**Problem 1** Consider the diffusion equation for $u(\Gamma, z, t)$ in a finite cylinder $0 \leq \Gamma \leq \alpha$, $0 \leq z \leq h$ with insulating boundaries, formulated as

$$u_t = D \left( u_{\Gamma \Gamma} + \frac{1}{\Gamma} u_{\Gamma} + u_{zz} \right), \quad 0 \leq \Gamma \leq \alpha, \quad 0 \leq z \leq h$$

where $D > 0$ is constant and

$$\begin{align*}
&BC \quad \begin{cases} u_{\Gamma}(\alpha, z, t) = u_z(\Gamma, 0, t) = u_z(\Gamma, h, t) = 0 \\
&u, u_{\Gamma} \text{ bounded as } \Gamma \to 0 \\
&IC \quad u(\Gamma, z, 0) = f(\Gamma, z)
\end{cases}
\end{align*}$$

(i) Find an eigenfunction expansion solution for $u(\Gamma, t)$.  
(ii) Calculate $\lim_{t \to \infty} u(\Gamma, z, t)$ and explain qualitatively what this limit corresponds to.

**Problem 2** Consider diffusion (non-radially symmetric) in a disk with bulk decay $\eta > 0$, formulated for $u(\Gamma, \varphi, t)$ by

$$u_t = D \left[ u_{\Gamma \Gamma} + \frac{1}{\Gamma} u_{\Gamma} + \frac{1}{\Gamma^2} u_{\varphi \varphi} \right] - \eta u \quad \text{in} \quad 0 \leq \Gamma \leq \alpha, \quad 0 \leq \varphi < 2\pi, \quad t > 0$$

with

$$\begin{align*}
&BC \quad \begin{cases} u(\alpha, \varphi, t) = 0; \quad u, u_{\Gamma} \text{ bounded as } \Gamma \to 0 \\
&u, u_{\varphi} \text{ are } 2\pi \text{ periodic in } \varphi
\end{cases}
\end{align*}$$

and

$$u(\Gamma, \varphi, 0) = 2 \left( 1 - \frac{\Gamma}{\alpha} \right) \cos^2 \varphi.$$

Here $D > 0$ and $\eta > 0$ are constants.
(i) Find an infinite series representation for $u(r, \psi, t)$.

(ii) Find a simple approximation for the solution valid for large $t$.

(Hint: Write $L(\phi, \psi) = \cos(2\phi) + 1$ and look at Problem 14 of Notes).

Problem 3: The modified Bessel equation of order $\nu$ with $\nu = 0, 1, 2, \ldots$ is

$$x^2 y'' + xy' - [x^2 + \nu^2] y = 0 \quad \text{for} \quad x > 0.$$  

(i) Show that as $x \to 0$ the two solutions to (iv) have the form $y \sim Cx^{\nu}$ for $\nu > 0$. (For $\nu = 0$, we have $y_1 \sim \text{constant}$ and $y_2 \sim \frac{1}{2} \ln x$ as $x \to 0$).

(ii) By eliminating the first derivative term by a change of variables (i.e., a Liouville transformation) find the behavior of the two solutions to (ix) for large $x$.

Note: The general solution to (ix) is

$$y = c_1 I_\nu(x) + c_2 K_\nu(x)$$

$I_\nu$, $K_\nu$ modified Bessel functions of first kind and second kind of order $\nu$.

$\nu = 0$, $I_0(0) = 0$ for $\nu > 0$ \hspace{1cm} $I_0(0) = 1$

$K_\nu \to +\infty$ as $x \to 0^+$ for $\nu > 0$.

AND $I_\nu \to +\infty$ as $x \to +\infty$ for any $\nu > 0$.

$K_\nu \to 0$ as $x \to +\infty$.\)
In terms of appropriate modified Bessel functions, solve the ODE problems for \( u(\Gamma) \) given by

(iii) \[ u'' + \frac{1}{\Gamma} u' - \alpha^2 u = 0 \quad \text{in } 0 < \Gamma < a \quad (\alpha > 0) \]
\[ u(a) = 1, \quad u, u' \text{ bounded as } \Gamma \to 0 \]

(iv) \[ u'' + \frac{1}{\Gamma} u' - \alpha^2 u = 0 \quad \text{in } \Gamma > a \]
\[ u(a) = 1, \quad u \text{ bounded as } \Gamma \to \infty \]

**Problem 4**

Consider the steady-state temperature distribution for \( u(\Gamma, \varphi, z) \) inside a finite cylinder formulated as

\[ u_{\Gamma\Gamma} + \frac{1}{\Gamma} u_\Gamma + \frac{1}{\Gamma^2} u_{\varphi\varphi} + u_{zz} = 0 \]

in \( 0 < \Gamma < a, \varphi \in [0, 2\pi], \quad 0 \leq z \leq H \).

With boundary conditions:

\[ u = 0 \text{ on } z = 0, H \]

\( u, u_\varphi \) are \( 2\pi \) periodic in \( \varphi \).

\( u(a, \varphi, z) = F(\varphi, z) \) (heating on the lateral side)

\( u, u_\Gamma \) bounded as \( \Gamma \to 0 \).

Find an infinite series representation for the solution in terms of appropriate modified Bessel functions.

(Hint: the SL problems come from \( \varphi \) and \( z \) directions.)