Problem 1: Put the following two problems in Sturm-Liouville form, identify the weight function \( w(x) \) and calculate the eigenvalues and eigenfunction explicitly. Also, what is the orthogonality relation for the eigenfunction?

(i) \( \phi_{xx} + \phi_x + \lambda \phi = 0 \) in \( 0 \leq x \leq 1 \); \( \phi(0) = \phi(1) = 0 \)

(ii) \( x \phi'' + \phi' + \lambda \phi/x = 0 \) in \( 1 \leq x \leq e \); \( \phi(1) = 0, \phi'(e) = 0, e = 2.7128 \)

Problem 2: Consider the eigenvalue problem for \( \phi(\Gamma) \):

\[
\phi'' + \frac{2}{\Gamma} \phi' + \lambda \phi = 0 \quad \text{in} \quad 1 \leq \Gamma \leq a \text{ with } a > 1
\]

(i) Suppose \( \phi(1) = 0 \) and \( \phi(a) = 0 \). Prove in two distinct ways that any eigenvalue \( \lambda \) must satisfy \( \lambda > 0 \). (Hint: One way is finding the explicit solution).

(ii) Now suppose that \( \phi(1) = 0 \), \( \phi'(a) = -\h \phi(a) \) with \( \h > 0 \). Prove that any eigenvalue \( \lambda \) must satisfy \( \lambda > 0 \) using any method you prefer.

Problem 3: Consider the problem of finding the eigenstates \( \phi \) for

\[
\psi'' + \left( \frac{\lambda}{x} - x^2 \right) \psi = 0 \quad \text{in} \quad -\infty < x < \infty
\]

with \( \psi \rightarrow 0 \) as \( x \rightarrow \pm \infty \) with \( \int_{-\infty}^{\infty} \psi^2 \, dx < \infty \).

(i) Introduce \( \phi(x) \) by \( \psi = \phi e^{-x^2/2} \) to convert (\#) to

\[
\phi_{xx} - 2x \phi_x + (\lambda - 1) \phi = 0 \quad \text{with} \quad \int_{-\infty}^{\infty} \phi^2 e^{-x^2} \, dx < \infty.
\]
(ii) By substituting \( \phi(x) = \sum_{m=0}^{\infty} q_m x^m \) into (+) derive the recursion relation for \( q_m \) and show that there are polynomial solutions for \( \phi(x) \) when \( \lambda = 1 + 2n \) where \( n = 0, 1, 2, \ldots \).

For such solutions, \( \int_{-\infty}^{\infty} x^2 e^{-x^2} dx < \infty \).

"Normalize" these polynomial solutions by imposing that

\( \phi(0) = 1, \ \phi'(0) = 0, \ \text{for} \ n = 0, 2, 4, 6, \ldots \)

and \( \phi(0) = 0, \ \phi'(0) = 1, \ \text{for} \ n = 1, 3, 5, \ldots \).

Label the polynomial \( \phi_n(x) \).

Derive explicit formulae for \( \phi_n(x) \) for \( n = 0, 1, 2, 3, 4, 5 \).

Remark: These solutions \( \phi_n(x) \), which exist when \( \lambda = \lambda_n = (2n) \)

are proportional to the classic Hermite polynomial arising in the study of quantum-mechanical oscillators. There

\( \psi_n(x) = e^{-x^2/2} \phi_n(x) \).

(iii) What is the orthogonality relation for the \( \phi_n(x) \)?

(iv) The quantum mechanical oscillator problem is to find eigenstates \( E_n, \psi_n(x) \) of the Schrödinger equation

\[
\begin{align*}
(x^2) - \frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{1}{2} \kappa x^2 \psi &= E \psi, \\
-\infty < x < \infty \quad \text{with} \quad \int_{-\infty}^{\infty} \psi^2 dx < \infty.
\end{align*}
\]

By rescaling \( x \) to map (iv) into (i), and then using (ii), write explicit formulae for the first three excited states \( E_n, \psi_n(x) \) for \( n = 0, 1, 2 \).

Note: In (iv) \( \hbar, \kappa, \alpha \) are positive constants.
Problem 4 (Separation of Variables) Find the separation of variables solution \( u(r, t) \) for heat flow within two concentric spheres, modelled by (for \( D > 0, \ k > 0 \) constants)

\[
\frac{\partial u}{\partial t} = D \left[ \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} \right] - ku \quad \text{in} \quad 1 \leq r \leq 2
\]

with boundary condition \( u(1, t) = u(2, t) = 0 \)

and initial condition \( u(r, 0) = f(r) \).

(Hint: Problem 2 is useful here). (Here \( D \) and \( k \) positive constants.)

Problem 5 (Separation of Variables) Now consider diffusion in the full sphere of radius \( a \) with an impermeable wall at \( r = a \), modelled by

\[
\frac{\partial u}{\partial t} = D \left[ \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} \right] - ku \quad \text{in} \quad 0 < r < a, \quad t > 0
\]

\( u, u_r \) bounded \( \rightarrow 0 \) \( r \rightarrow 0 \); \( u_r(a, t) = 0 \) (no flux)

and initial condition \( u(r, 0) = f(r) \). Here \( a > 0 \), and \( k, D \) positive constants.

(i) Using separation of variables, find an infinite series representation for \( u(r, t) \). (Hint: be careful! \( \lambda = 0 \) is an eigenvalue of the Sturm-Liouville problem).

(ii) By integrating the PDE directly over the domain, derive and solve a simple ODE for the "mass" \( M(t) \) defined by \( M(t) = \int_0^a \int_0^a u(r, t) \, dr \). Is this result consistent with integrating your separation of variables solution?