**Problem 1**

Consider the radially symmetric diffusion problem in a 2-D disk of radius \( r > 0 \) modeled by

\[
\begin{align*}
\frac{\partial u}{\partial t} &= D \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) - u \quad \text{in} \quad 0 \leq r \leq a, \quad t > 0 \\
\text{with} \quad u(r, 0) &= f(r) \quad \text{and} \quad u(r, t) = 0 \quad \text{for} \quad r > a.
\end{align*}
\]

Here \( D > 0 \) is a constant.

(i) Let \( M(t) = \int_0^a u(r, t) \, dr \). Derive an ODE for \( M(t) \) and solve it to determine \( M(t) \) in terms of \( f(r) \).

(ii) Determine an infinite series representation for the solution to (i). Is this infinite series consistent with the result in (i)?

**Problem 2** (Hanging Chain Problem)

(i) Consider the SL problem for \( \Phi(x) \) given by

\[
\begin{align*}
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial \Phi}{\partial x} + \lambda \Phi &= 0 \quad \text{for} \quad 0 < x < L \\
\Phi(L) &= 0, \quad \Phi, \Phi' \text{ bounded at} \quad x = 0.
\end{align*}
\]

Show that if we change variables with \( z = \sqrt{\alpha} x \) and \( \Psi(z) = \Phi \left( \frac{z^2}{\alpha} \right) \) that \( \Psi(z) \) satisfies Bessel's equation \( \frac{\partial^2 \Psi}{\partial x^2} + \frac{1}{z} \frac{\partial \Psi}{\partial x} + \Psi = 0 \) provided that \( \alpha \) is chosen appropriately. Determine the eigenvalues of (x) in terms of zeros of \( J_0(z) \).
(iii) The transverse deflection $u(x,t)$ of a hanging chain of length $L$ with gravity $g$ is known to satisfy the PDE

$$u_{tt} = g \left[ x u_{xx} + u_x \right], \quad 0 < x < L, \quad t > 0$$

$$u(L,t) = 0; \quad u, u_x \text{ bounded at } x \to 0$$

$$u(x,0) = f(x), \quad u_t(x,0) = 0.$$ 

By using (i) derive an eigenfunction expansion representation for the solution.

(iii) Give explicit formulae for the periods of oscillation of the first two terms in the infinite series representation of $u$ in (ii). Calculate them if $L = 1$ metres and $g = 9.8$ metre/sec$^2$

(Hint: Look up the first two zeroes of $J_0(z)$.)

Problem 3: Consider non radially symmetric diffusion in a disk modeled by the solution $u(r,\phi,t)$ to

$$u_t = D \left( u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\phi\phi} \right)$$

in $0 < r < 1$, $0 < \phi < 2\pi$, $t > 0$

with $D > 0$ constant and

$$\text{BC} \left\{ \begin{array}{l}
\quad u(1,\phi,t) = 0; \quad u, u_r \text{ bounded at } r \to 0
\end{array} \right.$$

$u, u_\phi$ are $2\pi$ periodic in $\phi$

and

$$\text{(IC)} \left\{ \begin{array}{l}
\quad u(r,\phi,0) = 2 \left( 1 - r^2 \right) \sin^2 \phi.
\end{array} \right.$$

(i) Find an infinite series representation for the solution

(Hint: $2 \sin^2 \phi = 1 - \cos 2\phi$. Use initial condition to your advantage).

(ii) Calculate a simple approximation for $u(r,t)$ when $t$ is large.
Problem 4
Consider the diffusion equation for \( u(\rho, z, t) \) in a finite cylinder \( 0 \leq \rho \leq a, \ 0 \leq z \leq H \) with insulating boundaries formulated as

\[
\dot{u} = D \left[ u_{\rho \rho} + \frac{1}{\rho} u_{\rho} + u_{zz} \right], \quad \text{in} \ 0 \leq \rho \leq a, \ 0 \leq z \leq H
\]

with \( D > 0 \) constant and

\[
\begin{align*}
(\text{BC}) & \quad u(\rho, 0, t) = u(\rho, H, t) = 0 \\
(\text{IC}) & \quad u(\rho, z, 0) = f(\rho, z)
\end{align*}
\]

We say that \( u \) is axisymmetric since it has no \( \phi \) dependence.

(i) Find an eigenfunction expansion solution for \( u(\rho, z, t) \).

(ii) Calculate the steady-state defined by \( \lim_{t \to \infty} u(\rho, z, t) \).

Problem 5
The modified Bessel equation of order \( \eta \) with \( \eta = 0, 1, 2, \ldots \) is

\[
x^2 y'' + xy' - \left( x^2 + \eta^2 \right) y = 0 \quad \text{for} \ x > 0.
\]

(i) Show that as \( x \to 0 \) the two solutions to (X) have the form \( y_1 \propto \rho^{1/2} \) for \( \eta > 0 \) this one is bounded and the other unbounded as \( x \to 0 \). (For \( \eta = 0 \), we have \( y_1 \) constant and \( y_2 \propto \log x \) as \( x \to 0 \)).

