Problem 1: Find series expansions for two linearly independent solutions of

\[ \frac{y''}{x} + \frac{(1-x)y'}{x^2} - \frac{y}{x^2} = 0 \] 

about \( x = 0 \).

Show that although the roots of the indicial equation differ by an integer, we can nevertheless find two Frobenius series solution.

Problem 2: Consider Bessel's equation of order \( \frac{1}{2} \) given by

\[ x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0. \]

(i) Find Frobenius series expansion for two linearly independent solutions of the form \( y = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \). Show that although the roots of the indicial equation differ by an integer, we can find \( y_1, y_2 \).

(ii) Show how to find the solutions \( y_1 \) and \( y_2 \) by substituting \( y = x^{-\frac{1}{2}}v \) into (i) and deriving the equation for \( v \).

Problem 3: Find the first three non-zero terms in a Frobenius series solution about \( x = 0 \) for the solution to the spherical Bessel's equation

\[ x^2y'' + 2xy' + (x^2 - \frac{5}{16})y = 0 \]

with \( y(0) = 0 \), \( \lim_{x \to 0^+} x^{3/4}y'(x) = 1 \).

(Hint: Use the initial value to identify which of the two linearly independent solutions is needed.)
**Problem 4** Find the form of two linearly independent Frobenius series expansions about the point \( x = 0 \) for
\[
\frac{y''}{\sin x} + \frac{1}{x} y' + \frac{(1-x)}{x^2} y = 0
\]
such that the series are real-valued on the positive real axis \( x > 0 \). Do not calculate the coefficients in these series expansions.

**Problem 5** In the notes, one solution of Bessel's equation of order zero
\[
x^2 y'' + x y' + x^2 y = 0
\]
is found to be
\[
J_0(x) = 1 - \frac{1}{2^2} x^2 + \frac{1}{2^2 4^2} x^4 - \frac{1}{2^2 4^2 6^2} x^6 + \ldots = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!^2} x^{2m}
\]
Find a second linearly independent solution of the form
\[
y_2 = J_0(x) \ln x + \tilde{y}(x)
\]
as follows:

(i) Show that if \( y_2 \) is to solve Bessel's equation then \( \tilde{y} \)
must solve
\[
(*) \quad x^2 \tilde{y}'' + x \tilde{y}' + x^2 \tilde{y} = -2x J_0'(x).
\]

(ii) Find the first three non-zero terms of a series expansion
\[
\tilde{y} = \sum_{n=1}^{\infty} b_n x^n
\]
for this ode for \( \tilde{y} \).

(Hint: Put series for \( J_0(x) \) into the right-hand side of (\( * \)) )