MATH 215: MIDTERM 1: February 6th 2015

Closed Book and Notes. 50 minutes. Total 50 points

PROBLEM 1: (28 Points) Solve the following initial value problems for $y(t)$ and then determine the interval of existence of the solution:

i) $ty' = y^3$, with $y(1) = 1$.

ii) $y' + 2y/t = -2/t^3$, with $y(1) = 1$.

iii) $ty' - y = y^3/t^2$, with $y(1) = 1$. For this problem find a substitution for y that turns this problem into i) and another substitution that turns this problem into ii). Then use either one to solve iii).

iv) $y' = 3t^2 - (t^3 - y)^{1/2}$, with $y(1) = 1$. (**Hint:** Let $u = t^3 - y$ be a new dependent variable). Is the solution to iv) unique? If not, find two different solutions.

PROBLEM 2: (12 Points) Let $N = N(t)$ satisfy

$$\frac{dN}{dt} = f(N) \qquad \text{where} \qquad f(N) = -6 + 5N - N^2 .$$

i) Plot $f(N)$ versus N, and then N versus t for various initial conditions.

ii) If $N(0) = 5/2$ what is $\lim_{t\to\infty} N(t)$?

iii) If $N(0) = 1$ determine the time $T > 0$ for which $N(T) = 0$.

PROBLEM 3: (10 Points) A 120 litre tank initially contains 90 grams of salt dissolved in 90 litres of water. Brine containing 2 grams per litre flows into the tank at a rate of 4 litres per minute, and the well-stirred mixture flows out of the tank at the rate of 3 litres per minute.

i) Find a formula for the amount of salt in the tank at any time before the tank starts to overflow.

ii) How much salt is in the tank at the instant the tank begins to overflow.

MATH 215 MIDTERM SOLUTIONS

PROBLEM 1

(i) $ty' = y^3$ WITH $y(1) = 1$.

SEPARABLE: $y^{-3}\,dy = dt/t$

SO $-\frac{1}{2}y^{-2} = \ln t + C \quad (t>0)$

$y(1)=1 \longrightarrow \quad C = -1/2$.

SO $y^{-2} = 1 - 2\ln t$, which gives

$$y = \frac{1}{\sqrt{1-2\ln t}} \quad \text{FOR } 0<t<e^{1/2}.$$

(ii) $y' + 2y/t = -2/t^3$ WITH $y(1) = 1$

integrating factor $= e^{2\ln t} = t^2$.

SO $(t^2 y)' = -2/t$

$\longrightarrow \quad t^2 y = -2\ln t + C$ FOR $t>0$.

NOW $y(1) = 1 \longrightarrow \quad 1 = C$

SO $y = \dfrac{-2\ln t + 1}{t^2}$ FOR $t>0$.

(iii) $ty' - y = y^3/t^2$ WITH $y(1) = 1$

<u>FIRST SUBSTITUTION</u> $\quad y' - y/t = y^3/t^3$.

Let $v = y/t$ so $y = vt \rightarrow y' = v't + v$.

SO $v't + v = v + v^3 \rightarrow \quad v't = v^3$ WITH $v(1)=1$.

THIS IS PROBLEM (i) WITH $v \leftrightarrow y$. SO

$$v = \frac{1}{\sqrt{1-2\ln t}} \qquad 0<t<e^{1/2}$$

THIS GIVES $\quad y = \dfrac{t}{\sqrt{1-2\ln t}}$ FOR $0<t<e^{1/2}$.

<u>SECOND SUBSTITUTION</u>

$y' - y/t = y^3/t^3$ WITH $y(1) = 1$.

THIS IS A BERNOULLI EQUATION WITH $q = 3$

SO LET $y = v^{-1/2}$. THEN $y' = -\frac{1}{2}v^{-3/2}v'$.

SUBSTITUTE: $-\frac{1}{2}v^{-3/2}v' - \dfrac{v^{-1/2}}{t} = \dfrac{v^{-3/2}}{t^3}$.

THUS $\quad v' + \dfrac{2}{t}v = -\dfrac{2}{t^3}$ WITH $v(1) = 1$

THIS IS PROBLEM (ii) SO

$v = \dfrac{1-2\ln t}{t^2}$ WHICH GIVES

$$y = \frac{t}{\sqrt{1-2\ln t}} \quad \text{FOR } 0<t<e^{1/2}.$$

(iv) $y' = 3t^2 - (t^3 - y)^{1/2}$ WITH $y(1) = 1$.

WE MAKE A SUBSTITUTION $u = t^3 - y$.

THEN $\quad y = t^3 - u$ SO $y' = 3t^2 - u'$.

THIS GIVES $3t^2 - u' = 3t^2 - u^{1/2}$.

WE OBTAIN $u' = f(u) = u^{1/2}$ WITH $u(1) = 0$.

<u>NOTE</u>: SINCE $f_u = 1/2\, u^{-1/2}$ IS UNDEFINED AT $u = 0$, PICARD'S THEOREM DOES <u>NOT</u> APPLY. <u>NO</u> GUARANTEE OF UNIQUE SOLUTION.

- BY INSPECTION $u = 0 \longrightarrow y = t^3$.
- NOW $u^{-1/2}\,du = dt \rightarrow 2u^{1/2} = t + C$.