(ii) Perform a Liouville transformation to eliminate the first derivative term in (X) by setting \( y = p \psi \). Show that for a certain choice of \( p(x) \) we get \( \psi'' - \left[ 1 + \frac{\eta^2 - 1/4}{x^2} \right] \psi = 0 \).
so that \[ u = A_1 e^x + A_2 e^{-x} \] \[ a \] \[ x \to \infty. \]

In this way, the general solution to (*) has the form
\[ y = c_1 I_0 (x) + c_2 K_0 (x) \]
where \( I_0 (x) \) and \( K_0 (x) \) are "modified Bessel functions" of the first and second kind, respectively, of order \( \eta \). They satisfy
\[ I_0 (0) = 0 \quad \text{for} \quad \eta > 0, \quad I_0 (0) = 1 \]
\[ I_n (x) \to +\infty \quad \text{as} \quad x \to +\infty \]
and
\[ K_n (x) \to 0 \quad \text{as} \quad x \to +\infty \]
\[ \left| K_n (x) \right| \to +\infty \quad \text{as} \quad x \to 0^+ \]

(iii) In terms of the appropriate modified Bessel function solve the following problem for \( u (\Gamma) \):

(A) \[ u '' + \frac{1}{\Gamma} u ' - 4 u = 0 \quad \text{in} \quad 0 < \Gamma < a \]
\[ u (a) = 1, \quad u, u' \text{ bounded as} \quad \Gamma \to 0 \]

(B) \[ u '' + \frac{1}{\Gamma} u ' - 4 u = 0 \quad \text{in} \quad \Gamma > a \]
\[ u (a) = 1, \quad u \to 0 \quad \text{as} \quad \Gamma \to +\infty. \]
PROBLEM 1

\[ U_t = D \left( \frac{U_{rr}}{r} + \frac{1}{r} U_r \right) - U \quad \text{in} \quad 0 < r < a, \quad t > 0 \]

\[ U_r(a,t) = 0; \quad U, U_r \quad \text{bounded as} \quad r \to 0 \]

\[ U(r, 0) = f(r). \]

(i) FIND \( M(t) = \int_0^a r \cdot U(r,t) \, dr. \)

\[ \text{MULTIPLY PDE BY } r: \quad r U_t = D \left( r U_r \right)_r - r U. \]

\[ \text{NOW INTEGRATE: } \int_0^a r \cdot U_t \, dr = \int_0^a D \left( r U_r \right)_r \, dr - \int_0^a r \cdot U \, dr \]

\[ \to \quad \frac{d}{dt} \left( \int_0^a r \cdot U \, dr \right) = D \left( \int_0^a r U_r \, dr \right)_0 - \left( \int_0^a r \cdot U \, dr \right)_0 \]

\[ \text{THUS, } \quad \frac{dM}{dt} = -M \quad \text{WITH} \quad M(0) = \int_0^a r \cdot U(r,0) \, dr = \left[ \int_0^a r \cdot f(r) \, dr \right]. \]

\[ \text{WE GET } \quad M(t) = M(0) e^{-t} \quad \text{SO} \quad M(t) = \left( \int_0^a r \cdot f(r) \, dr \right) e^{-t}. \quad (1) \]

(ii) NOW SEPARATE VARIABLES: \( U(r,t) = T(t) \Phi(r) \)

\[ T' \Phi = DT \left[ \Phi'' + \frac{1}{r} \Phi' - \lambda \Phi \right] \to \quad \frac{T' + T}{DT} = \frac{\Phi''}{\Phi'} + \frac{1}{r} \Phi' = -\lambda. \]

\[ \text{THUS THE SL PROBLEM IS} \]

\[ \begin{cases} 
\Phi'' + \frac{1}{r} \Phi' + \lambda \Phi = 0 & \text{in} \quad 0 < r < a \\
\Phi'(a) = 0, \quad \Phi, \Phi' \text{ bounded as} \quad r \to 0 
\end{cases} \quad (2) \]

AND \( T' = -[1 + DT] T \quad \text{SO} \quad T = e^{-(1+DT)t}. \quad (3) \]

\[ \text{NOW THE EIGENPAIRS OF } (2) \text{ ARE} \]

\[ \begin{cases} 
\lambda_0 = 0, \quad \Phi_0 = 1 \\
\lambda_n = \frac{1}{a^2}, \quad \Phi_n = J_0(\sqrt{\lambda_n} \cdot r) \quad \text{WHERE} \quad J_0(z_n) = 0, \quad n = 1, 2, \ldots \end{cases} \quad (4) \]

\[ \text{Diagram:} \]

\[ J_0(z) \]

\[ z_1, z_2, z_3 \]

\[ z \]
KEY POINT HERE IS THAT DUE TO NEUMANN CONDITION $\Phi'(0) = 0$, THEN $\exists I_0 = 0$ AND $I_0 I = 1$ EIGENPAIR. WE THEN USE SUPERPOSITION AND (3) TO WRITE

$$U(\gamma, t) = \sum_{\eta=0}^{\infty} C_\eta e^{-(1+IA_0)\eta} \Phi_\eta(\gamma) d\gamma.$$ \hspace{1cm} (5)

