MATH 257/316 HW 10

PROBLEM 1 FIND THE SOLUTION TO LAPLACE'S EQUATION

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = 0 \quad \text{IN} \quad 0 \le r \le 1, \quad 0 \le \varphi \le \pi$$

WITH $u(1, \varphi) = 2 \sin(\varphi/2) + 4 \sin(3\varphi/2)$; u BOUNDED AS $r \to 0$

AND $u(r, 0) = 0$, $u_\varphi(r, \pi) = 0$.

PROBLEM 2 SOLVE

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = 0 \quad \text{IN} \quad 0 \le r \le 2, \quad 0 \le \varphi \le 2\pi$$

WITH $u(2, \varphi) = 1 + 4 \cos^2 \varphi$; u BOUNDED AS $r \to 0$

AND u, u_φ ARE 2π PERIODIC IN φ.

PROBLEM 3 SOLVE THE EXTERIOR DOMAIN PROBLEM

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = 0 \quad \text{IN} \quad 1 \le r < \infty, \quad 0 \le \varphi \le 2\pi$$

WITH $u(1, \varphi) = 2 \cos^2 \varphi$; u BOUNDED AS $r \to +\infty$

u, u_φ ARE 2π PERIODIC IN φ.

PROBLEM 4 CONSIDER THE EIGENVALUE PROBLEM

$$(*) \begin{cases} \Phi'' + \lambda \Phi = 0 & \text{IN } 0 \le x \le L \\ \Phi'(0) - \Phi(0) = 0, \quad \Phi(L) = 0 & \text{WITH } L > 0. \end{cases}$$

(i) PROVE THAT ANY EIGENVALUE MUST BE REAL AND POSITIVE.

(ii) SHOW GRAPHICALLY FROM SOLVING $(*)$ THAT THERE ARE AN INFINITE NUMBER OF POSITIVE EIGENVALUES.

(iii) SHOW HOW TO SOLVE THE HEAT CONDUCTION PROBLEM

$u_t = u_{xx}$ IN $0 \le x \le L$, $t > 0$; WITH IC $u(x, 0) = f(x)$

AND WITH $u_x(0, t) = u(0, t) - A$ AND $u(L, t) = B$ WHERE A, B (CONSTANT).

PROBLEM 5 SOLVE THE WAVE EQUATION WHERE THE TENSION IN THE STRING VARIES WITH POSITION:

$$u_{tt} = c^2 \left[x^2 u_x \right]_x \quad \text{IN} \quad 1 \leq x \leq 2, \; t > 0$$

WITH IC: $u(x,0) = f(x), \quad u_t(x,0) = g(x)$

AND BC: $u(1,t) = 0, \quad u(2,t) = 0.$

(HINT: YOU NEED TO EXPLICITLY FIND THE EIGENVALUES OF THE EULER EQUATION $(x^2 \Phi')' + \lambda \Phi = 0$ IN $1 \leq x \leq 2$

WITH $\Phi(1) = 0, \quad \Phi(2) = 0$).

PROBLEM 6 (NOT TO BE TURNED IN).

SUPPOSE $u_{xx} + u_{yy} = 0$,

DEFINE $x = r\cos\varphi, \; y = r\sin\varphi$ AND $U(r,\varphi) \equiv u[r\cos\varphi, r\sin\varphi]$.

SHOW THAT $U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\varphi\varphi} = 0$.