**Problem 1**

Find series expansions for two linearly independent solutions of

\[
\frac{y'' + (1-x)y'}{x} - \frac{y}{x^2} = 0 \quad \text{about } x = 0.
\]

Show that although the roots of the indicial equation differ by an integer, we can nevertheless find two Frobenius series solutions.

**Problem 2**

Consider Bessel's equation of order \(1/2\) given by

\[
(x^2 y'' + xy' + (x^2 - 1/4))y = 0.
\]

(i) Find Frobenius series expansion for two linearly independent solutions of the form \(y = \sum_{n=0}^{\infty} a_n x^{n+\rho}\). Show that although the roots of the indicial equation differ by an integer, we can find \(y_1, y_2\).

(ii) Show how to find the solutions \(y_1\) and \(y_2\) by substituting \(y = x^{-1/2}v\) into (3) and deriving the equation for \(v\).

**Problem 3**

Find the first three non-zero terms in a Frobenius series solution about \(x = 0\) for the solution to the spherical Bessel's equation

\[
x^2 y'' + 2xy' + (x^2 - 5/16)y = 0
\]

with \(y(0) = 0\), \(\lim_{x \to 0^+} x^{3/4} y' = 1/4\).

(Hint: use the initial values to identify which of the two linearly independent solutions is needed.)
Problem 4 Find the form of two linearly independent Frobenius series expansions about the point $x = 0$ for

\[
y^{(1)} + \frac{1}{\sin x} y' + \frac{(1 - x)}{x^2} y = 0
\]

such that the series are real-valued on the positive real axis $x > 0$. Do not calculate the coefficients in these series expansions.

Problem 5 In the notes, one solution of Bessel's equation of order zero

\[
x^2 y^{(1)} + xy' + x^2 y = 0
\]

is found to be

\[
J_0(x) = 1 - \frac{1}{2^2} x^2 + \frac{1}{2^2 4^2} x^4 - \frac{1}{2^2 4^2 6^2} x^6 + \ldots = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m (m!)^2} x^{2m}.
\]

Find a second linearly independent solution of the form

\[\hat{y}_2 = J_0(x) J_0(x) + \hat{y}_1(x)\]

as follows:

(i) Show that if $\hat{y}_2$ is to solve Bessel's equation then $\hat{y}_2$ must solve

\[\ast \quad x^2 \hat{y}^{(1)} + x \hat{y}' + x^2 \hat{y} = -2x J_0'(x).
\]

(ii) Find the first three non-zero terms of a series expansion $\hat{y} = \sum_{n=1}^{\infty} b_n x^n$ for this ode for $\hat{y}$.

(Hint: put series for $J_0(x)$ into the right-hand side of $\ast$.)
Problem 1

\[ y'' + \left(1 - x\right) y' - \frac{y}{x} = 0. \]

Multiply by \( x^2 \):

\[ x^2 y'' + \left[ x - x^2 \right] y' - y = 0. \]

When \( x = 0 \) is a regular point,

Now put \( y = \sum_{n=0}^{\infty} a_n x^{n+\Gamma} \). Then

\[
\sum_{n=0}^{\infty} a_n (n+\Gamma)(n+\Gamma-1) x^{n+\Gamma} + \sum_{n=0}^{\infty} a_n (n+\Gamma) x^{n+\Gamma} - \sum_{n=0}^{\infty} a_n x^{n+\Gamma+1} - \sum_{n=0}^{\infty} a_n x^{n+\Gamma} = 0.
\]

This yields that

\[
\sum_{n=0}^{\infty} \left( a_n (n+\Gamma)^2 x^{n+\Gamma} - a_n \right) x^{n+\Gamma} = 0.
\]

Now

\[
x^{\Gamma - 1} q_0 \left[ x^{2-1} \right] + \sum_{n=1}^{\infty} x^n \left[ a_n \left( (n+\Gamma)^2 - 1 \right) - (n+\Gamma-1) a_{n-1} \right] = 0.
\]

