac{2}{2n+1} \]

From problem 4(ii).

Thus
\[
a_n = \frac{2n+1}{2} \int_0^\pi f(\theta) P_n(\cos \phi) \sin \phi \frac{d\phi}{2} \]

(iii) Now we find \( a_0 \) in a simple way when
\[
f(\phi) = 2 \cos^2 \phi.\]

We must find \( a_0 \) satisfying

\[
2 \cos^2 \phi = \sum_{n=0}^\infty a_n P_n(\cos \phi) \]

We know \( P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \) etc...

Put \( x = \cos \phi \) so that
\[
2x = \sum_{n=0}^\infty a_n P_n(x). \quad (x)
\]

Notice
\[
2P_2(x) = 3x^2 - 1 \quad \text{so} \quad x^2 = \left( \frac{2P_2(x) + 1}{3} \right)
\]

Or
\[
2x = \frac{2}{3} \left[ 2P_2(x) + 1 \right] = \frac{4}{3} P_2(x) + \frac{2}{3} P_0(x).
\]

Compare with \((x)\). This gives \( a_0 = 2/3, \quad a_2 = 4/3, \) all other \( a_n \) are zero. Thus from (i) on previous page

\[
U(\phi, t) = \frac{2}{3} P_0(\cos \phi) + \frac{4}{3} P_2(\cos \phi) e^{-\frac{6}{R^2}t}
\]

Since \( \phi \to 0 \) we conclude that \( U \to 2/3, \quad A \to + \infty. \)
(i) We begin with the generating function

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} p_n(x) t^n.$$ 

We differentiate w.r.t. $t$ to obtain

$$(x-t)(1 - 2xt + t^2)^{-3/2} = \sum_{n=0}^{\infty} P_n(x) n t^{n-1}.$$ 

Multiply by $(1 - 2xt + t^2)$ to obtain

$$(x-t)(1 - 2xt + t^2)^{-1/2} = (1 - 2xt + t^2) \sum_{n=0}^{\infty} P_n(x) n t^{n-1}.$$ 

But now use (x) on LHS

$$(x-t)\sum_{n=0}^{\infty} P_n(x) t^n = x\sum_{n=0}^{\infty} P_n(x) t^n - \sum_{n=0}^{\infty} P_n(x) t^{n+1}$$

$$= \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} P_n(x) t^{n-1} - 2x \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} P_n(x) t^n + \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} P_n(x) t^{n+1}.$$ 

Now we shift indices to obtain

$$x \sum_{n=0}^{\infty} P_n(x) t^n - \sum_{n=0}^{\infty} P_{n-1}(x) t^n = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} P_n(x) t^{n-1} - 2x \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} P_n(x) t^n + \sum_{n=1}^{\infty} \sum_{n=0}^{\infty} P_{n-1}(x) t^{n-1} (n-1).$$

Now equate coefficients of $t^n$.

$$x p_n(x) - p_{n-1}(x) = (n+1)p_{n+1}(x) - 2x n p_n(x) + (n-1)p_{n-1}(x).$$

Hence

$$(n+1)p_{n+1}(x) = + (2n+1) x p_n(x) - n p_{n-1}(x), \quad n \geq 1.$$ 

This is a recurrence relation.

For $n = 0$ we obtain $x p_0(x) = p_1(x)$, equating $t^0$. 

Hence if \( P_0(x) = 1 \)

then \( P_1(x) = x \)

\[
2 \cdot P_2(x) = 3 \cdot P_1(x) \quad \Rightarrow \quad P_2 = \frac{1}{2} (3x^2 - 1)
\]

Similarly, \( P_3(x) = \frac{1}{2} (5x^3 - 3x) \) etc.

