Problem 1: Consider the eigenvalue problem for $\phi(x)$:

$$\phi'' + \frac{2}{x} \phi' + \lambda \phi = 0 \quad \text{in} \quad 1 < x < a \quad \text{with} \quad a > 1$$

For the different boundary conditions given below.

(i) Suppose $\phi(1) = 0$ and $\phi'(1) = 0$. Prove in two different ways that any eigenvalue must satisfy $\lambda > 0$.

(Hint: one way is a "brute force" solution).

(ii) Now suppose that $\phi(1) = 0$ and $\phi'(1) = -h \phi(1)$ with $h > 0$. Prove that any eigenvalue satisfies $\lambda > 0$ using any method you prefer.

Problem 2: Consider the problem for finding eigenstates $\psi$ of the quantum-mechanical oscillator

$$\psi'' + \left[ \lambda - x^2 \right] \psi = 0, \quad -\infty < x < \infty$$

subject to $\psi \to 0$ as $x \to \pm \infty$ with $\int_{-\infty}^{\infty} \psi^2 \, dx < \infty$.

(i) Introduce the change of variables

$$\psi = \phi e^{-x^2/2}$$

to show that $\phi_{xx} - 2x \phi_x + (\lambda-1) \phi = 0$.

Where we impose that $\int_{-\infty}^{\infty} \phi^2 e^{-x^2} \, dx < \infty$.

(ii) Show by substituting $\phi(x) = \sum_{m=0}^{\infty} q_m X_m$, and deriving the recursion relation for $q_n$, that there are polynomial solutions for $\phi(x)$ when $\lambda - 1 = 2n$ with $n = 0, 1, 2, \ldots$. 
(iii) "Normalize" the polynomial solution by imposing that
\[ \phi(0) = 1 \quad \text{if} \quad n = 0, 2, 4, \ldots \]
\[ \phi'(0) = 1 \quad \text{if} \quad n = 1, 3, 5, \ldots \]

The resulting polynomial solutions are labeled Hermite polynomials \( H_n(x) \). Write formula for \( H_0(x), H_1(x), H_2(x), H_3(x), H_4(x), \) and \( H_5(x) \).

(Indeed we have \( H_0(x) = e^{-x^2/2} \) when \( \lambda = \lambda_0 = 2\pi + 1 \).

(iv) What is the orthogonality relation?

Problem 3 Let \( M \) be constant. Find an infinite series representation for the solution to
\[ u_t = D \left( u_{rr} + \frac{2}{r} u_r \right) + M \quad \text{in} \quad 1 < r < a, \quad t > 0 \quad (\text{with} \quad q > 1) \]
\[ u(r, 0) = f(r); \quad u(1, t) = u(a, t) = 0. \]

From this infinite series solution find an approximation that shows how the steady-state is attained as \( t \to \infty \).

Problem 4 Let \( k > 0 \) and consider the diffusion problem
\[ u_t = D \left( u_{rr} + \frac{2}{r} u_r \right) - ku \quad \text{in} \quad 0 < r < a, \quad t > 0 \quad (q > 0) \]
with \( u, u_r \) bounded as \( r \to 0 \); \( u_r(a, t) = 0 \) (no flux) and \( u(r, 0) = f(r) \).

(i) Using separation of variables find an infinite series representation for the solution.
(ii) Define the "mass" \( M(t) \) by
\[
M(t) = \int_0^a \Gamma^2 u(\Gamma, t) \, d\Gamma
\]
Calculate \( M(t) \) explicitly in terms of \( \int_0^a \Gamma^2 f(\Gamma) \, d\Gamma \).

(iii) Let \( u(\Gamma, t) = e^{-\kappa t} v(\Gamma, t) \) be a change of variables. Determine the problem for \( v(\Gamma, t) \).

Problem 5: Find in an explicit a form as you can the solution \( u(x, t) \) to:

\[
\begin{align*}
  & u_t = u_{xx}, \quad 0 < x < L, \quad t > 0 \\
  & u(0, t) = 0, \quad u(L, t) = e^{-t} \\
  & u(x, 0) = \frac{x}{L}
\end{align*}
\]

(Hint: To make homogeneous BC, write \( u(x, t) = \frac{x}{L} e^{-t} + v(x, t) \) and derive problem for \( v \), which is then solved by a generalized eigenfunction expansion.)
SOLUTION 1

(i) \[ \phi'' + \frac{2}{r} \phi' + \lambda \phi = 0 \quad \text{in } 1 < r < a \]
\[ \phi(1) = \phi(a) = 0. \]

In SL form this is \( (r^2 \phi')' + \lambda r^2 \phi = 0 \) in \( 1 < r < a \).
\[ \phi'(1) = \phi'(a) = 0. \]

Thus by SL theory \( \lambda \) is real.

