M316/257 HW 3

PROBLEM 1 FIND SERIES EXPANSIONS FOR TWO LINEARLY INDEPENDENT SOLUTIONS OF

$$y'' + \frac{(1-x)}{x} y' - \frac{y}{x^2} = 0 \quad \text{ABOUT } x = 0.$$

SHOW THAT ALTHOUGH THE ROOTS OF THE INDICIAL EQUATION DIFFER BY AN INTEGER, WE CAN NEVERTHELESS FIND TWO FROBENIUS SERIES SOLUTIONS.

PROBLEM 2 CONSIDER BESSEL'S EQUATION OF ORDER $1/2$ GIVEN BY

$$(*) \quad x^2 y'' + x y' + (x^2 - 1/4) y = 0.$$

(i) FIND FROBENIUS SERIES EXPANSIONS FOR TWO LINEARLY INDEPENDENT SOLUTIONS OF THE FORM $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. SHOW THAT ALTHOUGH THE ROOTS OF THE INDICIAL EQUATION DIFFER BY AN INTEGER WE CAN FIND y_1, y_2.

(ii) SHOW HOW TO FIND THE SOLUTIONS y_1 AND y_2 BY SUBSTITUTING $y = x^{-1/2} v$ INTO $(*)$ AND DERIVING THE EQUATION FOR v.

PROBLEM 3 FIND THE FIRST THREE NON-ZERO TERMS IN A FROBENIUS SERIES SOLUTION ABOUT $x = 0$ FOR THE SOLUTION TO THE SPHERICAL BESSEL'S EQUATION

$$x^2 y'' + 2x y' + (x^2 - 5/16) y = 0$$

WITH $\quad y(0) = 0, \quad \lim_{x \to 0^+} x^{3/4} y'(x) = 1.$

(HINT: USE THE INITIAL VALUES TO IDENTIFY WHICH OF THE TWO LINEARLY INDEPENDENT SOLUTIONS IS NEEDED).

PROBLEM 4 FIND THE FORMS OF TWO LINEARLY INDEPENDENT FROBENIUS SERIES EXPANSIONS ABOUT THE POINT $X=0$ FOR

$$y'' + \frac{1}{\sin x} y' + \frac{(1-x)}{x^2} y = 0$$

SUCH THAT THE SERIES ARE REAL-VALUED ON THE POSITIVE REAL AXIS $X>0$. DO NOT CALCULATE THE COEFFICIENTS IN THESE SERIES EXPANSIONS.

PROBLEM 5 IN THE NOTES ONE SOLUTION OF BESSEL'S EQUATION OF ORDER ZERO

$$x^2 y'' + x y' + x^2 y = 0$$

IS FOUND TO BE

$$J_0(x) = 1 - \frac{1}{2^2} x^2 + \frac{1}{2^2 4^2} x^4 - \frac{1}{2^2 4^2 6^2} x^6 + \dots = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m} (m!)^2} x^{2m}.$$

FIND A SECOND LINEARLY INDEPENDENT SOLUTION OF THE FORM

$$y_2 = J_0(x) \ln x + \hat{y}(x) \quad \text{AS FOLLOWS:}$$

(i) SHOW THAT IF y_2 IS TO SOLVE BESSEL'S EQUATION THEN $\hat{y}$ MUST SOLVE

$$(*) \quad x^2 \hat{y}'' + x \hat{y}' + x^2 \hat{y} = -2x J_0'(x).$$

(ii) FIND THE FIRST THREE NON-ZERO TERMS OF A SERIES EXPANSION $\hat{y} = \sum_{n=1}^{\infty} b_n x^n$ FOR THIS ODE FOR $\hat{y}$.

(HINT: PUT SERIES FOR $J_0(x)$ INTO THE RIGHT-HAND SIDE OF $(*)$).