NOW TO DETERMINE ORTHOGONALITY RELATION WE WRITE (2) IN SL FORM TO GET

$$(\gamma \Phi')' + \gamma \Phi = 0$$

THEN W: $I = 0$ AND WE HAVE

$$\int_0^q \gamma \Phi_m. \Phi_n d\gamma = 0 \text{ for } m \neq n \text{ AND } m, n \in \{0, 1, 2, \ldots\}$$

IN PARTICULAR, SETTING $M = 0$ WE GET

$$\int_0^q \gamma \Phi_0 d\gamma = 0 \text{ for } n = 1, 2, \ldots \text{ SINCE } \Phi_0 = 1.$$ \hspace{1cm} \text{NOW FIND THE } C_0 \text{ IN (5) FROM USING } U(\gamma, 0) = f(\gamma). \text{ WE GET EXTRACTING } n = 0 \text{ TERM}

$$f(\gamma) = C_0 + \sum_{n=1}^{\infty} C_n \Phi_n(\gamma) d\gamma$$

TO OBTAIN $C_0$ INTEGRATE TO GET

$$\int_0^q f(\gamma) \gamma d\gamma = C_0 \int_0^q \gamma d\gamma + \sum_{n=1}^{\infty} C_n \int_0^q \gamma \Phi_n d\gamma$$

SO

$$C_0 = \frac{2}{a^2} \int_0^q \gamma f(\gamma) d\gamma. \text{ TO FIND } C_0 \text{ FOR } n \geq 1 \text{ MULTIPLY BY } \gamma \Phi_m \text{ AND INTEGRATE. USING ORTHOGONALITY,}

\int_0^q \gamma f(\gamma) \Phi_m d\gamma = C_m \int_0^q \gamma \Phi_m^2 d\gamma. \text{ REPLACING } M \to n \text{ GIVES}

$$C_n = \frac{1}{\int_0^q \gamma \Phi_n^2 d\gamma} \int_0^q \gamma f(\gamma) d\gamma \text{ for } n = 1, 2, \ldots, \text{ } C_0 = \frac{2}{a^2} \int_0^q f(\gamma) d\gamma.$$ \hspace{1cm} (6)

THEN (6) BECOMES, EXTRACTING $n = 0$ TERM

$$U(\gamma, t) = \frac{2}{a^2} \left( \int_0^q f(\gamma) d\gamma \right) e^{-t} + \sum_{n=1}^{\infty} C_n e^{-(1+IA_0)\eta} \Phi_n(\gamma) \text{ with } \Phi_0 = J_0(\sqrt{A_0} \gamma)$$

IF WE INTEGRATE (7) BY MULTIPLYING BY $\gamma$ AND $\int_0^q$ WE GET USING $\int_0^q \gamma \Phi_n d\gamma = 0$ THAT

$$\int_0^q U(\gamma, t) \gamma d\gamma = \frac{2}{a^2} \left( \int_0^q f(\gamma) d\gamma \right) e^{-t} \int_0^q \gamma d\gamma = \left( \int_0^q f(\gamma) d\gamma \right) e^{-t}, \text{ WHICH IS SAME AS IN (1)}$$
(i) \[ x \ddot{x} + \ddot{x} + \lambda x = 0. \] Now in SL form \((x, \dot{x}), \dot{x} + \ddot{x} + \lambda x = 0\) so that \(W = 1\).

\[ \Phi(1) = 0 \]

we have \( \int_0^L \Phi_n \Phi_m \, dx = 0 \) \( n \neq m \).

\( \Phi, \Phi' \) bounded as \( x \to \infty \).

Now let \( z = \sqrt{a} x^{1/2} \), \( \Phi(z) = \Phi\left(\frac{z^2}{a}\right) \)

now \( \Phi_x = \Psi_z z_x \) \( \Phi_{xx} = \Psi_{zz} z_x^2 + \Psi_z z_{xx} \)

Substitute to get

\[ x \left[ \Psi_{zz} z_x^2 + \Psi_z z_{xx} \right] + \Psi_z z_x + \lambda \Psi = 0. \]

Thus

\[ x \int \Psi_{zz} \, \frac{a}{4} x^{-1} + \Psi_z \left( -\frac{\sqrt{a}}{4} x^{-3/2} \right) + \Psi_z \sqrt{a} x^{-1/2} + \lambda \Psi = 0. \]