(I) **METHOD 1** Multiply by \( \phi \) and integrate:
\[ \int_1^a \left( r^2 \phi' \right)' \phi \, dr + \lambda \int_1^a r^2 \phi^2 \, dr = 0. \]

Use IBP to get
\[ (x) \quad r^2 \phi' \big|_1^a = \int_1^a r^2 \phi' \, dr + \lambda \int_1^a r^2 \phi^2 \, dr = 0. \]

But \( \phi(1) = \phi(a) = 0 \) so \( \lambda \int_1^a r^2 \phi^2 \, dr = \int_1^a r^2 \phi'^2 \, dr. \)

Since \( \phi = \text{constant} \) is not an eigenfunction, we have \( \int_1^a r^2 \phi^2 \, dr > 0. \)
Thus \( \lambda > 0. \)

(II) **METHOD 2** Put \( \psi = \phi/r \) to get
\[ \left\{ \begin{array}{l}
\psi'' + \lambda \psi = 0 \quad \text{in } 1 < r < a \\
\psi(1) = 0 \quad \psi(a) = 0.
\end{array} \right. \]

Suppose \( \lambda > 0 \) then \( \psi = A \sin \left[ \sqrt{\lambda} (r-1) \right] \) satisfies ODE and \( \psi(1) = 0. \) Putting \( \psi(a) = 0 \) yields \( \sqrt{\lambda} (a-1) = n \pi \)
so \( \lambda_n = n^2 \pi^2 / (a-1)^2 \), \( n = 1, 2, \ldots \)
\[ \phi_n = \sin \left[ \frac{n \pi}{(a-1)} (r-1) \right]. \]

Suppose \( \lambda = 0 \) then \( \psi'' = 0 \) or \( \psi = A + Br. \)
Imposing \( \psi = 0 \) at \( r = 1, a \) \( \rightarrow A = B = 0 \) so \( \psi = 0 \). Thus \( \lambda = 0 \) is not an eigenvalue.
Suppose \( \lambda < 0 \) then

\[
\psi'' - (\lambda - 2) \psi = 0
\]

so \( \psi = A \sinh \left[ \sqrt{-\lambda} (r-1) \right] \) satisfies ODE and \( \psi(1) = 0 \).

Now \( \psi(2) = 0 \rightarrow A = 0 \) since \( \sinh x \neq 0 \) for \( x > 0 \).

Thus \( \lambda < 0 \rightarrow \psi \equiv 0 \), so \( \lambda < 0 \) is not an eigenvalue.

(ii) Now suppose \( \phi(1) = 0, \phi'(a) = -h \phi(a) \) with \( h > 0 \).

To prove \( \lambda > 0 \) we will method 1.

Returning to (i) we get

\[
a^2 \phi'(a) \phi(a) - \int_{1}^{a} r^2 \phi'' \, dr = -\lambda \int_{1}^{a} r^2 \phi^2 \, dr.
\]

But \( \phi'(a) = -h \phi(a) \) so that

\[
\lambda \int_{1}^{a} r^2 \phi^2 \, dr = \int_{1}^{a} r^2 \phi'^2 \, dr + \frac{h}{a^2} (\phi(a))^2
\]

\[
\rightarrow 0 \quad \rightarrow 0.
\]

So \( \lambda > 0 \).
We have \( \psi'' + [\lambda - V(x)] \psi = 0 \), for \( -\infty < x < \infty \).

\( \psi \to 0 \quad \text{as} \quad x \to \pm \infty \).

(i) Let \( \psi = \phi e^{-x^2/2} \). Then \( \psi'' = \phi'' e^{-x^2/2} - \phi e^{-x^2/2} \).

\( \psi'' = \phi'' e^{-x^2/2} - 2x \phi' e^{-x^2/2} - \phi \left[ e^{-x^2/2} - xe^{-x^2/2} \right] \).