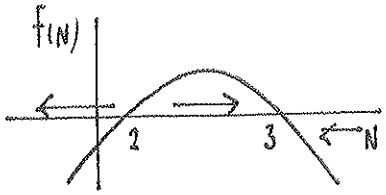
BUT $u(1) = 0$ SO $2u^{1/2} = t - 1 \rightarrow u = \dfrac{(t-1)^2}{4}$.

THUS $\quad y = t^3 - \dfrac{(t-1)^2}{4}$ IS <u>ANOTHER</u> SOLUTION.

PROBLEM 2

$$dN/dt = -(N^2 - 5N + 6) = -(N-2)(N-3) = f(N) \qquad N'' = f'(N)\, f(N)$$

(i)

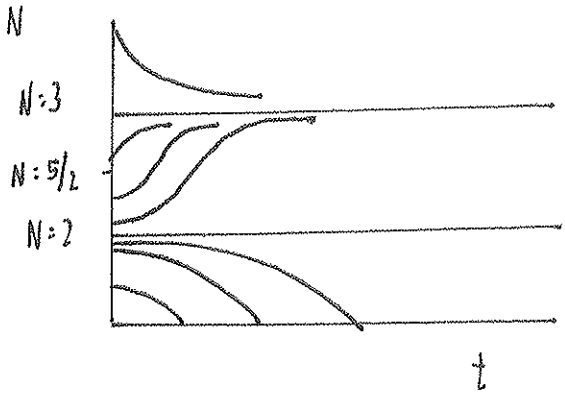


N	$N' = f(N)$	$N'' = f'(N)\, f(N)$
$-\infty < N < 2$	< 0	$(>0)(<0) = (<0)$
$2 < N < 5/2$	> 0	$(>0)(>0) = (>0)$
$5/2 < N < 3$	> 0	$(<0)(>0) = (<0)$
$N > 3$	< 0	$(<0)(<0) = (>0)$

EQ. POINTS ARE $N = 2$, $N = 3$.

$N = 3$ IS STABLE, $N = 2$ IS UNSTABLE.

$f'(N) = -2N + 5 = 0$ WHEN $N = 5/2$.



(ii) FROM THE PLOT IF $N(0) = 5/2$, THEN $\lim_{t \to \infty} N(t) = 3$.

(iii) IF $N(0) = 1$ FIND THE TIME $T > 0$ FOR WHICH $N(T) = 0$.

NOW $dN/dt = -(2-N)(3-N)$ SO $\dfrac{dN}{(N-2)(N-3)} = -dt$.

NOW $N(0) = 1$ SO $\displaystyle\int_1^N \frac{ds}{(s-2)(s-3)} = -t$. NOW LET $N(T) = 0$.

THEN $\displaystyle T = \int_0^1 \frac{ds}{(s-2)(s-3)}$ IS TIME FOR EXTINCTION.

LET'S INTEGRATE $\dfrac{A}{s-2} + \dfrac{B}{s-3} = \dfrac{1}{(s-2)(s-3)}$ SO $A(s-3) + B(s-2) = 1$

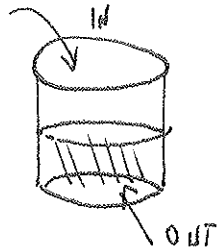
IF $s = 2 \to A = -1$

$s = 3 \to B = 1$

SO $\displaystyle T = \int_0^1 \left(\frac{-1}{s-2} + \frac{1}{s-3} \right) ds = -\log|s-2|\Big|_0^1 + \log|s-3|\Big|_0^1$

SO $\displaystyle T = \log\left|\frac{s-3}{s-2}\right|\Big|_0^1 = \log(2) - \log\left(\frac{3}{2}\right) = \log\left(4/3\right)$.

PROBLEM 3



let $Q(t)$ = AMOUNT OF SALT IN TANK

$$\frac{dQ}{dt} = \left(2 \ \frac{\text{gram}}{\text{litre}}\right)\left(4 \ \frac{\text{litres}}{\text{min}}\right) - \frac{Q}{V} \ \frac{3 \text{ litres}}{\text{min}}$$

where $V = 90 + (4-3)t = 90 + t$ LITRES.

THUS $Q' = 8 - \frac{3}{90+t} Q \qquad Q(0) = 90.$

FIRST ORDER ODE $Q' + \frac{3}{90+t} Q = 8$ integrating factor $e^{3\log(90+t)} = (90+t)^3.$

THUS $\left[(90+t)^3 Q\right]' = 8(90+t)^3.$

INTEGRATE $(90+t)^3 Q = 2(90+t)^4 + C. \qquad Q(0) = 90 \rightarrow 90^4 = 2 \cdot 90^4 + C$

$C = -90^4.$

(i) THUS $Q = 2(90+t) - (90)^4 / (90+t)^3.$ PROVIDED THAT $0 \le t \le 30$

AT $t = 30_{\text{MIN}}$ $V = 120$ litres which is capacity of tank.

(ii) AT $t = 30$ minutes, $Q = 240 - \frac{(90)^4}{(120)^3}.$

$$Q = 240 - \left(\frac{3}{4}\right)^4 120 = 120\left[2 - \frac{81}{256}\right]$$

$$Q = (120) \cdot \left(\frac{431}{256}\right) = 202.03 \text{ GRAMS salt.}$$