Thus

\[ \Psi_{zz} \frac{a}{4} + \Psi_z \left[ -\frac{\sqrt{a}}{4} x^{-1/2} + \sqrt{a} x^{-1/2} \right] + \lambda \Psi = 0. \]

Thus give

\[ \Psi_{zz} \frac{a}{4} + \Psi_z \left[ \sqrt{a} x^{-1/2} \right] + \lambda \Psi = 0. \]

we get

\[ \Psi_{zz} + \Psi_z \left( \frac{1}{\sqrt{a} x^{1/2}} \right) + \frac{\lambda a}{a} \Psi = 0. \]

Choose \( a = 4 \lambda \) and identify \( z = \sqrt{a} x^{1/2} \). Thus, since \( x = L \to z = \sqrt{aL} \),

\[ \Psi_{zz} + \frac{1}{z} \Psi_z + \Psi = 0 \]

\[ \Psi, \Psi_z \) bounded as \( z \to \infty \); \( \Psi (\sqrt{aL}) = 0. \)

The solution is \( \Psi = Aj_0(z) + B y_0(z) \) with \( B = 0 \) for bounded.

If \( z \to 0 \), then \( \sqrt{aL} = 2 \sqrt{aL} \). Thus

\[ J_0 \left( 2 \sqrt{aL} \right) = 0 \] or \( 2 \sqrt{aL} = Z_n, \) \( n = 1, 2, \ldots \)

where \( J_0 (Z_n) = 0 \)

\[ Z_1 \approx 2.4048 \]

\[ Z_2 \approx 5.5201 \]
Now solve the PDE:

\[ u_{tt} - g \left[ \frac{x}{L} u_{xx} + u_x \right], \quad 0 < x < L, \quad t > 0 \]

\[ u(L, t) = 0, \quad u, u_x \text{ bounded at } x \to 0 \]

\[ u(x, 0) = f(x), \quad u_t(x, 0) = 0. \]

We separate variables, \( u(x, t) = T(t) \Phi(x) \) so that

\[ \frac{T''}{g T} = \frac{\Phi''}{\Phi} = -\lambda \text{ so } \lambda \Phi'' + \Phi' + \lambda \Phi = 0 \]

\[ \Phi(L) = 0; \quad \Phi, \Phi' \text{ bounded at } x \to 0 \]

And

\[ T'' = -\lambda g T, \quad T'' + \lambda g T = 0 \to T = \text{span} \{ \cos(\sqrt{\lambda_g} t), \sin(\sqrt{\lambda_g} t) \} \]

From (i) we have

\[ \Phi_n(x) = J_0 \left( \frac{2 \sqrt{\lambda_n} \sqrt{x}}{L} \right) \]

Where

\[ \lambda_n = \frac{Z_n^2}{4L} , \quad n = 1, 2, ... \quad \text{and} \quad J_0 \left( \frac{Z_n}{L} \right) = 0. \]

Then

\[ T_n(t) = A_n \cos \left( \omega_n t + B_n \sin \left( \omega_n t \right) \right) \quad \text{where} \quad \omega_n = \sqrt{\lambda_n g} \]

Thus yield,

\[ u(x, t) = \sum_{n=1}^{\infty} \left[ A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right] J_0 \left( \frac{2 \sqrt{\lambda_n} \sqrt{x}}{L} \right) \]

With

\[ \omega_n = \sqrt{\lambda_n g} = \frac{Z_n \sqrt{g}}{2L} , \quad n = 1, 2, ... \quad \lambda_n = \frac{Z_n^2}{4L} \]

Now

\[ u_t(x, 0) = \sum_{n=1}^{\infty} -B_n \omega_n J_0 \left( \frac{2 \sqrt{\lambda_n} x}{L} \right) = 0 \to B_n = 0 \]

\[ u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n J_0 \left( \frac{2 \sqrt{\lambda_n} x}{L} \right). \]

By orthogonality with weight \( w \equiv 1 \),

\[ A_n = \frac{\int_0^L f(x) J_0 \left( \frac{2 \sqrt{\lambda_n} x}{L} \right) dx}{\int_0^L \left( J_0 \left( \frac{2 \sqrt{\lambda_n} x}{L} \right) \right)^2 dx}. \]
This gives,
\[ u(x,t) = \sum_{n=1}^{\infty} A_n \cos ( \omega_n t ) \sin \left( 2\sqrt{\lambda_n} x \right) \text{ with } A_n \text{ given in } (\star) \]

(iii) The frequencies are
\[ \omega_n = \frac{\pi}{L} \sqrt{\frac{g}{L}} \]

so the period is
\[ T_n = \frac{2\pi}{\omega_n} = 4\pi \sqrt{\frac{L}{g}} \frac{1}{Z_n} \text{ for } n = 1, 2, \ldots \]