Thus, \( \Gamma = \pm 1 \) are roots of indicial equation and

\[
q_n = \frac{(n+\Gamma-1) a_{n-1}}{(n+\Gamma)^2 - 1}, \quad n = 1, 2, \ldots, \quad \Gamma_1 = 1, \quad \Gamma_2 = -1, \quad |\Gamma_1 - \Gamma_2| = 2.
\]

Largest root \( (\Gamma_1 = 1) \), we have

\[
a_n = \frac{n}{n^2 + 2n} a_{n-1} = \frac{a_{n-1}}{n+2}.
\]

Thus

\[
a_1 = \frac{q_0}{3}, \quad a_2 = \frac{a_1}{4} = \frac{q_0}{3.4}, \quad a_3 = \frac{a_2}{5} = \frac{q_0}{3.45}
\]

Thus

\[
a_n = \frac{2 q_0}{(n+2)^{\Gamma_0}}, \quad n = 0, 1, 2, \ldots
\]

so

\[
y_1(x) = x \sum_{n=0}^{\infty} 2 q_0 \frac{x^n}{(n+2)^{\Gamma_0}} = 2 q_0 \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)^{\Gamma_0}}.
\]
or
\[
Y_1(x) = \frac{2 q_0}{x} \sum_{\eta = 2}^{\infty} \frac{x^\eta}{\eta_0^\eta} = \frac{2 q_0}{x} \left( \sum_{\eta = 2}^{\infty} \frac{x^\eta}{\eta_0^\eta} - 1 - x \right)
\]

we identify
\[
Y_1(x) = \frac{2 q_0}{x} \left[ e^x - 1 - x \right].
\]

Smaller root \( \Gamma_2 = -1 \). Then,
\[
q_\eta = \frac{(\eta - 2) q_{\eta-1}}{\eta_0^2 - 2\eta} = \frac{q_{\eta-1}}{\eta}, \quad \eta = 1, 2, \ldots
\]

thus
\[
q_\eta = \frac{q_0}{\eta_0^\eta}
\]
and
\[
Y_2(x) = x^{-1} \sum_{\eta = 0}^{\infty} \frac{q_0 x^\eta}{\eta_0^\eta}
\]

so
\[
Y_2(x) = \frac{q_0}{x} e^x.
\]

We have found \( Y_2 \) although \( \Gamma_1 - \Gamma_2 \) is not an integer.

Thus
\[
Y = \frac{C_1}{x} \left( e^x - 1 - x \right) + \frac{C_2}{x} e^x \quad \text{generally}
\]

Notice that this is (by linear combination)
\[
Y = \frac{d_1}{x} e^x + \frac{d_2}{x} \left( 1 + \frac{1}{x} \right)
\]

\( d_1, d_2 \) arbitrary.
Problem 2

\[ x^2 y'' + x y' + (x^2 - 1/4) y = 0. \]

(i) We put \( y = \sum_{n=0} a_n x^{n+\Gamma}. \) Then

\[ \sum_{n=0} (n+\Gamma)(n+\Gamma-1) a_n x^{n+\Gamma} + \sum_{n=0} (n+\Gamma) a_n x^{n+\Gamma} - \frac{1}{4} \sum_{n=0} a_n x^{n+\Gamma} + \sum_{n=0} a_{n-2} x^{n+\Gamma} = 0. \]

Shifting indices on last term:

\[ \sum_{n=0} (n+\Gamma)(n+\Gamma+1) a_n x^{n+\Gamma} + \frac{1}{4} \sum_{n=2} a_n x^{n+\Gamma} + \sum_{n=2} a_{n-2} x^{n+\Gamma} = 0. \]

Now extracting \( n=0,1 \) we get

\[ x^\Gamma \left( \Gamma^2 - \frac{1}{4} \right) a_0 + \left[ (\Gamma+1)^2 - \frac{1}{4} \right] a_1 x + \sum_{n=2} \left( \left( \Gamma + \frac{1}{4} \right)^2 - \frac{1}{4} \right) a_n + a_{n-2} \right) x^{\Gamma+n} = 0. \]

Thus if \( a_0 \) arbitrary \( \neq 0 \) then \( \Gamma = \pm 1/2 \) are roots of indicial equation. Then, \( a_1 = 0. \)

