PROBLEM 1  FIND THE RADIUS OF CONVERGENCE OF \( \sum_{n=0}^{\infty} \frac{x^n}{n^n} \).

PROBLEM 2  FIND THE TAYLOR SERIES OF \( F(x) = \frac{1}{1-x} \) ABOUT \( x_0 = 2 \).

PROBLEM 3  CONSIDER \( L(y) = x y'' + y' + xy \).

Let \( y_1 \) and \( y_2 \) solve \( L(y_1) = 0 \), \( L(y_2) = 0 \) with \( y_1(1) = 1 \), \( y_1'(1) = 0 \) and \( y_2(1) = 0 \), \( y_2'(1) = 1 \). Find the first 4 terms in each of these two fundamental solutions by expanding in powers of \( (x-1) \).

PROBLEM 4  DO THE SAME AS IN PROBLEM 3 FOR \( L(y) = (1-x)y'' + xy' - y \) but now expanding around \( x = 0 \), i.e. \( y_1(0) = 1 \), \( y_1'(0) = 0 \), \( y_2(0) = 0 \), \( y_2'(0) = 1 \).

PROBLEM 5  FIND THE FIRST FIVE TERMS IN THE TAYLOR SERIES SOLUTION ABOUT \( x = 0 \) FOR

\( y'' + xy' + 2y = 0 \) WITH \( y(0) = 4 \) AND \( y'(0) = -1 \).

PROBLEM 6  DETERMINE A LOWER BOUND ON THE RADIUS OF CONVERGENCE OF TAYLOR SERIES SOLUTIONS ABOUT \( x = x_0 \) FOR EACH OF THE FOLLOWING:

(i) \( (x^2 - 2x - 3)y'' + xy' + 4y = 0 \) WITH \( x_0 = 4 \)

(ii) \( (1 + x^3)y'' + 4xy' + y = 0 \) WITH \( x_0 = 2 \).

PROBLEM 7  FIND THE FIRST 3 TERMS IN EACH OF TWO LINEARLY INDEPENDENT SOLUTIONS FOR

\( y'' + \sin x \ y = 0 \)  (HINT: RECALL \( \sin x = x - x^3/3! + x^5/5! + \ldots \))
The equation \( y'' - 2xy' + \lambda y = 0 \) on \( -\infty < x < \infty \) with \( \lambda \) a constant is called Hermite's equation.

(i) By substituting \( y = \sum_{n=0}^{\infty} q_n x^n \) show that

\[
q_{n+2} = \frac{(2n - \lambda)}{(n+2)(n+1)} q_n \\
\text{for } n = 2, 3, 4, \ldots
\]

(ii) When \( \lambda = 2m \) with \( m \) a non-negative integer show that the corresponding solution is a polynomial.

(iii) Find the corresponding polynomials when \( \lambda = 0, 2, 4, 6, 8, 10 \).