PROBLEM 1  
Find the explicit solution to the heat equation

\[ U_t = 100 \, U_{xx}, \quad 0 < x < 1, \quad t > 0 \]

with

\[ U(0, t) = U(1, t) = 0 \]

and initial condition

\[ U(x, 0) = \sin(2\pi x) - 2\sin(5\pi x). \]

PROBLEM 2  
Find the explicit solution to the heat equation

\[ U_t = \frac{1}{4} \, U_{xx}, \quad 0 < x < 2, \quad t > 0 \]

with

\[ U(0, t) = U(2, t) = 0 \]

and initial condition

\[ U(x, 0) = 2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x). \]

PROBLEM 3 (i) Suppose that \( U(x, t) \) satisfies

\[ U_t = \kappa \, U_{xx} - bu \]

for some \( \kappa > 0 \) and \( b > 0 \), constants.

Show that if we write \( U(x, t) = e^{\alpha t} \, \psi(x, t) \) for some \( \alpha \) to be found, that we can get that \( \psi \) satisfies

\[ \psi_t = \kappa \, \psi_{xx}. \]

PROBLEM 4  
Use either the result in Problem 3 or standard separation of variables to find an infinite series representation for the solution to

\[ U_t = \kappa \, U_{xx} - U \]

in \( 0 < x < L, \quad t > 0 \)

with

\[ U(0, t) = U(L, t) = 0 \]

and initial condition

\[ U(x, 0) = x(L-x). \]
Problem 5: Consider the following heat equation with insulated boundary conditions:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \kappa \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0, \quad (\kappa > 0 \text{ constant}) \\
&
\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t) = 0 \\
&
\frac{u(x, 0)}{u(x, 0)} = 1 + 2 \cos^2 (\pi x) + 4 \cos^2 (2\pi x)
\end{align*}
\]

(i) Using the separation of variables technique find the form of an infinite series representation for the solution to (x).

(ii) Find the coefficient in the Fourier cosine series explicitly by using \( u(x, 0) \).

(Hint: you need a standard identity for \( \cos^2 (\alpha) \)).

(iii) Calculate \( \lim_{t \to \infty} u(x, t) = u_0 \). This is the steady-state temperature.

(iv) Define \( M(t) = \int_0^1 u(x, t) \, dx \).

By integrating the PDE over \( 0 < x < 1 \) show that \( M(t) = \int_0^1 u(x, 0) \, dx \) for all \( t \).

Thus, \( M(t) \) is the same for all time \( t \).