

POWER SERIES

(P1)

DEFINITION AN EXPRESSION OF THE FORM $\sum_{n=0}^{\infty} C_n X^n = C_0 + C_1 X + C_2 X^2 + \dots$

IS CALLED A POWER SERIES CENTERED AT $X=0$. AN EXPRESSION OF THE FORM

$\sum_{n=0}^{\infty} C_n (X-a)^n = C_0 + C_1 (X-a) + C_2 (X-a)^2 + \dots$ IS A POWER SERIES CENTERED AT $X=a$

THE GEOMETRIC SERIES $\sum_{n=0}^{\infty} X^n = 1 + X + \dots + X^n + \dots$ IS A POWER SERIES

CENTERED AT $X=0$. WE SHOWED EARLIER THAT IT CONVERGES TO $\frac{1}{1-X}$ FOR $|X| < 1$,

AND SO $\sum_{n=0}^{\infty} X^n = \frac{1}{1-X}$ FOR $-1 < X < 1$.

FOR A GENERAL POWER SERIES THERE ARE THREE POSSIBILITIES FOR $\sum_{n=0}^{\infty} C_n (X-a)^n$ WITH RESPECT TO CONVERGENCE.

THEOREM (CONVERGENCE OF POWER SERIES) FOR $\sum_{n=0}^{\infty} C_n (X-a)^n$, THERE ARE

THREE POSSIBILITIES:

1. THERE IS A POSITIVE NUMBER R SUCH THAT THE SERIES DIVERGES FOR $|X-a| > R$ BUT CONVERGES FOR $|X-a| < R$. THE SERIES MAY OR MAY NOT CONVERGE AT EITHER OF THE ENDPOINTS $X = a - R$ AND $X = a + R$.
2. THE SERIES CONVERGES FOR EVERY X (I.E. $R = \infty$)
3. THE SERIES CONVERGES AT $X = a$ AND DIVERGES ELSEWHERE (I.E. $R = 0$).

THE NUMBER R IS CALLED THE RADIUS OF CONVERGENCE OF THE SERIES AND THE SET OF ALL VALUES OF X FOR WHICH THE SERIES CONVERGES IS CALLED THE INTERVAL OF CONVERGENCE.

IN GENERAL, WE NEED TO EXAMINE (WHEN $R > 0$) THE ENDPOINTS $X = a + R$ AND $X = a - R$ TO SEE IF THE SERIES CONVERGES THERE.

A CONVENIENT WAY TO FIND THE RADIUS OF CONVERGENCE IS THE RATIO TEST.

FIND THE RADIUS OF CONVERGENCE R AND THE INTERVAL OF CONVERGENCE FOR EACH EXAMPLE BELOW.

EXAMPLE 1 $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} \dots$

SOLUTION DEFINE $a_n = (-1)^{n+1} \frac{x^n}{n}$ TEST FOR ABSOLUTE CONVERGENCE.

USING RATIO TEST, $|\frac{a_{n+1}}{a_n}| = |\frac{x^{n+1}}{n+1} \frac{n}{x^n}| = \frac{n}{n+1} |x| \rightarrow \lim_{n \rightarrow \infty} |\frac{a_{n+1}}{a_n}| = |x| \lim_{n \rightarrow \infty} (\frac{n}{n+1}) = |x|$

BY RATIO TEST, SERIES IS ABSOLUTELY CONVERGENT (AND HENCE CONVERGENT) IF $|x| < 1$ AND DIVERGENT IF $|x| > 1$. THUS, $R = 1$. NOW CHECK ENDPOINTS.

AT $x = 1$ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ THIS IS ALTERNATING SERIES THAT IS CONDITIONALLY CONVERGENT.

AT $x = -1$ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (-1)^n}{n} = - \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = - \sum_{n=1}^{\infty} \frac{1}{n} \rightarrow$ DIVERGES (P-SERIES WITH $p = 1$).

THUS, THE INTERVAL OF CONVERGENCE IS $-1 < x \leq 1$.

EXAMPLE 2 $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ DEFINE $a_n = \frac{x^n}{n!}$

SOLUTION TEST FOR ABSOLUTE CONVERGENCE.

