PROBLEM 1

Consider the radially symmetric diffusion problem in a 2-D disk of radius \( r > 0 \) modelled by

\[
\begin{aligned}
  \frac{\partial u}{\partial t} &= D \left( \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial u}{\partial r} \right) - u \\
  &\quad \text{in } 0 \leq r < a, \ t \geq 0 \\
  u_r (a, t) &= 0; \quad u, u_r \text{ bounded as } r \to 0 \\
  u(r, 0) &= f(r),
\end{aligned}
\]

Here \( D > 0 \) is a constant.

(i) Let \( M(t) = \int_0^a r u(r, t) \, dr \). Derive an ODE for \( M(t) \) and solve it to determine \( M(t) \) in terms of \( f(r) \).

(ii) Determine an infinite series representation for the solution to (i) Is this infinite series consistent with the result in (i)?

PROBLEM 2 (Hanging Chain Problem).

(i) Consider the Sturm-Liouville problem for \( \Phi(x) \) given by

\[
\begin{aligned}
  \Phi'' + \Phi' + \lambda \Phi &= 0 \quad \text{in } 0 < x < L \\
  \Phi(L) &= 0, \quad \Phi, \Phi' \text{ bounded as } x \to 0.
\end{aligned}
\]

Show that if we change variables with \( z = \sqrt{\alpha} x \) and \( \Psi(z) = \Phi \left( \frac{z^2}{\alpha} \right) \) that \( \Psi(z) \) satisfies Bessel's equation

\[
\Psi'' + \frac{1}{z} \Psi' + \Psi = 0
\]

provided that \( \alpha \) is chosen appropriately. Determine the eigenvalue of (i) in terms of zeroes of \( J_0(z) \).
(iii) The transverse deflection \( u(x, t) \) of a hanging chain of length \( L \) with gravity \( g \) is known to satisfy the PDE
\[
\frac{\partial^2 u}{\partial t^2} = g \left[ x u_{xx} + u_x \right], \quad 0 < x < L, \quad t > 0
\]
\( u(L, t) = 0; \quad u_x \) bounded as \( x \to 0 \)
\[
 u(x, 0) = f(x), \quad u_t(x, 0) = 0.
\]

By using (i) derive an eigenfunction expansion representation for the solution.

(iii) Give explicit formulae for the periods of oscillation of the first two terms in the infinite series representation of \( u \) in (ii). Calculate them if \( L = 1 \) metres and \( g = 9.8 \) metre/sec

(Hint: look up the first two zeroes of \( J_0(z) \).)

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**Problem 3**

Consider non radially symmetric diffusion in a disk modeled by the solution \( u(r, \theta, t) \) to
\[
\frac{\partial u}{\partial t} = D \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right), \quad 0 < r < 1, \quad 0 < \theta < 2\pi, \quad t > 0
\]
with \( D > 0 \) constant and
\[
\text{BC} \quad \left\{ \begin{array}{l}
u(1, \theta, t) = 0; \quad u, u_r \text{ bounded as } r \to 0 \\
u, u_\theta \text{ are } 2\pi \text{ periodic in } \theta
\end{array} \right.
\]
and
\[
\text{(IC)} \quad u(r, \theta, 0) = 2 \left( 1 - r^2 \right) \sin^2 \theta.
\]

(i) Find an infinite series representation for the solution

(Hint: \( 2 \sin^2 \theta = 1 - \cos 2 \theta \). Use initial condition to your advantage).

(ii) Calculate a simple approximation for \( u(r, t) \) when \( t \) is large.
**Problem 4** Consider the diffusion equation for \( u(\gamma, z, t) \) in a finite cylinder \( 0 \leq \gamma \leq \alpha, 0 \leq z \leq H \) with insulating boundaries formulated as

\[
\frac{\partial u}{\partial t} = D \left( \frac{\partial^2 u}{\partial \gamma^2} + \frac{1}{\gamma} \frac{\partial u}{\partial \gamma} + \frac{\partial^2 u}{\partial z^2} \right), \quad \text{in} \quad 0 \leq \gamma \leq \alpha, \ 0 \leq z \leq H
\]

with \( D > 0 \) constant and

\[
\begin{align*}
( BC ) & \quad u(0, z, t) = u(\alpha, z, t) = u(z, 0, t) = u(z, H, t) = 0 \\
( IC ) & \quad u(\gamma, z, 0) = f(\gamma, z)
\end{align*}
\]

We say that \( u \) is axi-symmetric since it has no \( \gamma \) dependence.

(i) Find an eigenfunction expansion solution for \( u(\gamma, z, t) \).

(ii) Calculate the steady-state defined by \( \lim_{t \to \infty} u(\gamma, z, t) \).

**Problem 5** The modified Bessel equation of order \( \eta \) with \( \eta = 0, 1, 2, \ldots \) is

\[
x^2 y'' + xy' - (x^2 + \eta^2)y = 0 \quad \text{for} \quad x > 0.
\]

(i) Show that as \( x \to 0 \) the two solutions to (X) have the form \( y \sim cx^{\pm \eta} \) for \( \eta > 0 \), thus one is bounded and the other unbounded as \( x \to 0 \). (For \( \eta = 0 \), we have \( y_1 \) constant and \( y_2 \sim \log x \) as \( x \to 0 \)).

(ii) Perform a Liouville transformation to eliminate the first derivative term in (X) by setting \( y = p(x) \). Show that for a certain choice of \( p(x) \) we get \[ \frac{\partial^2 \psi}{\partial x^2} - \left[ 1 + \frac{(\eta^2 - 1/4)}{x^2} \right] \psi = 0 \]
so that \( y = A_1 e^x + A_2 e^{-x} \) \( \forall x \to \infty \).

In this way, the general solution to (\( \star \)) has the form

\[
y = c_1 I_\nu (x) + c_2 K_\nu (x)
\]

where \( I_\nu (x) \) and \( K_\nu (x) \) are "modified Bessel functions" of the first and second kind, respectively, of order \( \nu \). They satisfy

\[
I_\nu (0) = 0 \quad \text{for} \quad \nu > 0, \quad I_\nu (0) = 1
\]

\[
I_\nu (x) \to +\infty \quad \text{as} \quad x \to +\infty
\]

and

\[
K_\nu (x) \to 0 \quad \text{as} \quad x \to +\infty
\]

\[
|K_\nu (x)| \to +\infty \quad \text{as} \quad x \to 0^+
\]

(iii) In terms of the appropriate modified Bessel function solve the following problem for \( u(\rho) \):

\( A \)

\[
u'' + \frac{\nu'}{\rho} - 4u = 0 \quad \text{in} \quad 0 < \rho < \alpha
\]

\[
u(\alpha) = 1, \quad u, u' \text{ bounded as} \quad \rho \to 0
\]

\( B \)

\[
u'' + \frac{\nu'}{\rho} - 4u = 0 \quad \text{in} \quad \rho > \alpha
\]

\[
u(\alpha) = 1, \quad u \to 0 \quad \text{as} \quad \rho \to \infty
\]