**Problem 1**

Consider the eigenvalue problem for $\phi(r)$:

$$\phi'' + \frac{2}{r} \phi' + \lambda \phi = 0 \quad \text{in } r < a \text{ with } a > 1$$

For the different boundary conditions given below:

(i) Suppose $\phi(1) = 0$ and $\phi(a) = 0$. Prove in two different ways that any eigenvalue must satisfy $\lambda > 0$.

(Hint: one way is a "brute force" solution.)

(ii) Now suppose that $\phi(1) = 0$ and $\phi'(a) = -h \phi(a)$ with $h > 0$. Prove that any eigenvalue satisfies $\lambda > 0$ using any method you prefer.

**Problem 2**

Consider the problem for finding eigenstates $\lambda$ of the quantum-mechanical oscillator:

$$\psi'' + \left(2 - x^2\right) \psi = 0, \quad -\infty < x < \infty$$

Subject to $\psi \to 0$ as $x \to \pm \infty$ with $\int_{-\infty}^{\infty} \psi^2 \, dx < \infty$.

(i) Introduce the change of variables $\psi = \phi e^{-x^2/2}$ to show that $\phi_{xx} - 2x \phi_x + (\lambda - 1) \phi = 0$.

Where we impose that $\int_{-\infty}^{\infty} \phi^2 \, e^{-x^2} \, dx < \infty$.

(ii) Show by substituting $\phi(x) = \sum m=0 \infty q_m x^m$ and deriving the recursion relation for $q_m$, that there are polynomial solutions for $\phi(x)$ when $\lambda = 2n$ with $n = 0, 1, 2, \ldots$.
(iii) "Normalize" the polynomial solution by imposing that
\[ \phi(0) = 1 \quad \text{if} \quad \eta = 0, 2, 4, \ldots \]
\[ \phi'(0) = 1 \quad \text{if} \quad \eta = 1, 3, 5, \ldots \]

The resulting polynomial solution are labelled Hermite polynomials
\( H_\eta(x) \). Write formulas for \( H_0(x), H_1(x), H_2(x), H_3(x), H_4(x), \) and \( H_5(x) \).

Indeed, we have \( H_\eta(x) = H_0(x) e^{-x^2/2} \) when \( \Delta = \Delta_0 = 2 \Delta + 1 \).

(iv) What is the orthogonality relation?

**Problem 3**
Let \( M \) be constant. Find an infinite series
representation for the solution to
\[ \begin{align*}
  U_t &= D \left( U_{rr} + \frac{2}{r} U_r \right) + M \quad \text{in} \quad 0 < r < a, \quad t > 0 \quad (\text{with} \quad q > 1) \\
  U(r, 0) &= f(r); \quad U(1, t) = U(a, t) = 0.
\end{align*} \]

From this infinite series solution find an approximation that shows how the steady-state is attained as \( t \to \infty \).

**Problem 4**
Let \( q > 0 \) and consider the diffusion problem
\[ \begin{align*}
  U_t &= D \left( U_{rr} + \frac{2}{r} U_r \right) - \kappa U \quad \text{in} \quad 0 < r < a, \quad t > 0 \quad (q > 0) \\
  \quad \text{with} \quad U, U_r \text{ bounded as} \quad r \to 0; \quad U_r(a, t) = 0 \quad (\text{no flux}) \\
  \quad \text{and} \quad U(r, 0) = f(r).
\end{align*} \]

(i) Using separation of variables find an infinite series representation for the solution.
(ii) Define the "major" $M(t)$ by 

$$M(t) = \int_0^a r^2 V(r, t) \, dr.$$ 

Calculate $M(t)$ explicitly in terms of 

$$\int_0^a r \, F(r) \, dr.$$ 

(iii) Let $V(r, t) = e^{-kt} U(r, t)$ be a change of variables. Determine the problem for $V(r, t)$.

\[\text{Problem 5: Find in an explicit a form as you can the solution } U(x, t) \text{ to:}\]

$$U_t = U_{xx}, \quad 0 < x < L, \quad t > 0$$

with 

$$U(0, t) = 0, \quad U(L, t) = e^{-t}$$

and 

$$U(x, 0) = \frac{x}{L}$$

(Hint: To make homogeneous BC write 

$$U(x, t) = \frac{x}{L} e^{-t} + V(x, t)$$

and derive problem for $V$, which is then solved by a generalized eigenfunction expansion.)