(ii) We begin with the generating function

\[
\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n \quad \text{for} \quad |t| < 1
\]

We square both sides

\[
\frac{1}{1 - 2xt + t^2} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_m(x) t^m \cdot P_n(x) t^n = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_m(x) \cdot P_n(x) \cdot t^{m+n}
\]

Now integrate with respect to \( x \)

\[
(1) \quad \int_{-1}^{1} \frac{dx}{1 - 2xt + t^2} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} t^{m+n} \int_{-1}^{1} P_m(x) P_n(x) \, dx = \sum_{n=0}^{\infty} \int_{-1}^{1} P_n(x) \, dx \cdot \frac{2^{2n}}{t^n}
\]

since \( \int_{-1}^{1} P_M P_0 \, dx = 0 \), \( M \neq 0 \)

Now integrate the left-hand side.

\[
I = \int_{-1}^{1} \frac{dx}{-2xt + (1+t^2)} = -\frac{1}{2t} \cdot \int_{-1}^{1} \frac{dx}{x + (1+t^2)/-2t} = -\frac{1}{2t} \cdot \left[ \ln \left( x + \frac{(1+t^2)}{-2t} \right) \right]_{-1}^{1}
\]

so \( I = -\frac{1}{2t} \left[ \ln \left( \frac{1+(1+t^2)}{-2t} \right) - \ln \left( -1 + \frac{(1+t^2)}{-2t} \right) \right] \)

\[
I = -\frac{1}{2t} \ln \left( \frac{(1-t)}{(1+t)^2} \right) = -\frac{1}{t} \ln \left( \frac{1-t}{1+t} \right)
\]

Hence (2) \[
\frac{1}{t} \ln \left( \frac{1+t}{1-t} \right) = \sum_{n=0}^{\infty} \int_{-1}^{1} (P_n(x))^2 \, dx \cdot \frac{2^n}{t^n}
\]
The final step is to expand LHS of (2) in a Taylor series for \( |t| < 1 \)

\[
\frac{1}{1-t} = 1 + t + t^2 + t^3 + t^4 \ldots \quad \frac{1}{1+t} = 1 - t + t^2 - t^3 + t^4 \ldots
\]

\[
\int_0^t (-t)^n dt = t^{n+1} \quad \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1}
\]

\[
\int_0^t \frac{1}{t} (1+t) dt = 2 \sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1}
\]

Hence (2) yield that

\[
\sum_{n=0}^{\infty} \frac{2}{2n+1} t^{2n} = \sum_{n=0}^{\infty} \left( \int_0^1 (P_n(x))^2 \, dx \right) t^{2n}
\]

Equating powers of \( t^{2n} \) gives

\[
\int_0^1 (P_n(x))^2 \, dx = \frac{2}{2n+1}, \quad n = 0, 1, 2, \ldots
\]

Now if \( x = \cos \varphi \)

\[
\int_0^\pi (P_n(\cos \varphi))^2 \sin \varphi \, d\varphi = \frac{2}{2n+1}, \quad n = 0, 1, 2, \ldots
\]
(iii) USE THE GENERATING FUNCTION.

- \( 2 P_2(x) = 3x P_1(x) - P_0(x) = 3x^2 - 1. \)
  
  \[ P_2(x) = \frac{1}{2} (3x^2 - 1). \]

- \( 3 P_3(x) = 5x P_2(x) - 2 P_1(x) = \frac{5}{2} (3x^3 - x) - 2x = \frac{15}{2} x^3 - \frac{9}{2} x. \)
  
  \[ P_3(x) = \frac{5}{2} x^3 - \frac{3x}{2}. \]

- \( 4 P_4(x) = 7x P_3(x) - 3 P_2(x) = \frac{35}{2} x^4 - \frac{21}{2} x^2 - \frac{3}{2} (3x^2 - 1). \)
  
  \[ 4 P_4(x) = \frac{35}{2} x^4 - 15 x^2 + \frac{3}{2}. \]
  
  \[ P_4(x) = \frac{35}{8} x^4 - \frac{15}{4} x^2 + \frac{3}{8}. \]
THE MOST GENERAL SOLUTION OF LAPLACE'S EQUATION IN SPHERICAL
COORDINATES WITH AZIMUTHAL SYMMETRY (AXISYMMETRIC) IS
\[
U = \sum_{n=0}^{\infty} \left( A_n \frac{\Gamma^n}{\Gamma^{1+n}} + \frac{B_n}{\Gamma^{1+n}} \right) P_n(\cos \varphi)
\]