Since \( g = 9.8 \text{ m/s}^2 \) and \( L = 1 \text{ m} \), we have using \( Z_1 \approx 2.4048 \) (from a table)
\[ Z_2 \approx 5.5701 \]

that \( T_1 \approx 1.67 \text{ sec} \) and \( T_2 \approx 0.727 \text{ sec} \). \{ HANGING CHAIN \}

Recall for a pendulum that \( T = 2\pi \sqrt{\frac{L}{g}} \)
PROBLEM 3

Let $u(\gamma, \phi, t)$ solve

$$u_t = D \left( u_{\gamma\gamma} + \frac{1}{\gamma} u_{\gamma} + \frac{1}{\gamma^2} u_{\phi\phi} \right) \quad \text{in} \quad 0 < \gamma < 1, \quad 0 < \phi < 2\pi, \quad t > 0$$

with $D > \text{constant}$ and

(B.C.) \quad u(1, \phi, t) = 0, \quad u, u_\gamma \text{ bounded at } \gamma \to 0

(I.C.) \quad u(\gamma, \phi, 0) = 2(1-\gamma^2) \sin^2 \phi = (1-\gamma^2)[1-\cos(2\phi)].

Rather than doing a full separation of variables in 3 coordinates, we will observe that initial conditions help to reduce the calculation to effectively 2 coordinates, i.e., $\gamma$ and $t$.

Separate variables $u = \Phi(\gamma) \Psi(\phi) T(t)$, then we get

$$\frac{T'}{DT} = \left( \frac{\Phi'' + \gamma \Phi'}{\Phi} \right) + \frac{1}{\gamma^2} \frac{\Psi''}{\Psi}$$

Now since L.H.S. depends on $t$, while R.H.S. depends on $\gamma, \phi$, we have our first separation constant $\lambda$ as

$$\frac{\Phi'' + \gamma \Phi'}{\Phi} + \frac{1}{\gamma^2} \frac{\Psi''}{\Psi} = -\lambda \quad \text{and} \quad \frac{T'}{DT} = -\lambda. \quad (1)$$

Now in first relation separate again:

$$\frac{\gamma^2 \Phi'' + \gamma \Phi'}{\Phi} + \frac{\Psi''}{\Psi} = -\lambda \gamma^2 \rightarrow \frac{\Phi'' + \gamma \Phi'}{\Phi} + \frac{\lambda \gamma^2}{\Psi} = -\Psi'' = \mu \quad (2)$$

This yield the second separation constant $\mu$. 
In this way we get two SL problems from (2):

\[ \begin{align*}
\psi'' + \mu \psi &= 0, \quad 0 \leq \varphi \leq 2\pi \\
\psi, \psi' &\text{ periodic in } \varphi
\end{align*} \]

(3)

\[ \begin{align*}
\phi'' + \frac{1}{\Gamma^2 - \eta^2} \phi &= 0 \quad 0 \leq \eta \leq \alpha \\
\phi(0) &= 0, \quad \phi, \phi' \text{ bounded at } \Gamma \to 0
\end{align*} \]

(4)

From (3) we obtain

\[ \begin{align*}
\psi_0 &= 0, \quad \mu_0 = 0 \\
\psi_n &= \text{span } \left\{ \cos(n\eta), \sin(n\eta) \right\}, \\
\mu_n &= n^2 \quad n = 1, 2, \ldots
\end{align*} \]

Therefore

\[ \begin{align*}
\phi'' + \frac{1}{\Gamma^2 - \eta^2} \phi &= 0 \quad 0 \leq \eta \leq \alpha \\
\phi(0) &= 0, \quad \phi, \phi' \text{ bounded at } \Gamma \to 0
\end{align*} \]

(5)

In SL form this is

\[ \left( \frac{\eta^2}{\Gamma} \frac{d^2}{d\eta^2} + \frac{1}{\Gamma^2 - \eta^2} \right) \phi = 0 \quad \text{so } \psi = \eta \text{ as weight.} \]

The SL problem (6) is the classic SL problem for \( J_0 \), \( Y_0 \). We get

\[ \phi = c_0 J_{\eta/2} (\sqrt{\lambda}) + c_1 Y_{\eta/2} (\sqrt{\lambda}) \quad \text{for } n = 0, 1, 2, \ldots \]

But \( Y_n \) unbounded at \( \Gamma \to 0 \) so need \( c_1 = 0 \). Then \( \phi(0) = 0 \), yield

\[ \lambda_{mn} = \frac{n^2}{m^2} \quad \text{where } J_{n/2} (\sqrt{\lambda_{mn}}) = 0 \]

Notice we need double subscript here since for each fixed \( n, \ell \)

an infinite \# of roots \( J_0 (Z) = 0 \), but these roots depend on the value of \( n \) chosen. Then WLOG we can set \( c_0 = 1 \) to get

\[ \bar{\phi}_{mn} = J_0 (\sqrt{\lambda_{mn}}) \]