This yields that

\[ \psi'' + (\lambda - x^2) \psi = 0 \rightarrow \phi'' - 2x \phi' + (\lambda - 1) \phi = 0. \]

This yields that

\[ \phi'' - 2x \phi' + (\lambda - 1) \phi = 0. \]

Now we want \( \int_{-\infty}^{\infty} \psi^2 \, dx < \infty \) and \( \psi \to 0 \quad \text{as} \quad x \to \pm \infty \).

Hence we look for solutions for which

\[ \begin{cases} 
\phi'' - 2x \phi' + (\lambda - 1) \phi = 0 \\
\int_{-\infty}^{\infty} \phi^2 e^{-x^2/2} \, dx < \infty \quad \text{and} \quad \phi e^{-x^2/2} \to 0 \quad \text{as} \quad x \to \pm \infty.
\end{cases} \]

Now solutions that are polynomial in \( x \) certainly would satisfy \( \int_{-\infty}^{\infty} \phi^2 e^{-x^2/2} \, dx < \infty \) and \( \phi e^{-x^2/2} \to 0 \).

(ii), (iii) We substitute \( \phi(x) = \sum_{n=0}^{\infty} a_n x^n \). We label \( \mu = \lambda - 1 \).

Then \( \phi'' - 2x \phi' + \mu \phi = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \mu \sum_{n=0}^{\infty} a_n x^n = 0. \)

Then we have

\[ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2 \sum_{m=0}^{\infty} a_m x^m + \mu \sum_{m=0}^{\infty} a_m x^m = 0. \]

Now since \( \sum_{m=0}^{\infty} a_n x^n = \sum_{m=0}^{\infty} a_m x^m \), we get

\[ \sum_{m=0}^{\infty} [a_m(1 + (m+2)(m+1) - 2m) a_m + \mu a_m] x^m = 0. \]
Setting the coefficient of \( x^m \) to zero give

\[
q_{m+2} = \frac{(2m - \mu) q_m}{(m + 2)(m + 1)}, \quad m = 0, 1, 2, \ldots \quad (*)
\]

Now to obtain a polynomial we need \( q_M = 0 \) at some \( M \).

To generate two independent solutions we choose

either \( q_0 = 1, \ q_1 = 0 \) \((I)\)

or \( q_0 = 0, \ q_1 = 1 \) \((II)\)

* for option \((I)\) we have \( q_3 = q_5 = q_7 = \ldots = 0 \) from \((*)\).

then if \( \mu = 2k \) for some \( k = 0, 2, 4, \ldots \) we conclude that

\( q_{k+2} = 0 \rightarrow q_k + 2 = q_{k+4} = \ldots = 0 \rightarrow \) so \( \phi \) is a polynomial

of degree \( k \).

* for instance if \( k = 0 \) then \( q_2 = q_4 = \ldots = 0 \) so

\( \phi = 1 \) and \( \mu = 0 \) are eigenpairs. \((I)\)

* if \( k = 2 \) then \( q_{m+2} = \frac{2(m - 2)q_m}{(m + 2)(m + 1)} q_{m+1} \), \( m = 0, 1, 2, \ldots \)

then \( q_2 = -4q_0 = -2q_2 = -2, \quad q_4 = q_6 = \ldots = 0 \).

so \( \phi = 1 - 2x^2 \), \( \mu = 4 \) is eigenpair. \((2)\)

* if \( k = 4 \) then \( q_{m+2} = \frac{2(m - 4)q_m}{(m + 2)(m + 1)} q_{m+1} \), \( m = 0, 1, 2, 3, 4 \).

we get \( q_2 = -8q_0 = -4q_2, \quad q_4 = \frac{2(2 - 4)q_2}{4 \cdot 3} = -\frac{1}{3} q_2 \)

so \( q_2 = 4, \ q_4 = 4 \)
so \( \phi = 1 - 4x^2 + 4\frac{1}{3}x^4 \) \( \mu = 8 \) is an eigenpair. \[(3)\]

For option (II) we have \( q_2 : q_4 : q_6 : \ldots = 0 \) from (k).