$|\frac{a_{n+1}}{a_n}| = |\frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n}| = \frac{|x|}{n+1}$ FOR CONVERGENCE WE NEED BY RATIO TEST

THAT $\lim_{n \rightarrow \infty} |\frac{a_{n+1}}{a_n}| = |x| \lim_{n \rightarrow \infty} (\frac{1}{n+1}) < 1 \implies$ SINCE $\lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$ WE HAVE THAT

THIS HOLDS FOR ALL $|x|$. THUS, $R = \infty$ AND SERIES CONVERGES FOR $-\infty < x < \infty$.

EXAMPLE 3 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots$

SOLUTION LET $a_n = (-1)^{n+1} \frac{x^{2n-1}}{2n-1}$ TEST FOR ABSOLUTE CONVERGENCE.

$|\frac{a_{n+1}}{a_n}| = |\frac{x^{2(n+1)-1}}{2(n+1)-1} \frac{(2n-1)}{x^{2n-1}}| = |\frac{2n-1}{2n+1} x^2|$ NOW $\lim_{n \rightarrow \infty} |\frac{a_{n+1}}{a_n}| = |x|^2 \lim_{n \rightarrow \infty} |\frac{2n-1}{2n+1}| = |x|^2$

BY RATIO TEST, SERIES CONVERGES ABSOLUTELY IF $|x|^2 < 1 \implies |x| < 1$ AND DIVERGES IF $|x| > 1$. THUS, $R = 1$.

NOW TEST THE ENDPOINTS :

AT $x=1$ $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ THIS SERIES CONVERGES CONDITIONALLY BY ALT. SERIES TEST.

AT $x=-1$ $\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^{2n-1}}{2n-1} = - \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ SINCE $(-1)^{2n-1} = -1$ FOR ALL n . THIS CONVERGES CONDITIONALLY.

THE INTERVAL OF CONVERGENCE IS $-1 \leq x \leq 1$.

EXAMPLE 4 $\sum_{n=0}^{\infty} n! x^n = 1 + x + 2! x^2 + 3! x^3 + \dots$

SOLUTION LET $a_n = n! x^n$. WE CALCULATE $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = (n+1) |x|$.

BY RATIO TEST SERIES CONVERGES IF AND ONLY IF $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x| \lim_{n \rightarrow \infty} (n+1) < 1$.

SINCE $\lim_{n \rightarrow \infty} (n+1) = \infty \rightarrow$ SERIES CONVERGES ONLY WHEN $x=0$. THUS, $R=0$.

EXAMPLE 5 $\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{2^n (n+2)}$

SOLUTION LET $a_n = \frac{(-1)^n (x-1)^n}{2^n (n+2)}$. TEST FOR ABSOLUTE CONVERGENCE.

$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(x-1)^{n+1}}{2^{n+1} (n+3)} \cdot \frac{2^n (n+2)}{(x-1)^n} \right| = \frac{|x-1|}{2} \left(\frac{n+2}{n+3} \right)$

NOW $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x-1|}{2} \lim_{n \rightarrow \infty} \left(\frac{n+2}{n+3} \right) = \frac{|x-1|}{2}$.

THUS BY RATIO TEST SERIES CONVERGES ABSOLUTELY IF $\frac{|x-1|}{2} < 1$

AND DIVERGES IF $\frac{|x-1|}{2} > 1$. THUS $R=2$. NOW TEST ENDPOINTS $x=3, -1$.

AT $x=3$ $\sum_{n=1}^{\infty} \frac{(-1)^n (3-1)^n}{2^n (n+2)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+2}$ WHICH CONVERGES CONDITIONALLY.

AT $x=-1$ $\sum_{n=1}^{\infty} \frac{(-1)^n (-2)^n}{2^n (n+2)} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n+2} = \sum_{n=1}^{\infty} \frac{1}{n+2}$ WHICH DIVERGES.

THUS, THE INTERVAL OF CONVERGENCE IS $-1 < x \leq 3$.

NEXT, WE STATE A RESULT FOR TERMWISE DIFFERENTIATION AND INTEGRATION OF A POWER SERIES WITHIN ITS INTERVAL OF CONVERGENCE.

