**Problem 1**
Find the solution to Laplace's equation
\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} = 0 \quad \text{in} \quad 0 < r \leq 1, \quad 0 < \phi < \pi
\]
with
\[
u(1, \phi) = 2 \sin\left(\frac{\phi}{2}\right) + 4 \sin\left(3\frac{\phi}{2}\right); \quad u \text{ bounded as } r \to 0
\]
and
\[
u(r, 0) = 0, \quad \nu(r, \pi) = 0.
\]

**Problem 2**
Solve
\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} = 0 \quad \text{in} \quad 0 \leq r \leq 2, \quad 0 < \phi < 2\pi
\]
with
\[
u(2, \phi) = 1 + 4 \cos^2 \phi; \quad u \text{ bounded as } r \to 0
\]
and
\[u, u_\phi \text{ are } 2\pi \text{ periodic in } \phi.
\]

**Problem 3**
Solve the exterior domain problem
\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} = 0 \quad \text{in} \quad 1 < r < \infty, \quad 0 < \phi < 2\pi
\]
with
\[
u(1, \phi) = 2 \cos^2 \phi; \quad u \text{ bounded as } r \to +\infty
\]
and
\[u, u_\phi \text{ are } 2\pi \text{ periodic in } \phi.
\]

**Problem 4**
Consider the eigenvalue problem
\[
\begin{cases}
\phi'' + \lambda \phi = 0 & \text{in} \quad 0 < x < L \\
\phi'(0) - \phi(L) = 0, \quad \phi(L) = 0 & \text{with} \quad L > 0.
\end{cases}
\]

(i) Prove that any eigenvalue must be real and positive.

(ii) Show graphically from solving (*) that there are an infinite number of positive eigenvalues.

(iii) Show how to solve the heat conduction problem
\[
ut = \nu_{xx} \quad \text{in} \quad 0 < x < L, \quad t > 0; \quad \text{with IC } \nu(x, 0) = \phi(x)
\]
and with \[
u(0, t) = u(0) - A \quad \text{and} \quad \nu(L, t) = B \quad \text{where} \quad A, B \text{ constant}.
\]
Problem 5. Solve the wave equation where the tension in the string varies with position:

\[ u_{tt} = c^2 \left[ x^2 u_x \right]_x \quad \text{in} \quad 1 \leq x \leq 2, \quad t > 0 \]

with IC: \( u(x, 0) = f(x), \quad u_t (x, 0) = g(x) \)

and BC: \( u(1, t) = 0, \quad u(2, t) = 0. \)

(Hint: you need to explicitly find the eigenvalues of the Euler equation \( x^2 \Phi'' + \Phi = 0 \) in \( 1 \leq x \leq 2 \))

with \( \Phi(1) = 0, \quad \Phi(2) = 0. \)

Problem 6. (Not to be turned in).

Suppose \( u_{xx} + u_{yy} = 0, \)

define \( x = \rho \cos \phi, \quad y = \rho \sin \phi \) and \( u(\rho, \phi) \equiv u[\rho \cos \phi, \rho \sin \phi]. \)

Show that \( u_{\rho \rho} + \frac{1}{\rho} u_{\rho} + \frac{1}{\rho^2} u_{\phi \phi} = 0. \)