Thus \( a_0 = \frac{-a_{n-2}}{(n+\Gamma)^2 - 1/4}, \quad n = 2, 3, \ldots. \)

Take root \( \Gamma = 1/2 \) : Then \( a_1 = a_3 = a_5 = \ldots = 0. \)

We have \( a_n = \frac{-a_{n-2}}{n^2 + \Gamma} = \frac{-a_{n-2}}{n(n+1)} \)

We get \( a_2 = -\frac{a_0}{2 \cdot 3}, \quad a_4 = -\frac{a_2}{4 \cdot 5} = \frac{a_0}{2 \cdot 3 \cdot 4 \cdot 5}, \quad a_6 = -\frac{a_0}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \)

Thus \( q_{2n} = \frac{(1-\Gamma^2)^n a_0}{(2n+1)!} \) and \( n = 0, 1, 2, \ldots. \)

Thus given

\[ y_1(x) = x^{1/2} \sum_{n=0} a_{2n} x^{2n} = a_0 x^{1/2} \sum_{n=0} \frac{(-1)^n}{(2n+1)!} x^{2n}. \]
\[ y_1(x) = q_0 x^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \]

AND IDENTIFY \[ y_1(x) = q_0 x^{-1/2} \sin x. \]

TAKE ROOT \[ \Gamma_1 = -\frac{1}{2} \]

THEN \[ q_1 = q_3 = q_5 = \ldots = 0. \]

WE HAVE \[ q_n = -\frac{q_{n-2}}{n(n-1)}, \quad n = 2, \ldots \]

SO \[ q_2 = -\frac{q_0}{1 \cdot 2}, \quad q_4 = -\frac{q_2}{3 \cdot 4} = \frac{q_0}{1 \cdot 2 \cdot 3 \cdot 4}, \quad q_6 = -\frac{q_2}{1 \cdot 2 \cdot \ldots \cdot 6} \]

THUS \[ q_{2n} = \frac{(-1)^n q_0}{(2n)!}, \quad n = 0, 1, 2, \ldots \quad q_{2n+1} = 0, \quad n = 0, 1, 2, \ldots \]

THUS GIVE \[ y_2 = x^{-1/2} q_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \]

WHICH WE IDENTIFY AS \[ y_2(x) = x^{-1/2} q_0 \cos x. \]

A J SUCH THE GENERAL SOLUTION IS \[ y(x) = c_1 x^{-1/2} \cos x + c_2 x^{-1/2} \sin x. \]

NO TROUBLE EVENTHOUGH \( \Gamma_1 - \Gamma_2 = \) POSITIVE INTEGER.

(iii) NOW PUT \[ y = x^{-1/2} V \] IN \[ x^2 y'' + xy' + (x^2 - 1/4) y = 0. \]

WE CALCULATE \[ y' = -\frac{1}{2} x^{3/2} V + x^{-1/2} V', \quad y'' = x^{-1/2} V'' + \frac{3}{4} x^{-5/2} V - x^{-3/2} V' \]

SUBSTITUTE TO GET \[ x^2 \left( x^{-1/2} V'' - x^{3/2} V' + \frac{3}{4} x^{-5/2} V \right) + x \left( x^{-1/2} V - \frac{1}{2} x^{-3/2} V' \right) + (x^{3} - \frac{1}{4}) x^{-1/2} V = 0. \]

THUS GIVE \[ x^{3/2} V''' + V' \left[ -x^{4/2} + x^{1/2} \right] + V \left[ \frac{3}{4} x^{-1/2} - \frac{1}{2} x^{1/2} + \frac{3}{4} x^{3/2} - \frac{1}{4} x^{5/2} \right] \]

CANCELING TERMS: \[ x^{3/2} V''' + x^{3/2} V = 0 \rightarrow V''' + V = 0. \]

\[ V = c_1 \cos x + c_2 \sin x \quad \rightarrow \quad y = c_1 x^{-1/2} \cos x + c_2 x^{-1/2} \sin x. \]
Problem 3

We have

\[ x^2 y'' + 2xy' + \left( x^2 - \frac{5}{16} \right) y = 0. \]