Then if we let \( \mu = 2k \) for \( k = 1, 3, 5, \ldots \) we get

\[
a_{m+2} = \frac{2 \left[ \frac{m - k}{m+1} \right] a_m}{(m+2)(m+1)}
\]

If \( k = 1 \) then \( q_3 : q_5 : q_7 : \ldots = 0 \) so

\[
\phi = x \text{ and } \mu = 2 \text{ are eigenpairs.} \quad (4)
\]

If \( k = 3 \) then \( q_5 : q_7 : \ldots = 0 \) and

\[
a_3 = \frac{2 \left( 1 - 3 \right) q_1}{3 \cdot 2} = -\frac{2}{3} q_1 = -\frac{2}{3}
\]

Thus \( \phi = x - \frac{2}{3} x^3 \) \( \mu = 6 \) is an eigenpair.

If \( k = 5 \) then \( q_7 : q_9 : \ldots = 0 \)

we get

\[
a_3 = \frac{2 \left[ 1 - 5 \right] q_1}{3 \cdot 2} = -\frac{4}{3}
\]

and

\[
a_5 = \frac{2 \left[ 3 - 5 \right] q_3}{5 \cdot 4} = \frac{4}{15}
\]

Thus \( \phi = x - \frac{4}{3} x^3 + \frac{4}{15} x^5 \text{ and } \mu = 10 \) is an eigenpair. \[(5)\]

Therefore combining these results for case I and II we claim that for \( k = 0, 1, 2, \ldots \)

\[
\phi(x) = H_k(x) \text{ when } \mu = \lambda - 1 = 2k \rightarrow \lambda = 2k + 1.
\]

Where \( H_k(x) \) is a polynomial of degree \( k \) in \( x \).
IN PARTICULAR, FROM (1) - (5) ABOVE WE HAVE

\[ \begin{align*}
H_0 (x) &= 1 \\
H_1 (x) &= x \\
H_2 (x) &= 1 - 2x^2 \\
H_3 (x) &= x - \frac{2}{3} x^3 \\
H_4 (x) &= 1 - 4x^2 + \frac{4}{3} x^4 \\
H_5 (x) &= x - \frac{4}{3} x^3 + \frac{4}{15} x^5.
\end{align*} \]

THE EIGENFUNCTIONS FOR \( y'' + [1 - x^2] y = 0 \)

ARE

\[ \begin{align*}
\psi_n (x) &= H_n (x) e^{-x^2/2} , \quad \Lambda_n = 2n + 1.
\end{align*} \]

NOTICE THAT \( \int_{-\infty}^{\infty} \psi_n^2 (x) e^{-x^2} \, dx \leq \infty \) SINCE \( \psi_n \) IS A POLYNOMIAL.

(iv) THE ORTHOGONALITY RELATION FOLLOW BY WRITING

\[ \psi'' - 2x \psi' + (1 - 1) \psi = 0 \rightarrow -\left(e^{-x^2} \psi'\right)' + e^{-x^2} \psi = \Lambda e^{-x^2} \psi. \]

THUS

\[ \int_{-\infty}^{\infty} \psi_n (x) \psi_m (x) e^{-x^2} \, dx = 0 \text{ if } n \neq m. \]

REMARK (CULTURAL) IT CAN BE SHOWN THAT IF WE DO NOT IMPOSE THAT \( \psi \) IS A POLYNOMIAL (BY RESTRICTING \( \lambda \) AS WE HAVE DONE), THEN WE WILL NOT HAVE

\[ \left| \int_{-\infty}^{\infty} \psi^2 e^{-x^2} \, dx \right| < \infty. \]
SOLUTION 3

WE CONSIDER
\[ U_t = D \left( \frac{U_{rr}}{r} + \frac{2}{r} U_r \right) + M \quad \text{in} \quad 1 < r < a, \quad t > 0 \]

with
\[ U(1, t) = U(a, t) = 0 \quad U(r, 0) = f(r) \]

WE FIRST FIND STEADY-STATE SOLUTION \( \overline{U}(r) \) SATISFYING
\[ \overline{U}'' + \frac{2}{r} \overline{U}' = -\frac{M}{D} \]
\[ \overline{U}(1) = \overline{U}(a) = 0. \]

THE PARTICULAR SOLUTION \( U_p \): \( b r^2 \) \text{ so } \( 2b + 4b = -\frac{M}{D} \) OR \( b = -\frac{M}{6D} \).