THEOREM (DIFFERENTIATION) IF $\sum_{n=0}^{\infty} c_n (x-a)^n$ CONVERGES IN $a-R < x < a+R$

FOR SOME $R > 0$ (POSSIBLY $R = \infty$), IT DEFINES A FUNCTION f VIA:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ FOR } a-R < x < a+R.$$

THEN WE HAVE,

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}, \quad f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2} \quad \square$$

THUS, WE ARE ALLOWED TO DIFFERENTIATE $f(x)$ TERM-BY-TERM IN $|x-a| < R$.

NOTICE THAT FOR $f'(x)$ THE SERIES STARTS AT $n=1$ SINCE $f(x) = c_0 + c_1(x-a) + \dots$

IMPLIES $f'(x) = c_1 + 2c_2(x-a) + \dots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$. SAME FOR $f''(x)$ WHEN WE START FROM $n=2$.

THEOREM (INTEGRATION) SUPPOSE $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ CONVERGES IN $a-R < x < a+R$.

THEN
$$\int f(x) dx = \sum_{n=0}^{\infty} \frac{c_n (x-a)^{n+1}}{n+1} + C \text{ IN } a-R < x < a+R \quad \square$$

HERE C IS A CONSTANT FIXED BY SOME VALUE OF $\int f dx$. THIS MEANS WE CAN INTEGRATE TERM BY TERM.

WE NOW CONSIDER SOME PROBLEMS RELATED TO THESE TWO KEY RESULTS.

WE RECALL THE GEOMETRIC SERIES

$$\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n \text{ FOR } |x| < 1.$$

NOW INTEGRATING WE GET

$$-\log(1-x) = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad -1 < x < 1$$

SINCE $\log(1) = 0$ WE EVALUATE AT $x=0$ TO GET $C = 0$.

THIS $\log(1-x) = - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -x - \frac{x^2}{2} - \frac{x^3}{3} \dots$ VALID FOR $-1 < x < 1$. (P5)

NOW REPLACE x BY $-x$ IN THIS FORMULA. WE GET

$$\log(1+x) = -(-x) - \frac{(-x)^2}{2} - \frac{(-x)^3}{3} \dots = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots$$

SO $\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ VALID FOR $-1 < x < 1$.

NOW DERIVE A POWER SERIES REPRESENTATION FOR $\arctan x$.

FIRST RECALL $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ FOR $|x| < 1$.

NOW LET $x = -y^2$. $\rightarrow \frac{1}{1+y^2} = 1 - y^2 + y^4 - y^6 + y^8 \dots$

REPLACE y BY x $\rightarrow \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 \dots$ FOR $|x| < 1$.

NOW INTEGRATE BOTH SIDES AND SET $\arctan 0 = 0$. THIS GIVES

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)} (-1)^n \text{ FOR } |x| < 1.$$

WE NOW CONSIDER A FEW SIMPLE EXAMPLES INVOLVING POWER SERIES MANIPULATION.

EXAMPLE THE INTERVAL OF CONVERGENCE OF $\sum_{n=0}^{\infty} n x^{n+1}$ IS $(-1, 1)$. (EASY TO SHOW WITH RATIO TEST AND EXAMINING ENDPOINTS AT $x = \pm 1$). FIND A COMPACT FORMULA FOR $\sum_{n=0}^{\infty} n x^{n+1}$.

SOLUTION WE WANT $S = \sum_{n=0}^{\infty} n x^{n+1} = x^2 + 2x^3 + 3x^4 + \dots$ FOR $-1 < x < 1$

WE START WITH $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ IN $-1 < x < 1$

DIFFERENTIATE WRT x : $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

MULTIPLY BY x^2 : $x^2/(1-x)^2 = x^2 + 2x^3 + 3x^4 + \dots$ ✓✓. THIS $S = \frac{x^2}{(1-x)^2}$.

EXAMPLE 2

FIND A SIMPLE COMPACT FORMULA FOR

$$\sum_{n=0}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \dots \quad (P6)$$

IN $-1 < x < 1$.

