Lecture 22: Interpreting D’Alembert’s Solution in Space-Time: characteristics, regions of influence and domains of dependence

(Compiled 25 October 2012)

In this lecture we discuss the physical interpretation of the D’Alembert solution in terms of space-time plots. In particular we identify the left and right-moving characteristics as well as the domain of dependence of a given point \((x_0, t_0)\) in space-time and the region of influence of a given initial value specified at the point \(x_1, 0\). We discuss the evolution of a few simple pulses and track the regions in space-time that are carved out by the intersecting characteristics.

**Key Concepts:** The one dimensional Wave Equation; D’Alembert’s Solution, Characteristics, Domain of Dependence, Region of Influence.

**Reference Section:** Boyce and Di Prima Section 10.7

22 Space-Time Interpretation of D’Alembert’s Solution

In this lecture we discuss the interpretation of D’Alembert’s solution

\[
    u(x, t) = \frac{1}{2} [u_0(x - ct) + u_0(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} v_0(s) \, ds
\]  

(22.1)

to the one dimensional wave equation

\[
    \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]  

(22.2)

22.1 Characteristics

In the \(x - t\) plane the lines

\[
    x - ct = x_0 \quad \text{and} \quad x + ct = x_0
\]  

(22.3)

are called the *characteristics* that emanate from the point \((x_0, 0)\) in space-time (see figure 1). Characteristics are the lines (or curves of more general hyperbolic problems) along which information is propagated by the equation. To interpret the characteristic lines in the \(x - t\) plane, it is useful to rewrite the characteristic equations in the form

\[
    \begin{align*}
    x - ct &= x_0 & \Rightarrow & & t &= \frac{1}{c} x - \frac{1}{c} x_0 \\
    x + ct &= x_0 & \Rightarrow & & t &= -\frac{1}{c} x + \frac{1}{c} x_0
    \end{align*}
\]  

(22.4)
22.2 Region of Influence and Domain of Dependence

Region of influence: The lines $x + ct = x_1$ and $x - ct = x_1$ bound the region of influence of the function values at the initial point $(x_1, 0)$. Thus all the solution values $u(x, t)$ are influenced by the value at the point $(x_1, 0)$.

Domain of Dependence: The lines $x = x_0 - ct_0$ and $x = x_0 + ct_0$ that pass through the point $(x_0, t_0)$ bound the domain of dependence. Thus the solution $u(x_0, t_0)$ depends on all the function values in the shaded region.
Example 22.1 A Rectangular pulse Pulse:

\[ u(x, 0) = \begin{cases} 
1 & |x| < 1 \\
0 & |x| > 1 
\end{cases} \]  

(22.5)

\[ u(x, t) = \frac{1}{2} [u_0(x - ct) + u_0(x + ct)] \]  

(22.6)

Let \( c = 1 \).

\( t = \frac{1}{2} \):

\[ x_r - \frac{1}{2} = 1 \quad \Rightarrow \quad x_r = \frac{3}{2} \quad x_R + \frac{1}{2} = 1 \quad \Rightarrow \quad x_R = \frac{1}{2} \]  

(22.7)

\[ x_\ell - \frac{1}{2} = -1 \quad \Rightarrow \quad x_\ell = -\frac{1}{2} \quad x_L + \frac{1}{2} = -1 \quad \Rightarrow \quad x_L = -\frac{3}{2} \]

\( t = 1 \):

\[ x_r - 1 = 1 \quad \Rightarrow \quad x_r = 2 \quad x_R + 1 = 1 \quad \Rightarrow \quad x_R = 0 \]  

\[ x_\ell - 1 = -1 \quad \Rightarrow \quad x_\ell = 0 \quad x_L + 1 = -1 \quad \Rightarrow \quad x_L = -2 \]  

(22.8)

\( t = 2 \):

\[ x_r - 2 = 1 \quad \Rightarrow \quad x_r = 3 \quad x_R + 2 = 1 \quad \Rightarrow \quad x_R = -1 \]  

\[ x_\ell - 2 = -1 \quad \Rightarrow \quad x_\ell = 1 \quad x_L + 2 = -1 \quad \Rightarrow \quad x_L = -3 \]  

(22.9)
Figure 3. Top: Space-time representation of the regions in which the solution takes on different values for the rectangular pulse (22.5). Bottom: Cross sections of the solution $u(x, t)$ at times $t = 0$, $1/2c$, $1/c$, and $t > 1/c$. 