THUS
\[ \overline{U}(r) = -\frac{M}{6D} \frac{r^2}{r} + \frac{C_0}{r} + C_1. \quad \left( 1 \right) \]

IMPOSING \( \overline{U}(1) = \overline{U}(a) = 0 \) YIE (D)
\[ C_0 + C_1 = \frac{M}{6D} \]
\[ \frac{C_0}{a} = \frac{Ma^2}{6D} \]

THE SOLUTION IS
\[ C_0 = -\frac{M}{6D} a (1 + a) \quad \left( 2 \right) \]
\[ C_1 = \frac{M}{6D} \left[ 1 + a (1 + a) \right] \]

NOW SUBTRACT OFF THE STEADY-STATE AS
\[ U(r, t) = \overline{U}(r) + V(r, t). \]

THEN
\[ \left\{ \begin{array}{l}
V_t = D \left( \frac{V_{rr}}{r} + \frac{2}{r} V_r \right) \quad \text{in} \quad 1 < r < a, \quad t > 0 \\
V(1, t) = V(a, t) = 0 \\
V(r, 0) = F(r) - \overline{U}(r)
\end{array} \right. \quad \left( 3 \right) \]
Now separate variables \( V = \bar{\Phi} (r) T (t) \).

Then,
\[
\frac{T'}{DT} = \frac{\bar{\Phi}'' + \frac{2}{r} \bar{\Phi}'}{\bar{\Phi}} = -\lambda.
\]

So
\[
\bar{\Phi}'' + \frac{2}{r} \bar{\Phi}' + \lambda \bar{\Phi} = 0 \quad \text{in} \quad 1 < r < a,
\]
\[
\bar{\Phi} (1) = \bar{\Phi} (a) = 0.
\]

From Problem 1 we have \( \bar{\Phi} = \frac{\sin \left[ \sqrt{\lambda} (r-1) \right]}{r} \)

with \( \sqrt{\lambda} (a-1) = n \pi \) so \( \lambda_n = \frac{n^2 \pi^2}{(a-1)^2} \), \( n = 1, 2, \ldots \)

Then \( \frac{T'}{DT} = -\lambda \) gives \( T = e^{-\lambda_n D \tau} \).

By superposition
\[
V (r, t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n D \tau} \bar{\Phi}_n (r)
\]

where \( \bar{\Phi}_n (r) = \frac{\sin \left[ \frac{n \pi}{(a-1)} (r-1) \right]}{r} \) and \( \lambda_n = \frac{n^2 \pi^2}{(a-1)^2} \) \( (3) \)

Finally
\[
V (r, 0) = -\bar{U} (r) + F (r) = \sum_{n=1}^{\infty} C_n \bar{\Phi}_n (r)
\]

Since
\[
\int_{1}^{a} r^2 \bar{\Phi}_n \bar{\Phi}_m \, dr = 0 \quad \text{for} \quad n \neq m
\]

we get
\[
C_n = \frac{\int_{1}^{a} [F (r) - U (r)] \bar{\Phi}_n (r) r^2 \, dr}{\int_{1}^{a} r^2 \bar{\Phi}_n^2 (r) \, dr}
\]
In this way we have the infinite series representation

$$U(\gamma, t) = \tilde{U}(\gamma) + \sum_{n=1}^{\infty} C_n e^{-\lambda_n D \frac{t}{r^2}} \tilde{\Phi}_n(\gamma)$$

where the steady state $\tilde{U}(\gamma)$ given by (1) and (2), the eigenvalues are given in (3) and the $C_n$ are given by (4).

As $t \to \infty$ we have upon keeping the $n=1$ term that

$$U(\gamma, t) \approx \tilde{U}(\gamma) + C_1 e^{-\lambda_1 D \frac{t}{r^2}} \tilde{\Phi}_1(\gamma)$$

which gives the approach to the steady state.

Remark: In (4) we can calculate the denominator

$$\int_{1}^{a} r^2 \tilde{\Phi}_1^2(\gamma) \, dr = \int_{1}^{a} \sin^2 \left( \frac{\pi (r-1)}{a-1} \right) \, dr = \frac{1}{2} \int_{1}^{a} \left[ 1 - \cos \left( \frac{2\pi (r-1)}{a-1} \right) \right] \, dr$$

$$= \frac{a}{2} - \frac{1}{2} \frac{(a-1)}{2\pi} \sin \left( \frac{2\pi (r-1)}{a-1} \right) \bigg|_{1}^{a} = \frac{a}{2}.$$

Thus

$$C_1 = \frac{2}{a} \int_{1}^{a} [\tilde{F}(\gamma) - \tilde{U}(\gamma)] r^2 \tilde{\Phi}_1(\gamma) \, dr.$$
Solution 4