Near \( x = 0 \) we have \( x^2 y'' + 2xy' - \frac{5}{16} y \approx 0. \) Put \( y = x^\Gamma \) so that

\[ \Gamma (\Gamma - 1) + 2\Gamma - \frac{5}{16} = 0 \rightarrow \Gamma^2 + \Gamma - \frac{5}{16} = 0 \rightarrow (\Gamma + \frac{5}{4})(\Gamma - \frac{1}{4}) = 0. \]

So \( \Gamma = -\frac{5}{4}, \Gamma = \frac{1}{4}. \) Thus we would look for

\[ y_1 = x^{\frac{1}{4}} \left[ 1 + a_1 x + a_2 x^2 + \ldots \right], \quad y_2 = x^{-\frac{5}{4}} \left[ 1 + b_1 x + b_2 x^2 + \ldots \right]. \]

And

\[ y = c_1 y_1 + c_2 y_2 = c_1 x^{\frac{1}{4}} \left[ 1 + a_1 x + \ldots \right] + c_2 x^{-\frac{5}{4}} \left[ 1 + b_1 x + b_2 x^2 + \ldots \right]. \]

Now we want

\[ \lim_{x \to 0^+} y = 0, \quad \lim_{x \to 0^+} x^{\frac{3}{4}} y' = 1. \]

The condition

\[ \lim_{x \to 0^+} y = 0 \rightarrow c_2 = 0. \]

Now

\[ \lim_{x \to 0^+} x^{\frac{3}{4}} y' = \lim_{x \to 0^+} x^{\frac{3}{4}} \left[ \frac{1}{4} c_1 x^{-\frac{3}{4}} + \ldots \right] = 1. \quad \text{Thus } c_1 = 4. \]

As such we need only find a solution of the form

\[ y = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}} \text{ with } \Gamma = \Gamma = \frac{1}{4} \text{ and } a_0 = 4. \]

This is the solution we want. We substitute to get

\[ \sum_{n=0}^{\infty} a_n \frac{(n+\frac{1}{2})(n+\frac{1}{2}-1)}{16} x^{n+\frac{1}{4}} + \sum_{n=0}^{\infty} 2a_0 (n+\frac{1}{2}) x^{n+\frac{1}{4}} = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}} + \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}} = 0. \]

We get

\[ \sum_{n=0}^{\infty} x^{n+\frac{1}{4}} \left[ a_0 \frac{n(n+1)}{2} x^{n+\frac{1}{4}} + \frac{5}{16} a_n \right] + \sum_{n=2}^{\infty} a_{n-2} x^{n+\frac{1}{4}} = 0. \]

Or

\[ \sum_{n=0}^{\infty} x^{n+\frac{1}{4}} \left[ (n+\frac{1}{2})(n+\frac{1}{2}+1) - \frac{5}{16} \right] a_n + \sum_{n=2}^{\infty} a_{n-2} x^{n+\frac{1}{4}} = 0. \]
\[ 0 = q_0 \left[ \frac{\Gamma(\Gamma+1)}{16} \right] x^{\Gamma} + q_1 \left[ (1+\Gamma)(2+\Gamma) - \frac{5}{16} \right] x^{\Gamma+1} + \sum_{n=2}^{\infty} x^{n+\Gamma} \left[ (n+\Gamma)(n+\Gamma+1) - \frac{5}{16} \right] q_n + q_{n-2} \]

Thus, \( \Gamma^2 + \Gamma - \frac{5}{16} = 0 \) \implies \( \Gamma_1 = -\frac{5}{4}, \Gamma_2 = \frac{1}{4} \)

We set \( \Gamma = \Gamma_2 = \frac{1}{4} \) and then \( q_1 \left[ (1 + \frac{1}{4})(2 + \frac{1}{4}) - \frac{5}{16} \right] = 0 \) \implies \( q_1 = 0 \).

The recursion is \( q_n = \frac{q_{n-2}}{(n+\frac{1}{4})(n+\frac{5}{4}) - \frac{5}{16}} \), \( n \geq 2 \).

So \( q_n = \frac{q_{n-2}}{n(n+\frac{3}{2})} \), \( n \geq 2 \).