FOR \( U \) BOUNDED AS \( \Gamma \to 0 \) AND AT \( \varphi \) WE REQUIRE
\[
U(\Gamma, \varphi) = \begin{cases} 
\sum_{n=0}^{\infty} A_n \frac{\Gamma^n}{\Gamma^{1+n}} P_n(\cos \varphi), & 0 < \Gamma < R \\
\sum_{n=0}^{\infty} \frac{B_n}{\Gamma^{n+1}} P_n(\cos \varphi), & \Gamma \geq R
\end{cases}
\]

NOTICE THAT \( U \to 0 \) AS \( \Gamma \to \infty \) IS SATISFIED.

NOW AT \( \Gamma = R \) WE HAVE THAT \( U(R, \varphi) = \cos 3 \varphi \)

SO \( \cos 3 \varphi = \sum_{n=0}^{\infty} A_n R^n P_n(\cos \varphi) \)

RECALL THAT \( \cos 3 \varphi = 4 \cos^3 \varphi - 3 \cos \varphi \)

HENCE (1) \( 4 \cos^3 \varphi - 3 \cos \varphi = \sum_{n=0}^{\infty} A_n R^n P_n(\cos \varphi) \).

NOW \( P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2} (3x^2 - 1) \)
\( P_3(x) = \frac{1}{2} (5x^3 - 3x) \).

HENCE \( \frac{8}{5} P_3(x) = 4x^3 - \frac{12}{5} x \)
\( \frac{8}{5} P_3(x) - \frac{3}{5} x = 4x^3 - 3x \).
HENCE \[ 4x^3 - 3x = \frac{8}{5} P_3(x) - \frac{3}{5} P_1(x) \]

THUS YIELD \[ 4\cos^3 \varphi - 3 \cos \varphi = \frac{8}{5} P_3(\cos \varphi) - \frac{3}{5} P_1(\cos \varphi) \]

HENCE IN (1), \[ A_1 R = -\frac{3}{5}, \quad A_3 R^3 = \frac{8}{5}, \quad A_\theta = 0, \quad \forall \neq 1, 3. \]

THUS YIELD \[ U(R, \varphi) = -\frac{3}{5} \left( \frac{R}{r} \right)^2 P_1(\cos \varphi) + \frac{8}{5} \left( \frac{R}{r} \right)^3 P_3(\cos \varphi) \]

IN \[ 0 < r < R. \]

SIMILARLY, SOLVING OUTSIDE \[ R: R \] WE GET \[ U(R, \varphi) = -\frac{3}{5} \left( \frac{R}{r} \right)^2 P_1(\cos \varphi) + \frac{8}{5} \left( \frac{R}{r} \right)^4 P_3(\cos \varphi) \] FOR \[ R > R. \]

FINALLY, \[ \frac{dU}{dr} \bigg|_{r=R^+} = \frac{4}{5} \frac{R^2}{R^3} P_1(\cos \varphi) - \frac{32}{5} \frac{R^5}{R^8} P_3(\cos \varphi) \]

\[ \frac{dU}{dr} \bigg|_{r=R^-} = -\frac{3}{5} \frac{R^2}{R^3} P_1(\cos \varphi) + \frac{24}{5} \frac{R^5}{R^8} P_3(\cos \varphi) \]

\[ \sigma(\varphi) = -\frac{9}{5R} P_1(\cos \varphi) - \frac{56}{5R} P_3(\cos \varphi) = -\sigma(\varphi) \]

\[ \sigma(\varphi) = -\frac{9}{5R} P_1(\cos \varphi) + \frac{56}{5R} P_3(\cos \varphi) \]

\[ \sigma(\varphi) \] is the SURFACE CHARGE DENSITY.