We consider
\[ u_t = D \left( \frac{u_{rr} + \frac{2}{r} u_r}{} \right) - \kappa u \quad \text{in} \quad 0 < r < a, \quad t > 0 \]
\[ u, u_r \text{ bounded at} \quad r \to 0; \quad u_r(a, t) = 0 \]
\[ u(\xi, 0) = f(\xi). \]

(i) We separate variables \( u(\xi, t) = \Phi(\xi)T(t) \)

We substitute
\[ T' \Phi = D T \left[ \Phi'' + \frac{2}{r} \Phi' \right] - \kappa \Phi T. \]

Thus,
\[ \frac{1}{D} \left( \frac{T'}{T} + \kappa \right) = \frac{\Phi'' + \frac{2}{r} \Phi'}{\Phi} = -\lambda. \]

So,
\[ \frac{T'}{T} + \kappa = -\lambda D \]

or \( T' = -(\kappa + \lambda D)T. \)

Thus,
\[ \left\{ \begin{array}{l}
\Phi'' + \frac{2}{r} \Phi' + \lambda \Phi = 0 \quad \text{in} \quad 0 < r < a \\
\Phi, \Phi' \text{ bounded at} \quad r \to 0 \\
\Phi'(a) = 0
\end{array} \right\} \quad \rightarrow \quad (r^2 \Phi')' + \lambda r^2 \Phi = 0 \]

The solution is \( \Phi = \sin \left[ \sqrt{\lambda} \right] \) with \( \lambda > 0 \)

\[ \Phi = 1 \quad \text{if} \quad \lambda = 0. \]

Thus if \( \lambda > 0, \quad \Phi'(a) = 0 \rightarrow \sqrt{\lambda} \left( \cos \left( \sqrt{\lambda} a \right) - \frac{\sin \left( \sqrt{\lambda} a \right)}{a} \right) = 0 \]

which yields \( \tan \left( \sqrt{\lambda} a \right) = \sqrt{\lambda} a. \)

There is an infinite number \( Z > 0 \) with \( D = 1, 2, ... \) to \( \tan (Z) = Z \).

The eigenvalues are
\[ \lambda_n = \frac{Z_n^2}{a^2}, \quad n = 1, 2, ... \]

Thus,
\[ \Phi_0 = 1, \quad T_0 = e^{-\kappa t} \]
\[ \Phi_n = \sin \left( \sqrt{\lambda_n} r \right) / r, \quad T_n = e^{-(\kappa + \lambda_n) t}, \quad n = 1, ... \]
\[ u(r, t) = c_0 e^{-\lambda \tau} + \sum_{n=1}^{\infty} c_n e^{-\lambda \tau - \lambda n^2 \tau} \Phi_n(r). \]  

Now \( f(r) = u(r, 0) = c_0 + \sum_{n=1}^{\infty} c_n \Phi_n(r). \)

Using orthogonality \( \int_0^a \tau^2 \Phi_m \Phi_n \, d\tau = 0 \quad m \neq n \quad m, n = 3, 6, 9, \ldots \)

Then
\[ \int_0^a \tau^2 f(r) \, d\tau = c_0 \int_0^a \tau^2 \, d\tau + \sum_{n=1}^{\infty} \left( \int_0^a \tau^2 \Phi_n(r) \, d\tau \right) \]

So
\[ c_0 = \frac{2}{a^3} \int_0^a \tau^2 f(r) \, d\tau. \]  

Similarly
\[ c_n = \frac{1}{\int_0^a \tau^2 \Phi_n^2(r) \, d\tau} \left( \int_0^a \tau^2 \Phi_n(r) \, d\tau \right). \]  

The solution is given by (3) with coefficients given in (4) and (4)_2.