We have \( q_1 = q_3 = q_5 = \ldots = 0 \).

Now \( q_2 = \frac{-q_0}{2 \left( \frac{7}{2} \right)} = -\frac{q_0}{7} \)

\( q_4 = \frac{-q_2}{4 \left[ \frac{11}{2} \right]} = -\frac{q_2}{22} = \frac{q_0}{7 \cdot 22} \)

We must take \( q_0 = 4 \) as explained on previous page. So

\[ y = x^{\frac{1}{4}} q_0 \left[ 1 - \frac{1}{7} x^1 + \frac{1}{7 \cdot 22} x^4 + \ldots \right] \] with \( q_0 = 4 \).

Or \[ y = 4 x^{\frac{1}{4}} - \frac{4}{7} x^{\frac{9}{4}} + \frac{2}{7 \cdot 11} x^{17/4} + \ldots \]
PROBLEM 4

\[ y'' + \frac{1}{\sin x} y' + \frac{(1-x)}{x^2} y = 0 \]

Multiply by \( x^2 \):

\[ x^2 y'' + x \cdot p(x) y' + q(x) y = 0 \]

with \( p = \frac{x}{\sin x}, \quad q = 1-x \).

Thus

\[ x^2 y'' + x p(x) y' + q(x) y = 0 \quad \text{as} \quad x \to 0 \]

Now, as \( x \to 0 \),

\[ \lim_{x \to 0} p(x) = \lim_{x \to 0} \left( \frac{x}{\sin x} \right) = 1, \quad \lim_{x \to 0} q(x) = 1. \]

Thus

\[ x^2 y'' + x y' + y = 0. \]

Put \( y = x^\Gamma \), so \( \Gamma (\Gamma - 1) + \Gamma + 1 = 0 \)

which gives \( \Gamma^2 + 1 = 0 \), so \( \Gamma = \pm i \). Therefore

\[ y = x^\Gamma = \cos \left( \ln x \right) + i \sin \left( \ln x \right) \]

Therefore, we derive a solution of the form

\[ y = x^{\frac{1}{2}} \sum_{n=0}^{\infty} C_n x^n \]

where \( C_n \) are complex constants to be found.

i.e., we have \( C_0 = a_0 + i b_0 \).

Now define

\[ y_1 = \text{RE} \left[ y \right] = \text{RE} \left[ x^{\frac{1}{2}} \sum_{n=0}^{\infty} C_n x^n \right] = \text{RE} \left[ (\cos (\ln x) + i \sin (\ln x)) \sum_{n=0}^{\infty} (a_n + i b_n) x^n \right] \]

\[ y_2 = \text{IM} \left[ y \right] = \text{IM} \left[ (\cos (\ln x) + i \sin (\ln x)) \sum_{n=0}^{\infty} (a_n + i b_n) x^n \right] \]

Thus yield

\[ y_1 = \cos \left( \ln x \right) \sum_{n=0}^{\infty} a_n x^n - \sin \left( \ln x \right) \sum_{n=0}^{\infty} b_n x^n \]

\[ y_2 = \cos \left( \ln x \right) \sum_{n=0}^{\infty} b_n x^n + \sin \left( \ln x \right) \sum_{n=0}^{\infty} a_n x^n \]

And general solution is \( y = d_1 y_1 + d_2 y_2 \). Equivalently, we can look for 2 independent solutions of the form

\[ \widetilde{y}_1 = \cos \left( \ln x \right) \sum_{n=0}^{\infty} \tilde{a}_n x^n \quad \text{and} \quad \widetilde{y}_2 = \sin \left( \ln x \right) \sum_{n=0}^{\infty} \tilde{b}_n x^n, \quad \tilde{a}_n, \tilde{b}_n \quad \text{real} \]

Then general solution is \( y = e_1 \widetilde{y}_1 + e_2 \widetilde{y}_2 \).
PROBLEM 5

\[ x^2 y'' + xy' + x^2 y = 0. \]

(i) We know one solution is \( y = J_0(x) \). We look for the second solution as \( y = J_0(x) \ln x + \tilde{y} \).