SOLUTION

RECALL FROM EXAMPLE 1 THAT

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

MULTIPLY BY x : $\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \dots$ IN $|x| < 1$

NOW DIFFERENTIATE: $\frac{d}{dx} \left[\frac{x}{(1-x)^2} \right] = 1 + 4x + 9x^2 + 16x^3 + \dots$

THUS $\frac{d}{dx} \left[\frac{x}{(1-x)^2} \right] = 1 + 4x + 9x^2 + 16x^3 + \dots$

MULTIPLY BY x : $x \left[\frac{d}{dx} \left(\frac{x}{(1-x)^2} \right) \right] = x + 4x^2 + 9x^3 + 16x^4 + \dots$ IN $|x| < 1$.

$$\begin{aligned} x \frac{d}{dx} \left(\frac{x}{(1-x)^2} \right) &= x \frac{d}{dx} [x(1-x)^{-2}] = x \left[(1-x)^{-2} + 2x(1-x)^{-3} \right] \\ &= x(1-x)^{-3} [(1-x) + 2x] \\ &= \frac{x(1+x)}{(1-x)^3} \end{aligned}$$

SO $\frac{x(1+x)}{(1-x)^3} = x + 4x^2 + 9x^3 + 16x^4 + \dots = \sum_{n=0}^{\infty} n^2 x^n$ IN $-1 < x < 1$.

EXAMPLE 3

CALCULATE $L = \lim_{x \rightarrow 0} \frac{2x - \log(1+2x)}{x^2}$

SOLUTION

RECALL $\frac{1}{1-x} = 1 + x + x^2 + \dots$ FOR $|x| < 1$.

REPLACE $x \mapsto -x$: $\frac{1}{1+x} = 1 - x + x^2 - \dots$

INTEGRATE $\log(1+x) = x - \frac{x^2}{2} + \dots$

REPLACE $x \mapsto 2x$ $\log(1+2x) = 2x - \frac{(2x)^2}{2} + \dots$

THUS $\frac{2x - \log(1+2x)}{x^2} = \frac{2x - (2x - \frac{(2x)^2}{2} + \dots)}{x^2} = \frac{2x^2}{x^2} + \dots$

THIS GIVES

$$\lim_{x \rightarrow 0} \frac{2x - \log(1+2x)}{x^2} = \lim_{x \rightarrow 0} \frac{2x^2 + \dots}{x^2} = 2.$$

(P7)

EXAMPLE 4

CALCULATE

$$\lim_{x \rightarrow \infty} x [\log(x+3) - \log(x+1)] = L$$

SOLUTION

FOR LARGE x

WE WRITE

$$\begin{aligned} \log(x+3) - \log(x+1) &= \log\left[x\left(1+\frac{3}{x}\right)\right] - \log\left[x\left(1+\frac{1}{x}\right)\right] \\ &= \log x + \log\left(1+\frac{3}{x}\right) - \log x - \log\left(1+\frac{1}{x}\right) \end{aligned}$$

$$\text{THIS WE HAVE } \log(x+3) - \log(x+1) = \log\left(1+\frac{3}{x}\right) - \log\left(1+\frac{1}{x}\right). \quad (*)$$

NOW RECALL

$$\frac{1}{1-h} = 1 + h + h^2 + \dots \text{ FOR } |h| < 1.$$

REPLACE $h \mapsto -h$:

$$\frac{1}{1+h} = 1 + (-h) + (-h)^2 + \dots = 1 - h + h^2 + \dots \text{ FOR } |h| < 1$$

$$\text{INTEGRATE: } \log(1+h) = h - \frac{h^2}{2} + \frac{h^3}{3} - \dots \text{ FOR } |h| < 1.$$

NOW SETTING $h = 3/x$ AND $h = 1/x$ FOR x LARGE WE GET FROM (*) THAT

$$\log(x+3) - \log(x+1) = \log\left(1+\frac{3}{x}\right) - \log\left(1+\frac{1}{x}\right) = \left(\frac{3}{x} + \dots\right) - \left(\frac{1}{x} + \dots\right) = \frac{2}{x}.$$

THIS

$$\lim_{x \rightarrow \infty} x (\log(x+3) - \log(x+1)) = \lim_{x \rightarrow \infty} x \left(\frac{2}{x} + \dots\right) = 2. \text{ SO } L = 2.$$

(HARD)

EXAMPLE 5

SHOW

THAT THE IMPROPER INTEGRAL

$$I = \int_1^{\infty} (\log(x^2+1) - 2\log x) dx$$

IS CONVERGENT AND CALCULATE ITS VALUE.