In (3) we have \( \lambda n^2 = \lambda n^2 a^2 \) where \( \lambda n^2, n = 1, 2, \ldots \) are roots of \( \tan \tau = \tau. \)

(i) We use (3), multiply by \( \tau^2 \) and integrate
\[ \int_0^a \tau^2 u(r, t) \, d\tau = e^{-\lambda \tau} c_0 \int_0^a \tau^2 \, d\tau + \sum_{n=1}^{\infty} c_n e^{-\lambda \tau - \lambda n^2 \tau} \int_0^a \tau^2 \Phi_n(r) \, d\tau \]

So
\[ M(t) = \frac{a^3}{2} e^{-\lambda \tau} c_0 \]

where \( M(t) \equiv \int_0^a \tau^2 u(r, t) \, d\tau. \)

\[ M(t) = \frac{a^3}{2} e^{-\lambda \tau} \left( \int_0^a \tau^2 f(r) \, d\tau \right) = e^{-\lambda \tau} \left( \int_0^a \tau^2 f(r) \, d\tau \right) = M(0) \]

Thus
\[ M(t) = M(0) e^{-\lambda \tau} \quad \text{with} \quad M(0) = \int_0^a \tau^2 f(r) \, d\tau. \]
Alternatively, we could have derived this by integrating the PDE directly.

\[
\int_0^q \frac{\partial}{\partial t} U_r \, r^2 \, dr = D \int_0^q r^2 (U_{rr} + \frac{2}{r} U_r) \, dr - \kappa \int_0^q r^2 U \, dr
\]

\[
\frac{d}{dt} \left|_0^q U_r \, r^2 \, dr = D \int_0^q (r^2 U_r) \, dr - \kappa \int_0^q r^2 U \, dr
\]

\[
= D \left( r^2 U_r \right) \bigg|_0^q - \kappa \int_0^q r^2 U \, dr
\]

DEFINING \[ M(t) = \int_0^q U(r,t) \, r^2 \, dr \]

so

\[
\frac{dM}{dt} = -\kappa M \quad \text{with} \quad M(0) = \int_0^q U(r,0) \, r^2 \, dr
\]

\[
M(0) = \int_0^q r^2 f(r) \, dr
\]

then

\[
M(t) = M(0) \, e^{-\kappa t}
\]

\[
M(t) = e^{-\kappa t} \int_0^q r^2 f(r) \, dr.
\]

(iii) Let \( U(r,t) = e^{-\kappa t} V(r,t) \) into the PDE. This gives,

\[
-\kappa e^{-\kappa t} V + e^{-\kappa t} V_t = e^{-\kappa t} D \left( V_{rr} + \frac{2}{r} V_r \right) - \kappa e^{-\kappa t} V
\]

then

\[
V_t = D \left( V_{rr} + \frac{2}{r} V_r \right) \quad \text{in} \quad 0 \leq r \leq q, \quad t > 0
\]

\[
V, V_r \hspace{1em} \text{bounded at} \quad r \to 0; \hspace{1em} V_r = 0 \quad \text{on} \quad r = q
\]

\[
V(r,0) = U(r,0) = f(r)
\]
SOLUTION 5

We want to solve

\[ U_t = U_{xx}, \quad 0 < x < L, \quad t > 0 \]

with \( U(0, t) = 0, \quad U(L, t) = e^{-t} \)

and \( U(x, 0) = x/L. \)

We let \( U(x, t) \) satisfy \( U_{xx} = 0 \) so \( U = Ax + B. \)

Now impose \( U = 0 \) at \( x = 0 \) \( \Rightarrow B = 0 \)

\( U = e^{-t} \) at \( x = L \) \( \Rightarrow e^{-t} = AL \) \( \Rightarrow A = e^{-t}/L. \)

Thus \( U = x/L e^{-t}. \)

We now put \( U(x, t) = U + V(x, t) \) so that

\[ V_t = V_{xx} - (U_t - U_{xx}) \]

\[ V = 0 \] at \( x = 0, L \)

\[ V(x, 0) = U(x, 0) - U(x, 0) = x/L - x/L = 0. \]

So with \( U_{xx} = 0 \), \( U_t = -x/L e^{-t} \) we have that

\[
\begin{align*}
    V_t &= V_{xx} + x/L e^{-t}, \quad 0 < x < L, \quad t > 0 \\
    V(0, t) &= 0 \\
    V(L, t) &= 0 \\
    V(x, 0) &= 0
\end{align*}
\]

We consider the homogeneous problem with \( V = \Phi T \) so

\[
\begin{align*}
    \Phi_{xx} + \lambda \Phi &= 0, \quad 0 < x < L \\
    \Phi &= 0 \] at \( x = 0, L \)
\]

Thus \( \Phi_n(x) = \sin \left( \frac{n \pi x}{L} \right) \) and \( \lambda = \lambda_n = \frac{n^2 \pi^2}{L^2}. \)
We then expand
\[ \psi(x, t) = \sum_{\eta=1}^{\infty} B_{\eta} e^{-t} \Phi_{\eta}(x). \]  