We substitute after calculating \( y' = \frac{1}{x} J_0 + J_0' \ln x + \tilde{y}' \) and \( y'' = -x^2 J_0 + 2x' J_0' + J_0'' \ln x + \tilde{y}'' \).

This gives \( x^2 \left[ -x^2 J_0 + 2x' J_0' + J_0'' \ln x + \tilde{y}'' \right] + x \left[ \frac{1}{x} J_0 + J_0' \ln x + \tilde{y}' \right] + x^2 \left[ J_0' \ln x + \tilde{y} \right] = 0. \)

Grouping terms:

\[ x^2 \tilde{y}'' + x \tilde{y}' + x^2 \tilde{y} + (\ln x) \left[ J_0'' x^2 + x J_0' + x^2 J_0 \right] + J_0' \ln x + \tilde{y} = 0. \]

Now \( x^2 J_0'' + x J_0' + x^2 J_0 = 0 \) gives that

\[ (\tilde{y}) \quad x^2 \tilde{y}'' + x \tilde{y}' + x^2 \tilde{y} = -2x J_0'(x). \]

(ii) Now we put \( J_0(x) \) into the differential equation:

\[ J_0(x) = 1 - \frac{1}{2} x^2 + \frac{1}{2 \cdot 4} x^4 - \frac{1}{2^2 \cdot 4^2} x^6 + \ldots. \]

Now put \( J_0' = -\frac{1}{2} x + \frac{1}{2 \cdot 4} x^3 - \frac{1}{2^2 \cdot 4^2} x^5 \)

Thus

\[ x^2 \tilde{y}'' + x \tilde{y}' + x^2 \tilde{y} = x^2 - \frac{1}{8} x^4 + \frac{x^6}{2^6 \cdot 3} + \ldots. \]

Now we put in a series

\[ \tilde{y} = \sum_{n=0}^{\infty} b_n x^n. \]

Since \( \tilde{y} \) is even in \( x \), we must have that \( \tilde{y} \) is even in \( x \).

We put

\[ \tilde{y} = b_0 + b_2 x^2 + b_4 x^4 + b_6 x^6 + \ldots. \]
We substitute:

\[ x^3 \left[ 2b_1 + 12b_4x^2 + 30b_6x^4 + \ldots \right] + x \left[ 2b_2x + 4b_4x^3 + 6b_6x^5 + \ldots \right] + x^4b_0 + x^4b_2 + b_4x^6 + \ldots = x^2 - \frac{1}{8}x^4 + \frac{x^6}{2^6.3} \]

Equating \( x^2 \):

\[ 2b_2 + 2b_2 + b_0 = 1 \]
\[ 12b_4 + 4b_4 + b_2 = -\frac{1}{8} \]
\[ 30b_6 + 6b_6 + b_4 = \frac{1}{2^6.3} \]

Now wlog we take \( b_0 = 0 \). Then \( b_2 = \frac{1}{4} \).

\[ 16b_4 = -\frac{1}{8} - \frac{1}{4} = -\frac{3}{8} \]

Thus, \( b_4 = -\frac{3}{8} \cdot \frac{1}{16} = -\frac{3}{2^7} \)

Then \( 36b_6 = \frac{1}{2^6.3} + \frac{3}{2^7} = \frac{2}{2^7.3} + \frac{9}{2^7.3} = \frac{11}{2^7.3} \)

Thus, \( b_6 = \frac{11}{2^7.3.36} = \frac{11}{2^9.3^3} \)

Thus, \( \tilde{y} = b_2x^2 + b_4x^4 + b_6x^6 = x^2/4 - \frac{3}{2^7}x^4 + \frac{11}{2^9.3^3}x^6 + \ldots \)

This gives:

\[ y = J_0(x) \ln x + \tilde{y} = (\ln x) \left[ 1 - \frac{x^2}{4} + \frac{x^4}{2^2.4^2} - \frac{1}{2^2.4^2}x^6 + \ldots \right] + \frac{x^2}{4} - \frac{3}{2^7}x^4 + \frac{11}{2^9.3^3}x^6 + \ldots \]