SOLUTION

LET $f(x) = \log(x^2+1) - 2\log x$. WE NEED TO SHOW THAT

$f(x) \rightarrow 0$ AS $x \rightarrow \infty$ "FAST ENOUGH".

$$\text{WE WRITE } f(x) = \log\left[x^2\left(1+\frac{1}{x^2}\right)\right] - 2\log x = \log x^2 + \log\left(1+\frac{1}{x^2}\right) - 2\log x$$

SO $f(x) = \log\left(1+\frac{1}{x^2}\right)$ EXACTLY.

NOW RECALL

$$\log(1+h) = h - \frac{h^2}{2} + \dots \text{ FOR } |h| < 1$$

THEREFORE

FOR $x \rightarrow +\infty$,

$$f(x) \approx \frac{1}{x^2} - \frac{1}{2x^4} + \dots$$

SINCE $f(x) \approx 1/x^p$ WITH $p=2$ FOR LARGE $x \rightarrow$

WE MUST HAVE THAT $\int_1^{\infty} f(x) dx < \infty$.

NEXT, WE CALCULATE DIRECTLY:

$$I = \lim_{L \rightarrow \infty} \int_1^L (\log(x^2+1) - 2 \log x) dx.$$

$$\begin{aligned} u &= \log x & du &= \frac{1}{x} dx \\ dv &= dx & v &= x \end{aligned}$$

USE INTEGRATION BY PARTS: IN FIRST INTEGRAL

$$\begin{aligned} u &= \log(x^2+1) \rightarrow & du &= \frac{2x}{x^2+1} dx \\ dv &= dx \rightarrow & v &= x \end{aligned}$$

$$\text{THU} \quad I = \lim_{L \rightarrow \infty} \left[x \log(x^2+1) \Big|_1^L - 2 \int_1^L \frac{x^2}{x^2+1} dx - 2(x \log x - x) \Big|_1^L \right]$$

$$= \lim_{L \rightarrow \infty} \left[x \log(x^2+1) \Big|_1^L - 2 \int_1^L \left(1 - \frac{1}{x^2+1}\right) dx - 2(x \log x - x) \Big|_1^L \right]$$

$$I = \lim_{L \rightarrow \infty} \left[x \log(x^2+1) \Big|_1^L - 2(x - \arctan x) \Big|_1^L - 2x \log x \Big|_1^L + 2x \Big|_1^L \right]$$

WE NOW CANCEL TERMS:

$$I = \lim_{L \rightarrow \infty} \left[L \log(L^2+1) - \log 2 - 2(L-1) + 2 \arctan x \Big|_1^L - 2L \log L + 2(L-1) \right]$$

$$= \lim_{L \rightarrow \infty} \left[L \log(L^2+1) - 2L \log L + 2 \arctan L - 2 \arctan(1) - \log 2 \right]$$

$$= \lim_{L \rightarrow \infty} \left[L \log\left(\frac{L^2+1}{L^2}\right) + 2 \arctan L - 2 \arctan 1 - \log 2 \right]$$

$$= \lim_{L \rightarrow \infty} \left[L \log\left(1 + \frac{1}{L^2}\right) + 2 \arctan L - 2 \arctan 1 - \log 2 \right]$$

$$= \lim_{L \rightarrow \infty} \left[L \left(\frac{1}{L^2} - \frac{1}{2L^4} + \dots \right) \right] + 2 \left(\frac{\pi}{2} - \frac{\pi}{4} \right) - \log 2$$

$$= 0 + \frac{\pi}{2} - \log 2 \quad \text{THU} \quad I = \frac{\pi}{2} - \log 2 \approx .87765$$