Substitute into (1) to get
\[ \sum_{\eta=1}^{\infty} B_{\eta} \Phi_{\eta}^\prime = \sum_{\eta=1}^{\infty} B_{\eta} \Phi_{\eta}^\prime + e^{-t} x/L. \]

Now expand
\[ x = \sum_{\eta=1}^{\infty} C_{\eta} \sin \left( \frac{\eta \pi x}{L} \right) \]
so
\[ C_{\eta} = \int_{0}^{L} \sin \left( \frac{\eta \pi x}{L} \right) dx = \frac{2}{L} \int_{0}^{L} x \sin \left( \frac{\eta \pi x}{L} \right) dx \]

Now integrate by parts:
\[ \int_{0}^{L} x \sin \left( \frac{\eta \pi x}{L} \right) dx = -\frac{Lx}{\eta \pi} \cos \left( \frac{\eta \pi x}{L} \right) \bigg|_{0}^{L} + \frac{L}{\eta \pi} \int_{0}^{L} \cos \left( \frac{\eta \pi x}{L} \right) dx \]
\[ = -\frac{L^2}{\eta \pi} \sin \left( \frac{\eta \pi}{L} \right) + \frac{L^2}{\eta^2 \pi^2} \sin \left( \frac{\eta \pi x}{L} \right) \bigg|_{0}^{L} \]
\[ = \frac{L^2}{\eta \pi} \left( -1 \right)^{n}. \]

Thus, we have
\[ x = -\sum_{\eta=1}^{\infty} \frac{2L}{\eta \pi} \left( -1 \right)^{n} \sin \left( \frac{\eta \pi x}{L} \right) \]

Substitute into (4) to obtain:
\[ \sum_{\eta=1}^{\infty} B_{\eta} \Phi_{\eta}^\prime = \sum_{\eta=1}^{\infty} B_{\eta} \Phi_{\eta}^\prime + \frac{-2L}{\pi} \left( -\frac{2L}{\pi} \right) \sum_{\eta=1}^{\infty} \left( -1 \right)^{n} \Phi_{\eta}(x) \]

But \[ \Phi_{\eta}^\prime = -\Lambda_{\eta} \Phi_{\eta} \]
so that
\[ \sum_{\eta=1}^{\infty} \left( B_{\eta}^\prime + \Lambda_{\eta} B_{\eta} \right) \Phi_{\eta} = -\frac{2L}{\pi} \sum_{\eta=1}^{\infty} \left( -1 \right)^{n} \Phi_{\eta}(x) \]

Now by orthogonality, we conclude that
\[ B_{\eta}^\prime + \Lambda_{\eta} B_{\eta} = -\frac{2L}{\pi} \left( -1 \right)^{n}. \]
\[ V(x, t) = \sum_{n=1}^{\infty} B_n(0) \phi_n(x) = 0. \text{ Thus, } B_n(0) = 0, \ n = 1, 2, \ldots. \]

So we have
\[ B_n' + \lambda_n B_n = -\gamma_n e^{-t} \quad \text{with} \quad \gamma_n = \frac{2(-1)^n}{n\pi}. \]

With initial condition \[ B_n(0) = 0. \]

Now the integrating factor is \[ e^{\lambda_n t} \] so that
\[ (B_n e^{\lambda_n t})' = -\gamma_n (e^{(\lambda_n-1)t}). \]

Thus
\[ B_n e^{\lambda_n t} = -\gamma_n e^{(\lambda_n-1)t} + C. \]

But \[ B_n(0) = 0 \rightarrow C = \frac{\gamma_n}{\lambda_n - 1}. \]

Thus, get
\[ B_n = \frac{\gamma_n}{\lambda_n - 1} \left[ e^{-\lambda_n t} - e^{-t} \right]. \]

Thus we obtain that
\[ V(x, t) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi [\lambda_n - 1]} \left( e^{-\lambda_n t} - e^{-t} \right) \sin \left( \frac{n\pi x}{L} \right). \]

In terms of \( u(x, t) \) we have
\[ u(x, t) = \frac{x}{L} e^{-t} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi [\lambda_n - 1]} \left( e^{-\lambda_n t} - e^{-t} \right) \sin \left( \frac{n\pi x}{L} \right). \]

Where \[ \lambda_n = n^2\pi^2/L^2. \]