Notice that the roots are labelled \( Z_{mn} \) with \( m = 1, 2, \ldots \) in picture.
Recalling (1) we get \( T_{mn} = e^{-\lambda_{mn} dt} \) and from (5)

\[
\psi_n = A \cos(n \varphi) + B \sin(n \varphi)
\]

By using superposition we must let \( A \to A_{mn} \) and \( B \to B_{mn} \) so that

\[
U(\varpi, \varphi, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} e^{-\lambda_{mn} dt} \left[ A_{mn} \cos(n \varphi) + B_{mn} \sin(n \varphi) \right] J_n \left( \sqrt{\lambda_{mn}} \varpi \right)
\]

Notice that \( B_{mn} \) is arbitrary since \( \sin(0) = 0 \). Finally we put \( U(\varpi, \varphi, 0) = \)

\[
U(\varpi, \varphi, 0) = (1 - r^2) \cos(2 \varphi) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left[ A_{mn} \cos(n \varphi) + B_{mn} \sin(n \varphi) \right] J_n \left( \sqrt{\lambda_{mn}} \varpi \right)
\]

It is here that we see that due to the specific form of the initial condition the doubly infinite sum collapses to two separate singly infinite sums. We only need the \( \varpi = 0 \) and \( \varpi = 2 \) modes (with \( \cos(2 \varphi) \)).

Thus,

\[
A_{mn} = 0 \quad \text{for} \quad n = 1, 3, 4, 5, 6, \ldots
\]

\[
B_{mn} = 0 \quad \text{for} \quad m = 0, 1, 2, \ldots
\]

In this way we get, using orthogonality w.r.t \( \varpi \):

\[
(1 - r^2) = \sum_{m=1}^{\infty} A_{m} J_{0} \left( \sqrt{\lambda_{m}} \varpi / a \right) \quad \rightarrow \quad A_{m} = \frac{\int_{0}^{\varpi} (1 - r^2) J_{0} \left( \sqrt{\lambda_{m}} r / a \right) r \, dr}{\int_{0}^{\varpi} r \left[ J_{0} \left( \sqrt{\lambda_{m}} r / a \right) \right]^2 \, dr}
\]

\[
-(1 - r^2) = \sum_{m=1}^{\infty} A_{m} J_{2} \left( \sqrt{\lambda_{m}} \varpi / a \right) \quad \rightarrow \quad A_{m} = -\frac{\int_{0}^{\varpi} (1 - r^2) J_{2} \left( \sqrt{\lambda_{m}} r / a \right) r \, dr}{\int_{0}^{\varpi} r \left[ J_{2} \left( \sqrt{\lambda_{m}} r / a \right) \right]^2 \, dr}
\]

The solution is then

\[
U(\varpi, \varphi, t) = \sum_{n=1}^{\infty} e^{-\lambda_{mn} dt} A_{mn} J_{0} \left( \sqrt{\lambda_{mn}} \varpi \right) + \sum_{m=1}^{\infty} e^{-\lambda_{mn} dt} A_{m2} J_{2} \left( \sqrt{\lambda_{m2}} \varpi \right)
\]

(7)
In (6) we have $A_{mn} = \frac{Z_{mn}}{a}$ for $n = 0, 2$ where $J_0(Z_{mn}) = 0$ for $n = 0, 2$ and $m = 1, 2, 3, \ldots$

Now to obtain an approximation for $t$ large we need to identify the smallest $A_{mn}$. As a first step, taking first terms in infinite sum of (7) we get for $t$ large

$$u(r, \theta, t) \approx A_{10} J_0 \left( \sqrt{A_{10}} \ r \right) e^{-A_{10} \frac{r}{a^2}} + A_{12} J_2 \left( \sqrt{A_{12}} \ r \right) e^{-A_{12} \frac{r}{a^2}}$$

(8)

Now we look up first zero of $J_0(z) = 0$ \rightarrow $Z_{10} \approx 2.4048$

and the first zero of $J_2(z) = 0$ \rightarrow $Z_{12} \approx 5.1356$

Since $Z_{10} < Z_{12}$ we have $A_{10} < A_{12}$. As such the first term in (8) decay more slowly than the second. We conclude that for $t \rightarrow \infty$

$$u(r, \theta, t) \sim A_{10} J_0 \left( \frac{Z_{10} r}{a} \right) e^{-z_{10} 2 \frac{r}{a^2}}$$

where $Z_{10} \approx 2.4048$ and $A_{10} = \frac{\int_0^a r (1-r^2) J_0 \left( \frac{Z_{10} r}{a} \right) \, dr}{\int_0^a r \left[ J_0 \left( \frac{Z_{10} r}{a} \right) \right]^2 \, dr}$
**Solution 4**}

We separate variables:

\[ u(r, z, t) = R(r) \chi(z) T(t) \]

Thus given

\[ \frac{T'}{DT} = \frac{R'' + \frac{1}{r} R'}{R} + \frac{\chi''}{\chi} = -\Lambda. \]

We take

\[ \frac{\chi''}{\chi} = -\mu \quad \rightarrow \quad \chi'' + \mu \chi = 0 \quad \rightarrow \quad \chi_n = \cos\left( \frac{n\pi Z}{H} \right) \]

\[ \chi'_n(0) = \chi'_n(H) = 0 \]

\[ \mu_n = \frac{n^2 \pi^2}{H^2}, \]

for \( n = 0, 1, 2, \ldots \).

**Notice** \( \chi_0 = 1 \) and \( \mu_0 = 0 \); zero eigenvalue.

Then

\[ \frac{R'' + \frac{1}{r} R'}{R} = \mu - \Lambda \]

Let \( \kappa = \Lambda - \mu \).

Then

\[ R'' + \frac{1}{r} R' + (\mu - \kappa) R = 0 \]

\[ R'(a) = 0, \quad R(0) = \text{nice}. \]

**Notice** \( \kappa = 0 \) is an eigenvalue \( \rightarrow R = 1 \) is eigenfunction.

Then

\[ R_n(r) = J_0\left( \sqrt{\kappa_n} r \right) \]

\[ \kappa_n = \frac{\sigma^2}{a^2} J_0'(\sigma_n) = 0, \quad m = 1, 2, 3, \ldots \]

Then with \( \Lambda = \kappa_n + \mu_n \) we get

\[ \frac{T'}{DT} = -\Lambda_n \quad \rightarrow \quad T = e^{-\Lambda_n t} \]

**Notice** that \( \Lambda_{00} = 1 \) \( \rightarrow \) \( T_{00} = 1 \).

Hence by superposition over \( m \) and \( n \) we obtain

\[ u(r, z, t) = A_{00} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n} \cos\left( \frac{n\pi Z}{H} \right) J_0\left( \frac{\sigma_m \Gamma}{a} \right) e^{-\Lambda_{m,n} t} \]

**Case** \( U(r, z, t) = A_{00} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n} \cos\left( \frac{n\pi Z}{H} \right) J_0\left( \frac{\sigma_m \Gamma}{a} \right) e^{-\Lambda_{m,n} t} \)

\[ (m, n) \neq (0, 0) \]
Now since \( \kappa = 0 \) and \( \kappa = K_m \) are eigenvalues corresponding to different eigenfunctions \( R = 1 \) and \( R = J_0(\sqrt{K_m} \; r) \), we have by orthogonality that

\[
(1) \quad \int_0^q 1 \cdot J_0(\sqrt{K_m} \; r) \; r \; dr = 0
\]

Now let \( U(r, z, \phi) = \bar{F}(r, z) \). This gives

\[
\bar{F}(r, z) = A_{00} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m0} \cos \left( \frac{n \pi z}{H} \right) J_0 \left( \frac{\sigma_m r}{\alpha} \right)
\]

To find \( A_{00} \), multiply by \( r \) and integrate using (1).

\[
\int_0^H \int_0^q f(r, z) \; r \; dr \; dz = A_{00} \int_0^H \int_0^q r \; dr \; dz + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m0} \int_0^H \cos \left( \frac{n \pi z}{H} \right) \int_0^q J_0 \left( \frac{\sigma_m r}{\alpha} \right) \; r \; dr \; dz \to = 0 \leftarrow
\]

\[
= A_{00} \frac{H}{2} \frac{q^2}{2} + 0
\]

Hence,

\[
A_{00} = \frac{2}{a^2 H} \int_0^H \int_0^q r \; f(r, z) \; dr \; dz = \frac{1}{\text{Vol}(\Omega)} \int_{\Omega} U_0 \; d\Omega,
\]

which is the average of the initial data over the cylinder.

If \( \Omega \) is cylinder \( 0 < \phi < 2\pi, 0 < z < H, 0 < r < a \to \text{Vol}(\Omega) = \pi a^2 H \).

And,

\[
\int_{\Omega} U_0 \; d\Omega = \int_0^{2\pi} \int_0^a \int_0^H f(r, z) \; r \; dr \; dz = 2\pi \int_0^q \int_0^H f(r, z) \; r \; dr \; dz.
\]

To find \( A_{mn} \) use orthogonality

\[
\int_0^q \int_0^H r \; f(r, z) \; J_0 \left( \sqrt{K_m} \; r \right) \cos \left( \frac{n \pi z}{H} \right) \; dr \; dz = A_{ij} \int_0^q r \; J_0 \left( \sqrt{K_i} \; r \right) \; dr \; dz
\]

Hence,

\[
A_{mn} = \frac{2}{H} \left( \int_0^q r \; J_0 \left( \sqrt{K_m} \; r \right) \; dr \right)^{-1} \int_0^q \int_0^H r \; f(r, z) \; J_0 \left( \sqrt{K_m} \; r \right) \cos \left( \frac{n \pi z}{H} \right) \; dr \; dz.
\]
\[ x^2 y'' + x y' - \left[ x^2 + n^2 \right] y = 0 \]

(i) Put \( y = x^d \) for \( x \to 0 \). Then

\[
\frac{d}{d\xi} \left( x^d \right) + \frac{d}{d\xi} x^d - \left[ n^2 x^d + x^{2+n} \right] = 0.
\]

So

\[
x^d \left[ d^2 - n^2 \right] + x^{d+2} \left[ \ldots \right] = 0.
\]

The indicial equation is \( d^2 - n^2 = 0 \) so \( d = \pm n \) with \( n > 0 \).

Thus for \( n = 1, 2, \ldots \) we have \( y_1 \sim c_0 x^n \) and \( y_2 \sim c_1 x^{-n} \) as \( x \to 0^+ \).

For \( n = 0 \) we have \( y_1 \sim c_0 \) and \( y_2 \sim \left( \log x \right) c_1 \) as \( x \to 0 \)

by Frobenius theory.

(ii) Now to eliminate \( y' \) proceed as in the notes.

We put \( y = p \psi \) to get

\[
x^2 \left[ p \psi'' + 2 p' \psi' + \frac{1}{x^2} \psi \right] + x \left[ \psi' + \frac{p'}{p} \psi \right] - \left( x^2 + n^2 \right) \frac{p}{x} \psi = 0.
\]

So

\[
x^3 p \psi'' + \psi' \left[ 2 p' x^2 + (n^2 + n^2) \frac{x}{x} \right] + \psi \left[ \frac{1}{x^2} p'' + \frac{p'}{x} \right] - \left( x^2 + n^2 \right) \frac{p}{x} \psi = 0.
\]

Divide by \( x^3 p \):

\[
\psi'' + \psi' \left[ \frac{2 p'}{p} + \frac{1}{x} \right] + \psi \left[ \frac{p''}{p} + \frac{p'}{x} \right] - \left( \frac{1}{x^2} + n^2 \right) \frac{p}{x} \psi = 0.
\]

We choose \( p(x) \) so that \( \frac{2 p'}{p} + \frac{1}{x} = 0 \) or \( p' + \frac{1}{2x} \frac{x}{p} = 0 \).

(Which gives \( p = x^{-1/2} \)). Then

\[
\left( \frac{p'' + p'/x}{p} \right) = \left( \frac{3/4 x^{-5/2} - 1/2 x^{-3/2}}{x^{-1/2}} \right) = \frac{1}{4 x^2}.
\]
We conclude that if \( y = x^{-\frac{1}{2}} \psi \) then \((X)\) becomes exactly
\[
\psi'' = \left[1 + \left(\frac{n^2 - 1/4}{x^2}\right)\right] \psi = 0.
\]
Now for \( x \to \infty \), \( \psi'' - \psi = 0 \) so \( \psi \cong e^{\pm x} \).

We conclude for \( x \to +\infty \) that
\[
y = \frac{\psi}{x^{1/2}} \sim \text{span } \begin{pmatrix} e^x \sqrt{x} \text{ grow} \end{pmatrix}, \begin{pmatrix} e^{-x} \sqrt{x} \text{ decay} \end{pmatrix}
\]

The general solution to \((X)\) is
\[
y = c_1 \, J_n(x) + c_2 \, K_n(x)
\]
where \( J_n(z) \), \( K_n(z) \) are (alled modified Bessel functions) of the first and second kind of order \( n \). The plots are

\[\begin{array}{c}
\text{Plot of } J_0(x) \\
\text{Plot of } K_0(x)
\end{array}\]

\[\begin{array}{c}
\text{Plot of } J_1(x) \\
\text{Plot of } K_1(x)
\end{array}\]
THE GENERAL SOLUTION TO
\( u'' + \frac{1}{\gamma} u' - \lambda u = 0 \) \( u = c_1 I_0(2\gamma) + c_2 K_0(2\gamma) \).

(A) IF \( u(\gamma) = 1 \) AND \( u, u' \) BOUNDED \( \gamma \rightarrow 0 \) WE MUST SET \( c_2 = 0 \) SINCE \( K_0(2\gamma) \) UNBOUNDED \( \gamma \rightarrow 0 \). THEN \( u = c_1 I_0(2\gamma) \).

SINCE \( u(\gamma) = 1 \) WE GET \( u = \frac{I_0(2\gamma)}{I_0(2\alpha)} \).

(B) IF \( u(\gamma) = 1 \) BUT \( u \rightarrow 0 \) AS \( \gamma \rightarrow \infty \) WE SET \( c_1 = 0 \) SINCE \( I_0(2\gamma) \) UNBOUNDED \( \gamma \rightarrow \infty \). THEN \( u(\gamma) = 1 \) YIELDS

\[ u = \frac{I_0(2\gamma)}{K_0(2\alpha)} \]